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Invariant Submanifolds of Hyperbolic Sasakian Manifolds and η-Ricci-Bourguignon Solitons

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Abstract. We set the goal to study the properties of invariant submanifolds of the hyperbolic Sasakian manifolds. It is proven that a three-dimensional submanifold of a hyperbolic Sasakian manifold is totally geodesic if and only if it is invariant. Also, we discuss the properties of η-Ricci-Bourguignon solitons on invariant submanifolds of the hyperbolic Sasakian manifolds. Finally, we construct a non-trivial example of a three-dimensional invariant submanifold of five-dimensional hyperbolic Sasakian manifold and validate some of our results.

1. Introduction

The concept of Ricci-Bourguignon flow as an extension of Ricci flow [22] has been introduced by J. P. Bourguignon [10] based on some unprinted article of Lichnerowicz and a paper of Aubin [2]. Ricci-Bourguignon flow is an intrinsic geometric flow on Riemannian manifolds, whose fixed points are solitons. Therefore, the Ricci-Bourguignon solitons generate self-similar solution to the Ricci-Bourguignon flow [10]:

$$\frac{\partial g}{\partial t} = -2(Ric - \rho R g), \quad g(0) = g_0, \tag{1}$$

where Ric is the Ricci curvature tensor, R is the scalar curvature with respect to the semi-Riemannian metric g and ρ is a non-zero real constant. It should be noticed that for special values of the constant ρ in equation (1), we obtain the following situations for the tensor $Ric - \rho R g$ appearing in equation (1). In particular [10], we have

(i) for $\rho = \frac{1}{2}$, the Einstein tensor $Ric - \frac{R}{2}g$ (for Einstein soliton [5]),
(ii) for $\rho = \frac{1}{n}$, the traceless Ricci tensor $Ric - \frac{R}{n}g$,
(iii) for $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $Ric - \frac{R}{2(n-1)}g$ (for Schouten soliton [10]),

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(iv) for $\rho = 0$, the Ricci tensor Ric (for Ricci soliton [22]).

In dimension two, the first three tensors are zero, hence the flow is static, and in higher dimensions, the value of ρ is strictly ordered as above, in descending order.

Short time existence and uniqueness for the solutions of this geometric flow have been proved in [10]. In fact, for sufficiently small t, the equation (1) has a unique solution for $\rho < \frac{1}{2(n-1)}$.

On the other hand, quasi-Einstein metrics or Ricci solitons serve as solutions to Ricci flow equation ($\frac{\partial g}{\partial t} = -2Ric, \quad g(0) = g_0$) [11, 22]. Aubin [2] has given the notion of Ricci-Bourguignon flow in a complete Riemannian manifold. Recently, De et al. [19] and Danish [30] have studied the properties of Ricci-Bourguignon solitons. A (semi-)Riemannian manifold of dimension $n \geq 3$ is said to be Ricci-Bourguignon soliton [2] if

$$\frac{1}{2}\mathcal{L}_V g + Ric + (\lambda - \rho R)g = 0, \tag{2}$$

where $\mathcal{L}_V$ denotes the Lie derivative operator along the vector field V (called soliton or potential vector field), ρ is a non-zero constant and λ is a real constant. Similar to the Ricci soliton, a Ricci-Bourguignon soliton (M, g, V, λ, ρ) is called expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

Perturbing the equation that define (2) Ricci-Bourguignon solitons by multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$, we obtain a slightly more general notion, namely η-Ricci-Bourguignon solitons [30] such as

$$\mathcal{L}_V g + 2Ric + 2(\lambda - \rho R)g + 2\omega\eta \otimes \eta = 0, \tag{3}$$

where ω is a real constant and η is 1-form. Particularly, if we choose $\rho = 0$ in equation (3), then the η-Ricci Bourguignon soliton reduces to the η-Ricci soliton (see [6], [7], [12]-[14], [18], [20], [24], [31], [32], [34]). We say that (M, g, f, λ, ρ) is a gradient Ricci-Bourguignon soliton if the potential vector field V, defined in (2), is the gradient of some smooth function f on M. Here, the soliton equation (2) takes the following form as:

$$Hess f + Ric + (\lambda - \rho R)g = 0, \tag{4}$$

where $Hess\, f$ is the Hessian of f.

