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Algorithmic and Analytical Approach to the Proximal Split Feasibility Problem and Fixed Point Problem

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Abstract. In this paper, we investigate the proximal split feasibility algorithm and fixed point problem in Hilbert spaces. We propose an iterative algorithm for finding a common element of the solution of the proximal split feasibility algorithm and fixed point of an L- Lipschitz pseudocontractive operator. We demonstrate that the considered algorithm converges strongly to a common point of the investigated problems under some mild conditions.

1. Introduction

It is well known that the split feasibility problem can be a model for numerous inverse problems where constraints are imposed on the solutions in the domain of a bounded linear operator as well as in its range ([2, 23, 31, 32, 38]). The prototype of the split feasibility problem proposed by Censor and Elfving [5] came out of phase retrieval problems and the intensity-modulated radiation therapy. Now, the split feasibility problem has a large number of specific applications in real world such as medical care, image reconstruction and signal processing, see [2, 6, 37, 39] for more details. Since then, the split problems have been studied extensively by many authors, see, for instance, [8, 15, 20, 22, 28].

In this paper, we are interested in the following more general case of the proximal split feasibility problem:

$$\min_{x^\dagger \in H_1} \{\varphi(x^\dagger) + \psi_\tau(Ax^\dagger)\}, \tag{1}$$

where H_1 and H_2 are two real Hilbert spaces, $\varphi : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi : H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper and lower semi-continuous convex functions, $A : H_1 \to H_2$ is a bounded linear operator and ψ_τ is the Moreau envelope of ψ of index $\tau(\tau > 0)$, known as the Moreau[9]-Yosida[33] approximate defined as

$$\psi_\tau(x) = \min_{y \in H_2} \Big\{\psi(y) + \frac{1}{2\tau}\|x - y\|^2\Big\}, x \in H_2. \tag{2}$$

Use Γ to denote the solution set of (1).

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Special case: Setting $\varphi = \delta_C$ and $\psi = \delta_Q$, the indicator functions of two nonempty closed convex sets $C \subset \mathcal{H}_1$ and $Q \in \mathcal{H}_2$, respectively, the proximal split feasibility problem (1) can be transformed into

$$\min_{x^\dagger \in \mathcal{H}_1} \{\delta_C(x^\dagger) + (\delta_Q)_\tau(Ax^\dagger)\},$$

which is equivalent to

$$\min_{x^\dagger \in C} \Big\{\frac{1}{2\tau}\|(I - proj_Q)(Ax^\dagger)\|^2\Big\}, \tag{3}$$

where $proj_Q : H_2 \to Q$ is the orthogonal projection.

It is obviously that solving (3) is exactly to solve the following split feasibility problem ([2, 7, 17]) of finding $x^\dagger$ such that

$$x^\dagger \in C \text{ and } Ax^\dagger \in Q. \tag{4}$$

Thus, the proximal split feasibility problem (1) includes the split feasibility problem (4) as a special case.

Let $\psi : H_2 \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous convex function. Recall that the subdifferential $\partial\psi(x^\dagger)$ of ψ at $x^\dagger$ is defined as follows

$$\partial\psi(x^\dagger) = \{x^* \in H_2 : \psi(u^\dagger) \geq \psi(x^\dagger) + \langle x^*, u^\dagger - x^\dagger\rangle, \forall u^\dagger \in H_2\}. \tag{5}$$

The proximity operator $prox_{\tau\psi}$ ([1, 10, 12]) of ψ is defined by

$$prox_{\tau\psi}(x) = \arg\min_{u\in H_2} \Big\{\psi(u) + \frac{1}{2\tau}\|u - x\|^2\Big\}, \ x \in H_2. \tag{6}$$

Based on (5) and (6), we can deduce

$$0 \in \partial\psi(x^\dagger) \Longleftrightarrow x^\dagger = prox_{\tau\psi}(x^\dagger). \tag{7}$$