Motivated by the contact structure, Upadhyay and Dube [33] introduced the notion of an almost hyperbolic contact (f, g, η, ξ)-structure. A $(2n+1)$-dimensional differentiable manifold of class C^∞ equipped with the structure (f, g, η, ξ) is known as an almost hyperbolic contact manifold. Further, it was studied by number of authors [1, 3, 25, 28]. Let $T_p(\tilde{M})$ denote the tangent space of the almost hyperbolic contact manifold $\tilde{M}$ at point p. Then a vector field $\nu \in T_p(\tilde{M})$, $\nu \neq 0$, is said to be time-like (resp., null, space-like, and non-space-like) if it satisfies $g_p(\nu, \nu) < 0$ (resp., $= 0$, > 0, and ≤ 0) ([17, 27]). If $\{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\}$ represents a local orthonormal basis of $\tilde{M}$, then the Ricci tensor Ric and scalar curvature R of an almost hyperbolic contact metric manifold, respectively, are defined as follows:

$$Ric(X, Y) = \sum_{i=1}^{2n+1} \varepsilon_i g(\tilde{\mathcal{R}}(e_i, X)Y, e_i) = \sum_{i=1}^{2n} \varepsilon_i g(\tilde{\mathcal{R}}(e_i, X)Y, e_i) - g(\tilde{\mathcal{R}}(\xi, X)Y, \xi), \tag{5}$$

$$R = \sum_{i=1}^{2n+1} \varepsilon_i Ric(e_i, e_i) = \sum_{i=1}^{2n} \varepsilon_i Ric(e_i, e_i) - Ric(\xi, \xi) \tag{6}$$

for all $X, \ Y \in T\tilde{M}$, where $\varepsilon_i = g(e_i, e_i)$, ξ is a unit time-like vector field, $\tilde{\mathcal{R}}$ represents the curvature tensor of $\tilde{M}$ and $T\tilde{M}$ denotes the tangent bundle of $\tilde{M}$.

We structure our work as follows: Section 2 gathers the basic information of hyperbolic Sasakian manifolds whereas in Section 3 we give some basic tools of submanifolds of the hyperbolic Sasakian manifolds. The properties of invariant submanifolds of the hyperbolic Sasakian manifolds are studied in Section 4. Sections 5 and 6 deal with the study of η-Ricci-Bourguignon solitons on invariant submanifolds of hyperbolic Sasakian manifolds. We give a non-trivial example of an invariant submanifold of hyperbolic Sasakian manifold in Section 7.

2. Hyperbolic Sasakian manifolds

Let $\tilde{M}$ be an $(n = 2m + 1)$-dimensional differentiable manifold of differentiability class C^∞. Then the structure (ϕ, ξ, η) satisfying

$$\phi^2 = I + \eta \otimes \xi, \quad \eta \circ \phi = 0 \tag{7}$$

is said to be an almost hyperbolic contact structure [33], where I denotes the identity transformation and ϕ, η, ξ and $\otimes$ are the tensor fields of type $(1, 1)$, $(0, 1)$, $(1, 0)$ and tensor product, respectively. The manifold $\tilde{M}$ equipped with the structure (ϕ, ξ, η) is called an almost hyperbolic contact manifold. From (7), it is noticed that

$$\phi\xi = 0, \quad \eta(\xi) = -1 \text{ and } rank(\phi) = n - 1. \tag{8}$$

If the associated semi-Riemannian metric g of $\tilde{M}$ satisfies

$$g(X, \xi) = \eta(X) \text{ and } g(\phi X, \phi Y) + g(X, Y) + \eta(X)\eta(Y) = 0 \tag{9}$$

for all $X, Y \in T\tilde{M}$, then the structure (ϕ, ξ, η, g) is called an almost hyperbolic contact metric structures and $\tilde{M}$ with the structure (ϕ, ξ, η, g) is known as an almost hyperbolic contact metric manifold. From (9), it is obvious that $g(\phi X, Y) = -g(X, \phi Y)$, $\forall\ X, Y \in T\tilde{M}$.