By applying this equivalent relation (7), we can solve the proximal split feasibility problem (1) by using fixed point techniques. In fact, since the Yosida-approximate ψ_τ (2) is differentiable ([13]), we get $\partial(\psi_\tau(Ax^\dagger)) = A^*\big(\frac{I-prox_{\tau\psi}}{\tau}\big)(Ax^\dagger)$. So,

$$\partial(\varphi(x^\dagger) + \psi_\tau(Ax^\dagger)) = \partial\varphi(x^\dagger) + A^*\Big(\frac{I - prox_{\tau\psi}}{\tau}\Big)(Ax^\dagger). \tag{8}$$

Note that the optimality condition of (1) is $0 \in \partial\varphi(x^\dagger) + A^*\big(\frac{I-prox_{\tau\psi}}{\tau}\big)(Ax^\dagger)$, i.e.,

$$0 \in \mu\tau\partial\varphi(x^\dagger) + \mu A^*(I - prox_{\tau\psi})(Ax^\dagger). \tag{9}$$

From (7) and (9), we deduce

$$x^\dagger \text{ solves (1)} \Leftrightarrow x^\dagger = prox_{\mu\tau\varphi}(x^\dagger - \mu A^*(I - prox_{\tau\psi})(Ax^\dagger)). \tag{10}$$

By applying fixed point techniques (7) and (10), several iterative algorithms for solving the proximal split feasibility problem (1) have been proposed and the convergence analysis of the suggested algorithms were demonstrated, see [11, 14].

At the same time, we also interested in iterative algorithms for solving fixed point problems. It is well known that fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications ([24–27]. Especially, fixed point iterative algorithm comes to be useful in many mathematical formulations and theorems ([3, 4, 18, 19, 29, 30]). Often, approximations and solutions to iterative guess strategies utilized in dynamic engineering problems are sought using this method. Recently, fixed point algorithms have attracted so much attention, see [34–36, 40].

In the present paper, our main purpose is to investigate the proximal split feasibility algorithm (1) and fixed point problem. We propose an iterative algorithm for finding a common element of the solution of the proximal split feasibility algorithm (1) and fixed point of an L- Lipschitz pseudocontractive operator. We demonstrate that the considered algorithm converges strongly to a common point of the investigated problems.

2. Preliminaries

Let H_1 be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let $\phi : H_1 \to H_1$ be an operator. Denote the fixed point set of ϕ by $Fix(\phi)$. ϕ is said to be

(i) L-Lipschitz, if

$$\|\phi(x) - \phi(y)\| \le L\|x - y\|, \forall x, y \in H_1,$$

where $L > 0$ is a constant.

If $L = 1$, ϕ is called nonexpansive. If $L < 1$, ϕ is called L-contractive.

(ii) firmly nonexpansive if

$$\|\phi(x) - \phi(y)\|^2 \le \langle \phi(x) - \phi(y), x - y\rangle, \forall x, y \in H_1.$$

(iii) pseudocontractive if

$$\|\phi(x) - \phi(y)\|^2 \le \|x - y\|^2 + \|(I - \phi)x - (I - \phi)y\|^2, \ \forall x, y \in H_1.$$

Remark 2.1. *(i) The proximal operators $prox_{\tau\psi}$ and $prox_{\tau\varphi}$ are firmly nonexpansive; (ii) $I - prox_{\tau\psi}$ and $I - prox_{\tau\varphi}$ are also firmly nonexpansive. So,*

$$\|(I - prox_{\tau\varphi})(u) - (I - prox_{\tau\varphi})(v)\|^2 \le \langle (I - prox_{\tau\varphi})(u) - (I - prox_{\tau\varphi})(v), u - v\rangle, \forall u, v \in H_1. \tag{11}$$

and

$$\|(I - prox_{\tau\psi})(x) - (I - prox_{\tau\psi})(y)\|^2 \le \langle (I - prox_{\tau\psi})(x) - (I - prox_{\tau\psi})(y), x - y\rangle, \forall x, y \in H_2. \tag{12}$$

Let C be a nonempty closed convex subset of H_1. For any $x \in H_1$, there exists a unique nearest point $proj_C(x)$ in C satisfying

$$\|x - proj_C(x)\| \le \|x - y\|, \forall y \in C.$$

It is well known that $proj_C$ is firmly nonexpansive and has the following characterization

$$\langle x - proj_C(x), y - proj_C(x)\rangle \le 0 \tag{13}$$

for all $x \in H_1$ and $y \in C$.

In the sequel, we use the following symbols.

- $\rightharpoonup$ denotes the weak convergence.
- $\to$ denotes the strong convergence.
- $\omega_w(x_n)$ denotes the set of all weak cluster points of the sequence $\{x_n\}$, i.e., $\omega_w(x_n) = \{x^\dagger : \exists\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x^\dagger$ as $i \to \infty\}$.

Lemma 2.2 ([21, 42]). *Let H_1 be a real Hilbert space. Let $\phi : H_1 \to H_1$ be an L-Lipschitz pseudocontractive operator. Let γ be a constant in $(0, \frac{1}{\sqrt{1+L^2}+1})$. Then,*

$$\|\phi[(1 - \gamma)x + \gamma\phi(x)] - \hat{p}\|^2 \le \|x - \hat{p}\|^2 + (1 - \gamma)\|\phi[(1 - \gamma)x + \gamma\phi(x)] - x\|^2,$$

for all $x \in C$ and $\hat{p} \in Fix(\phi)$.

Lemma 2.3 ([41]). *Let H_1 be a real Hilbert space. Let $\phi : H_1 \to H_1$ be a continuous pseudocontractive operator. Then ϕ is demi-closed, namely,*

$$\left.\begin{array}{l} \{u_n\}_{n=0}^\infty \subset H_1 \\ u_n \rightharpoonup \tilde{u} \in H_1 \\ \phi(u_n) \to u^\dagger \end{array}\right\} \Rightarrow \phi(\tilde{u}) = u^\dagger.$$

Lemma 2.4 ([16]). *Let $\{z_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\}$ be three real number sequences. If $z_{n+1} \le (1 - \alpha_n)z_n + s_n, \forall n \ge 0$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\limsup_{n\to\infty} s_n/\alpha_n \le 0$ or $\sum_{n=1}^\infty |s_n| < \infty$, then $\lim_{n\to\infty} z_n = 0$.*

3. Main results

Now, we are in a position to propose an iterative algorithm for solving the proximal split feasibility algorithm (1) and fixed point problem of L-Lipschitz pseudocontractive operators.

Let H_1 and H_2 be two real Hilbert spaces. Let $\varphi : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $\psi : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semi-continuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* be the adjoint of A. Let $f : H_1 \to H_1$ be a λ-contractive operator. Let $\phi : H_1 \to H_1$ be an L-Lipschitz pseudocontractive operator with $L > 1$. Assume that $\Gamma \cap Fix(\phi) \neq \emptyset$. Let $\{\lambda_n\} \subset (0, +\infty)$, $\{\varsigma_n\} \subset (0, 1)$, $\{\gamma_n\} \subset (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\vartheta_n\} \subset (0, 1)$ be five real number sequences.

Now, we present our algorithm below.

Algorithm 3.1. *Let $x_0 \in H_1$ be an initial value. Set $n = 0$.*

Step 1. For known x_n, compute

$$w_n = A^*(I - prox_{\tau\psi})Ax_n + (I - prox_{\tau\varphi})x_n. \tag{14}$$

If $w_n = 0$, then set $z_n = x_n$ and go to Step 2. Otherwise, compute

$$z_n = x_n - \frac{\lambda_n(u_n + v_n)}{\|w_n\|^2} w_n, \tag{15}$$

where $u_n = \frac{1}{2}\|(I - prox_{\tau\psi})Ax_n\|^2$ and $v_n = \frac{1}{2}\|(I - prox_{\tau\varphi})x_n\|^2$.

Step 2. Compute

$$y_n = (1 - \varsigma_n)z_n + \varsigma_n\phi[(1 - \gamma_n)z_n + \gamma_n\phi(z_n)]. \tag{16}$$

Step 3. Compute

$$x_{n+1} = (1 - \vartheta_n)[\alpha_n f(x_n) + (1 - \alpha_n)x_n] + \vartheta_n y_n. \tag{17}$$

Step 4. Set $n := n + 1$ and return to Step 1.