An almost hyperbolic contact metric manifold is said to be an almost hyperbolic Sasakian manifold if the 2-form defined as $\Phi(X, Y) = g(\phi X, Y)$ satisfies $-2\Phi = d\eta$, which is equivalent to

$$(\tilde{\nabla}_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X. \text{ Then } \tilde{\nabla}_X\xi = -\phi X, \tag{10}$$

for $X, Y \in T\tilde{M}$. Here $\tilde{\nabla}$ represents the Levi-Civita connection of $\tilde{M}$. It is obvious from (7)-(10) that a hyperbolic Sasakian manifold satisfies the following relations

$$\tilde{\mathcal{R}}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{11}$$

$$\eta(\tilde{\mathcal{R}}(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{12}$$

$$\tilde{\mathcal{R}}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{13}$$

for X, Y, $Z \in T\tilde{M}$.

3. Submanifolds of hyperbolic Sasakian manifolds

Let M be a submanifold immersed in a hyperbolic Sasakian manifold $\tilde{M}$. We use the same notation g for the induced metric of M. Let TM be a set of all vector fields tangent to M, and $T^\perp M$ is a set of all vector fields normal to M. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{14}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{15}$$

for X, $Y \in TM$ and $N \in T^\perp M$, where ∇ and $\nabla^\perp$ are the connections in TM and $T^\perp M$, respectively. The second fundamental form h and the shape operator A_N are connected by the relation

$$g(A_N X, Y) = g(h(X, Y), N), \quad \forall\ X,\ Y \in TM,\ N \in T^\perp M. \tag{16}$$

A submanifold M of an n-dimensional hyperbolic Sasakian manifold $\tilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M everywhere on M and ϕX is tangent to M for any vector field X tangent to M at every point of M, that is, $\phi(TM) \subset TM$ (see [15], [23], [29], [35]).

A submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is said to be *totally umbilical* [15] if

$$h(X, Y) = g(X, Y)H, \tag{17}$$

where H is the mean curvature on M for $X, Y \in TM$. Moreover, if $h(X, Y) = 0$ for all $X, Y \in TM$, then M is *totally geodesic* and if $H = 0$, then M is *minimal* in $\tilde{M}$.

4. Invariant submanifold of a hyperbolic Sasakian manifold

This section is dedicated to study the properties of invariant submanifolds of hyperbolic Sasakian manifolds.

Lemma 4.1. *If M is an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$, then there exists the distributions D and $D^{\perp}$ on M such that*

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle, \qquad \phi(D) \subset D^{\perp} \text{ and } \phi(D^{\perp}) \subset D.$$

Proof. Since the characteristic vector field ξ is tangent to the invariant submanifold M of $\tilde{M}$, we have $TM = D^1 \oplus \langle \xi \rangle$. Let $X \in D^1$. Then $g(\xi, X) = 0$ and $g(X, \phi X) = 0$. This shows that ϕX is orthogonal to ξ and X. If possible, we suppose that $D^1 = D \oplus D^{\perp}$, where $X \in D \subset D^1$ and $\phi X \in D^{\perp} \subset D^1$. If $\phi X \in D^{\perp}$, then from equation (7) we have $\phi(\phi X) = \phi^2 X = X + g(\xi, X)\xi = X \in D$. Again if $X \in D$ and $\phi X = Y \in D^{\perp}$. Then we can easily show that for $X \in D$, $\phi X \in D^{\perp}$ and for $Y \in D^{\perp}$, $\phi Y \in D$. Hence, the statement of proposition is proved. □