Proposition 3.2. *If $w_n = 0$, then $x_n \in Fix(prox_{\tau\varphi})$ and $Ax_n \in Fix(prox_{\tau\psi})$, i.e., $x_n \in \Gamma$.*

Proof. Let $\tilde{x} \in \Gamma$. Then, we have $\tilde{x} = prox_{\tau\varphi}\tilde{x}$ and $A\tilde{x} = prox_{\tau\psi}A\tilde{x}$. Since $I - prox_{\tau\psi}$ and $I - prox_{\tau\varphi}$ are firmly-nonexpansive, applying (11) and (12), we have

$$\begin{aligned}\langle (I - prox_{\tau\varphi})x_n, x_n - \tilde{x}\rangle &= \langle (I - prox_{\tau\varphi})x_n - (I - prox_{\tau\varphi})\tilde{x}, x_n - \tilde{x}\rangle \\ &\geq \|(I - prox_{\tau\varphi})x_n\|^2,\end{aligned} \tag{18}$$

and

$$\begin{aligned}\langle (I - prox_{\tau\psi})Ax_n, Ax_n - A\tilde{x}\rangle &= \langle (I - prox_{\tau\psi})Ax_n - (I - prox_{\tau\psi})A\tilde{x}, Ax_n - A\tilde{x}\rangle \\ &\geq \|(I - prox_{\tau\psi})Ax_n\|^2.\end{aligned} \tag{19}$$

From (14), we have

$$\begin{aligned}\langle w_n, x_n - \tilde{x}\rangle &= \langle A^*(I - prox_{\tau\psi})Ax_n + (I - prox_{\tau\varphi})x_n, x_n - \tilde{x}\rangle \\ &= \langle A^*(I - prox_{\tau\psi})Ax_n, x_n - \tilde{x}\rangle + \langle (I - prox_{\tau\varphi})x_n, x_n - \tilde{x}\rangle \\ &= \langle (I - prox_{\tau\psi})Ax_n, Ax_n - A\tilde{x}\rangle + \langle (I - prox_{\tau\varphi})x_n, x_n - \tilde{x}\rangle.\end{aligned} \tag{20}$$

This together with (18)-(20) implies that

$$\|(I - prox_{\tau\varphi})x_n\|^2 + \|(I - prox_{\tau\psi})Ax_n\|^2 \leq \langle w_n, x_n - \tilde{x}\rangle. \tag{21}$$

If $w_n = 0$, then $\langle w_n, x_n - \tilde{x}\rangle = 0$. It follows from (21) that

$$(I - prox_{\tau\varphi})x_n = 0 \text{ and } (I - prox_{\tau\psi})Ax_n = 0.$$

Therefore, $x_n \in Fix(prox_{\tau\varphi})$ and $Ax_n \in Fix(prox_{\tau\psi})$, i.e., $x_n \in \Gamma$. □

Theorem 3.3. *Suppose that the sequences $\{\lambda_n\}$, $\{\varsigma_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$ and $\{\vartheta_n\}$ satisfy the following conditions:*

(C1): $\liminf_{n\to\infty} \lambda_n(4-\lambda_n) > 0$;

(C2): $0 < \underline{\varsigma} < \varsigma_n < \overline{\varsigma} < \gamma_n < \overline{\gamma} < \frac{1}{\sqrt{1+L^2}+1} (\forall n \geq 0)$;

(C3): $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(C4): $0 < \liminf_{n\to\infty} \vartheta_n \leq \limsup_{n\to\infty} \vartheta_n < 1$.

Then sequence $\{x_n\}$ generated by Algorithm 3.1 strongly converges to the solution $p^\dagger = proj_{\Gamma\cap Fix(\phi)}(f(p^\dagger))$.

Proof. Pick up any $x^* \in \Gamma \cap Fix(\phi)$. By (15), we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|x_n - x^* - \frac{\lambda_n(u_n+v_n)}{\|w_n\|^2} w_n\| \\ &= \|x_n - x^*\|^2 - 2\frac{\lambda_n(u_n+v_n)}{\|w_n\|^2}\langle w_n, x_n - x^*\rangle + \frac{\lambda_n^2(u_n+v_n)^2}{\|w_n\|^2}. \end{aligned} \tag{22}$$

According to (21), we have

$$2(u_n+v_n) \leq \langle w_n, x_n - x^*\rangle. \tag{23}$$

Combining (22) and (23) to get

$$\begin{aligned} \|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \frac{4\lambda_n(u_n+v_n)^2}{\|w_n\|^2} + \frac{\lambda_n^2(u_n+v_n)^2}{\|w_n\|^2} \\ &= \|x_n - x^*\|^2 - \lambda_n(4-\lambda_n)\frac{(u_n+v_n)^2}{\|w_n\|^2} \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{24}$$

For any $u, v \in H_1$ and $\varsigma \in \mathbb{R}$, we have

$$\|\varsigma u + (1-\varsigma)v\|^2 = \varsigma\|u\|^2 + (1-\varsigma)\|v\|^2 - \varsigma(1-\varsigma)\|u-v\|^2. \tag{25}$$

Using (16) and (25), we obtain

$$\begin{aligned} \|y_n - x^*\|^2 &= \|(1-\varsigma_n)(z_n - x^*) + \varsigma_n(\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - x^*)\|^2 \\ &= (1-\varsigma_n)\|z_n - x^*\|^2 + \varsigma_n\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - x^*\|^2 \\ &\quad - \varsigma_n(1-\varsigma_n)\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - z_n\|^2. \end{aligned} \tag{26}$$

Applying Lemma 2.2, we have

$$\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - x^*\|^2 \leq (1-\gamma_n)\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - z_n\|^2 + \|z_n - x^*\|^2. \tag{27}$$

It follows from (26), (27) and condition ($C2$) that

$$\begin{aligned} \|y_n - x^*\|^2 &\leq (1-\varsigma_n)\|z_n - x^*\|^2 - \varsigma_n(1-\varsigma_n)\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - z_n\|^2 \\ &\quad + \varsigma_n(\|z_n - x^*\|^2 + (1-\gamma_n)\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - z_n\|^2) \\ &= \|z_n - x^*\|^2 + \varsigma_n(\varsigma_n - \gamma_n)\|\phi[(1-\gamma_n)z_n + \gamma_n\phi(z_n)] - z_n\|^2 \\ &\leq \|z_n - x^*\|^2. \end{aligned} \tag{28}$$

Set $q_n = \alpha_n f(x_n) + (1-\alpha_n)x_n$ for all $n \geq 0$. Then, we have

$$\begin{aligned} \|q_n - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(x_n - x^*)\| \\ &\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq [1-(1-\lambda)\alpha_n]\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|. \end{aligned} \tag{29}$$

Thus, from (17), (24), (28) and (29), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|(1-\vartheta_n)(q_n-x^*)+\vartheta_n(y_n-x^*)\|\\ &\le (1-\vartheta_n)\|q_n-x^*\|+\vartheta_n\|y_n-x^*\|\\ &\le \alpha_n(1-\vartheta_n)\|f(x^*)-x^*\|+[1-(1-\lambda)\alpha_n(1-\vartheta_n)]\|x_n-x^*\|\\ &\le \max\{\frac{\|f(x^*)-x^*\|}{1-\lambda},\|x_n-x^*\|\}.\end{aligned}$$

Then, $\|x_n-x^*\|\le\max\{\frac{\|f(x^*)-x^*\|}{1-\lambda},\|x_0-x^*\|\}$ and $\{x_n\}$ is bounded. Subsequently, $\{f(x_n)\}$, $\{Ax_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded.

Observe that

$$\begin{aligned}\|q_n-x^*\| &= \|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(x_n-x^*)\|^2\\ &= (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f(x_n)-f(x^*),x_n-x^*\rangle+\alpha_n^2\|f(x_n)-x^*\|^2\\ &\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,x_n-x^*\rangle\\ &\le (1-\alpha_n)^2\|x_n-x^*\|^2+2\alpha_n\lambda\|x_n-x^*\|^2+\alpha_n^2\|f(x_n)-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,x_n-x^*\rangle\\ &= [1-2(1-\lambda)\alpha_n]\|x_n-x^*\|^2+\alpha_n^2(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,x_n-x^*\rangle.\end{aligned}\tag{30}$$

and

$$\|x_{n+1}-x^*\|^2\le(1-\vartheta_n)\|q_n-x^*\|^2+\vartheta_n\|y_n-x^*\|^2.\tag{31}$$

On account of (24), (28), (30) and (31), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\vartheta_n)\Big\{[1-2(1-\lambda)\alpha_n]\|x_n-x^*\|^2+\alpha_n^2(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)\\ &\quad+2\alpha_n(1-\alpha_n)\langle f(x^*)-x^*,x_n-x^*\rangle\Big\}+\vartheta_n\Big\{\|x_n-x^*\|^2-\lambda_n(4-\lambda_n)\frac{(u_n+v_n)^2}{\|w_n\|^2}\\ &\quad+\varsigma_n(\varsigma_n-\gamma_n)\|\phi[(1-\gamma_n)z_n+\gamma_n\phi(z_n)]-z_n\|^2\Big\}\\ &= [1-2(1-\lambda)(1-\vartheta_n)\alpha_n]\|x_n-x^*\|^2+2(1-\lambda)(1-\vartheta_n)\alpha_n\\ &\quad\times\Big\{\frac{\alpha_n}{2(1-\lambda)}(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)+\frac{1-\alpha_n}{1-\lambda}\langle f(x^*)-x^*,x_n-x^*\rangle\\ &\quad-\frac{\vartheta_n\lambda_n(4-\lambda_n)}{2(1-\lambda)(1-\vartheta_n)}\frac{(u_n+v_n)^2}{\alpha_n\|w_n\|^2}+\frac{\vartheta_n\varsigma_n(\varsigma_n-\gamma_n)}{2(1-\lambda)(1-\vartheta_n)}\frac{\|\phi[(1-\gamma_n)z_n+\gamma_n\phi(z_n)]-z_n\|^2}{\alpha_n}\Big\}.\end{aligned}\tag{32}$$

Set $\delta_n=\|x_n-z\|^2$, $\mu_n=2(1-\lambda)(1-\vartheta_n)\alpha_n$ and

$$\begin{aligned}\sigma_n &= \frac{\alpha_n}{2(1-\lambda)}(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)+\frac{1-\alpha_n}{1-\lambda}\langle f(x^*)-x^*,x_n-x^*\rangle\\ &\quad-\frac{\vartheta_n\lambda_n(4-\lambda_n)}{2(1-\lambda)(1-\vartheta_n)}\frac{(u_n+v_n)^2}{\alpha_n\|w_n\|^2}+\frac{\vartheta_n\varsigma_n(\varsigma_n-\gamma_n)}{2(1-\lambda)(1-\vartheta_n)}\frac{\|\phi[(1-\gamma_n)z_n+\gamma_n\phi(z_n)]-z_n\|^2}{\alpha_n}.\end{aligned}\tag{33}$$

for all $n\ge1$.

By virtue of (32) and (33), we obtain

$$\delta_{n+1}\le(1-\mu_n)\delta_n+\mu_n\sigma_n, n\ge1.\tag{34}$$

Taking into account (33), we get

$$\begin{aligned}\sigma_n &\le \frac{\alpha_n}{2(1-\lambda)}(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)+\frac{1-\alpha_n}{1-\lambda}\langle f(x^*)-x^*,x_n-x^*\rangle\\ &\le \frac{\alpha_n}{2(1-\lambda)}(\|x_n-x^*\|^2+\|f(x_n)-x^*\|^2)+\frac{1-\alpha_n}{1-\lambda}\|f(x^*)-x^*\|\|x_n-x^*\|.\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, it follows that $\limsup_{n\to\infty}\sigma_n < +\infty$. Next we show $\limsup_{n\to\infty}\sigma_n \geq -1$. If not, we have $\limsup_{n\to\infty}\sigma_n < -1$. Then, there exists positive integer N_0 such that $\sigma_n \leq -1$ for all $n \geq N_0$. In the light of (34), we conclude

$$\delta_{n+1} \leq \delta_n - \mu_n, \forall n \geq N_0.$$

It follows that

$$\delta_{n+1} \leq \delta_{N_0} - \sum_{k=N_0}^{n} \mu_k. \tag{35}$$

Note that $\sum_{n=0}^{\infty}\mu_n = \sum_{n=0}^{\infty} 2(1-\lambda)(1-\vartheta_n)\alpha_n = +\infty$. In (35), taking $\limsup_{n\to\infty}$, we have

$$\limsup_{n\to\infty}\delta_{n+1} \leq \delta_{N_0} - \lim_{n\to\infty}\sum_{k=N_0}^{n}\mu_k = -\infty,$$

which results in a contradiction. Thus,

$$-1 \leq \limsup_{n\to\infty}\sigma_n < +\infty.$$

Hence, $\limsup_{n\to\infty}\sigma_n$ exists. In the meantime, the sequence $\{x_n\}$ is bounded. We take a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z^\dagger (k\to\infty)$ and

$$\begin{aligned}
\limsup_{n\to\infty}\sigma_n &= \lim_{k\to\infty}\sigma_{n_k}\\
&= \lim_{k\to\infty}\Big[\frac{\alpha_{n_k}}{2(1-\lambda)}(\|x_{n_k}-x^*\|^2 + \|f(x_{n_k})-x^*\|^2) + \frac{1-\alpha_{n_k}}{1-\lambda}\langle f(x^*)-x^*, x_{n_k}-x^*\rangle\\
&- \frac{\vartheta_{n_k}\lambda_{n_k}(4-\lambda_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{(u_{n_k}+v_{n_k})^2}{\alpha_{n_k}\|w_{n_k}\|^2} + \frac{\vartheta_{n_k}\varsigma_{n_k}(\varsigma_{n_k}-\gamma_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{\|\phi[(1-\gamma_{n_k})z_{n_k}+\gamma_{n_k}\phi(z_{n_k})]-z_{n_k}\|^2}{\alpha_{n_k}}\Big]\\
&= \lim_{k\to\infty}\Big[\frac{1}{1-\lambda}\langle f(x^*)-x^*, z^\dagger - x^*\rangle - \frac{\vartheta_{n_k}\lambda_{n_k}(4-\lambda_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{(u_{n_k}+v_{n_k})^2}{\alpha_{n_k}\|w_{n_k}\|^2}\\
&- \frac{\vartheta_{n_k}\varsigma_{n_k}(\gamma_{n_k}-\varsigma_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{\|\phi[(1-\gamma_{n_k})z_{n_k}+\gamma_{n_k}\phi(z_{n_k})]-z_{n_k}\|^2}{\alpha_{n_k}}\Big],
\end{aligned} \tag{36}$$

which implies that

$$\lim_{k\to\infty}\frac{\vartheta_{n_k}\lambda_{n_k}(4-\lambda_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{(u_{n_k}+v_{n_k})^2}{\alpha_{n_k}\|w_{n_k}\|^2} \text{ exists} \tag{37}$$

and

$$\lim_{k\to\infty}\frac{\vartheta_{n_k}\varsigma_{n_k}(\gamma_{n_k}-\varsigma_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})}\frac{\|\phi[(1-\gamma_{n_k})z_{n_k}+\gamma_{n_k}\phi(z_{n_k})]-z_{n_k}\|^2}{\alpha_{n_k}} \text{ exists.} \tag{38}$$

Since $\liminf_{k\to\infty}\frac{\vartheta_{n_k}\lambda_{n_k}(4-\lambda_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})} > 0$, $\liminf_{k\to\infty}\frac{\vartheta_{n_k}\varsigma_{n_k}(\gamma_{n_k}-\varsigma_{n_k})}{2(1-\lambda)(1-\vartheta_{n_k})} > 0$ and $\lim_{k\to\infty}\alpha_{n_k} = 0$, it follows from (37) and (38) that

$$\lim_{k\to\infty}\frac{(u_{n_k}+v_{n_k})^2}{\|w_{n_k}\|^2} = 0 \tag{39}$$

and

$$\lim_{k\to\infty}\|\phi[(1-\gamma_{n_k})z_{n_k}+\gamma_{n_k}\phi(z_{n_k})]-z_{n_k}\|^2 = 0. \tag{40}$$

Since w_{n_k} is bounded, by (39), we obtain $\lim_{k\to\infty}(u_{n_k}+v_{n_k})=0$. Therefore,

$$\lim_{k\to\infty} u_{n_k} = \lim_{k\to\infty} v_{n_k} = 0. \tag{41}$$

The weak lower semicontinuity of the norm gives

$$0 \le \|(I - prox_{\tau\varphi})z^{\dagger}\| \le \liminf_{k\to\infty} \|(I - prox_{\tau\varphi})x_{n_k}\| = 0,$$

and

$$0 \le \|(I - prox_{\tau\psi})Az^{\dagger}\| \le \liminf_{k\to\infty} \|(I - prox_{\tau\psi})Ax_{n_k}\| = 0.$$

Thus, we conclude that $z^{\dagger} \in Fix(prox_{\tau\varphi})$ and $Az^{\dagger} \in Fix(prox_{\tau\psi})$, i.e., $z^{\dagger} \in \Gamma$.

By the L-Lipschit continuity of ϕ, we derive

$$\begin{aligned}\|\phi(x_{n_i}) - x_{n_i}\| &\le \|\phi(x_{n_i}) - \phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})]\| + \|\phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})] - x_{n_i}\| \\ &\le L\gamma_{n_i}\|\phi(x_{n_i}) - x_{n_i}\| + \|\phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})] - x_{n_i}\|,\end{aligned}$$

which leads to

$$\|\phi(x_{n_i}) - x_{n_i}\| \le \frac{1}{1-L\gamma_{n_i}}\|\phi[(1-\gamma_{n_i})x_{n_i} + \gamma_{n_i}\phi(x_{n_i})] - x_{n_i}\|.$$

Which together with (32) implies that

$$\lim_{i\to\infty} \|\phi(x_{n_i}) - x_{n_i}\| = 0.$$

At the same time, noting that $x_{n_i} \rightharpoonup z^{\dagger}$, by Lemma 2.3 and the last equality, we deduce that $z^{\dagger} \in Fix(\phi)$. Therefore, $z^{\dagger} \in \Gamma \cap Fix(\phi)$. So, $\omega_w(x_n) \subset \Gamma \cap Fix(\phi)$.

With the help of (36), we have

$$\limsup_{n\to\infty} \sigma_n = \lim_{k\to\infty} \sigma_{n_k} \le \frac{1}{1-\lambda}\langle f(p^{\dagger}) - p^{\dagger}, z^{\dagger} - p^{\dagger}\rangle \le 0. \tag{42}$$

From (32), we obtain

$$\|x_{n+1} - p^{\dagger}\|^2 \le [1 - 2(1-\lambda)(1-\vartheta_n)\alpha_n]\|x_n - p^{\dagger}\|^2 + 2(1-\lambda)(1-\vartheta_n)\alpha_n \times \frac{1-\alpha_n}{1-\lambda}\langle f(p^{\dagger}) - p^{\dagger}, x_n - p^{\dagger}\rangle. \tag{43}$$

Finally, applying Lemma 2.4 to (43) to deduce that $x_n \to p^{\dagger}$. This completes the proof. □

Algorithm 3.4. *Let $x_0 \in H_1$ be an initial value. Set $n = 0$.*

Step 1. For known x_n, compute

$$w_n = A^*(I - prox_{\tau\psi})Ax_n + (I - prox_{\tau\varphi})x_n.$$

If $w_n = 0$, then set $z_n = x_n$ and go to Step 2. Otherwise, compute

$$z_n = x_n - \frac{\lambda_n(u_n + v_n)}{\|w_n\|^2} w_n,$$

where $u_n = \frac{1}{2}\|(I - prox_{\tau\psi})Ax_n\|^2$ and $v_n = \frac{1}{2}\|(I - prox_{\tau\varphi})x_n\|^2$.

Step 2. Compute

$$x_{n+1} = (1-\vartheta_n)[\alpha_n f(x_n) + (1-\alpha_n)x_n] + \vartheta_n z_n.$$

Step 3. Set $n := n+1$ and return to Step 1.

Corollary 3.5. *Suppose that $\Gamma \neq \emptyset$. Suppose that the sequences $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\vartheta_n\}$ satisfy the conditions* (C1), (C3) *and* (C4). *Then sequence $\{x_n\}$ generated by Algorithm 3.4 strongly converges to the solution $q^{\dagger} = proj_{\Gamma}(f(q^{\dagger}))$.*

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