Lemma 4.2. *The second fundamental form h on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ satisfies*

$$h(X, \xi) = 0 \text{ and } h(X, \phi Y) = h(\phi X, Y) = \phi h(X, Y).$$

Proof. In view of equations (10) and (14), we immediately get the first result. Taking the covariant derivative of ϕY along the vector field X and making use of equation (14), we get

$$(\tilde{\nabla}_X \phi)(Y) = (\nabla_X \phi)(Y) + h(X, \phi Y) - \phi h(X, Y).$$

This equation along with equation (10), after taking normal part, give the second part of Lemma 4.2. □

Lemma 4.3. *An invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ satisfies the following relations:*

$$\nabla_X \xi = -\phi X,$$

$$\mathcal{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$\mathcal{R}(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$Q\xi = (n-1)\xi, \quad Ric(X, \xi) = (n-1)\eta(X).$$

Proof. The straight forward calculation follows Lemma 4.3. □

Lemma 4.4. *[26] An invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is also hyperbolic Sasakian.*

Now, we prove the following:

Theorem 4.5. *Every 3-dimensional invariant submanifold M of a hyperbolic Sasakian manifold is totally geodesic.*

Proof. Let M be a 3-dimensional invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then from Lemma 4.4, it is obvious that M is also a hyperbolic Sasakian manifold. We have from (7)

$$\phi^2 h(X, Y) = h(X, Y) + \eta(h(X, Y))\xi.$$

From Lemma 4.1, we have

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle.$$

Let $X_1,\ Y_1 \in D$. Then from the Lemma 4.2, we have

$$h(X_1, \phi Y_1) = \phi\, h(X_1, Y_1).$$

Also, Lemma 4.2 tells that

$$\phi h(X_1, \phi Y_1) = \phi^2 h(X_1, Y_1).$$

Since $h(X_1, Y_1) \in T^{\perp}M$, $h(X_1, Y_1)$ is orthogonal to $\xi \in TM$. Thus, the above equation gives

$$\phi h(X_1, \phi Y_1) = h(X_1, Y_1) = h(\phi X_1, \phi Y_1) = \phi^2 h(X_1, Y_1). \tag{18}$$

Next, we suppose that $X_2,\ Y_2 \in D^{\perp}$ and $X_2 = \phi X_1$, $Y_2 = \phi X_2$. We have

$$h(X_2, Y_2) = h(\phi X_1, \phi Y_1) = h(X_1, Y_1),$$

where equation (18) is used. As we know that h is bilinear, therefore for $X_1,\ Y_1\ \in D$ and $X_2,\ Y_2 \in D^{\perp}$, we get

$$h(X_1 + X_2 + \xi, Y_1) = h(X_1, Y_1) + h(X_2, Y_1) + h(\xi, Y_1),$$

$$h(X_1 + X_2 + \xi, -Y_2) = -h(X_1, Y_2) - h(X_2, Y_2) - h(\xi, Y_2)$$

and

$$h(X_1 + X_2 + \xi, \xi) = h(X_1, \xi) + h(X_2, \xi) + h(\xi, \xi).$$

It is well-known that on a hyperbolic Sasakian manifold, $h(X, \xi) = 0,\ \forall\ X \in TM$. By considering this result together with the last expressions, we have

$$h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_2, Y_1) - h(X_1, Y_2).$$

If $U,\ W \in TM$, then we can write U and W as

$$U = X_1 + X_2 + \xi,\ \textit{and}\ W = Y_1 - Y_2 + \xi.$$

We have

$$h(U, W) = h(X_1 + X_2 + \xi, Y_1 - Y_2 + \xi) = h(X_2, Y_1) - h(X_1, Y_2),$$

$$\phi h(U, W) = h(X_2, \phi Y_1) - h(\phi X_1, Y_2) = 0,$$

$$\phi^2(h(U, W)) = 0 \implies h(U, W) = 0.$$

Then follows the statement. □

Theorem 4.6. *Every totally geodesic submanifold M of a hyperbolic Sasakian manifold is invariant.*

Proof. We suppose that the submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is totally geodesic, that is,

$$h(X, Y) = 0,\ \forall\, X,\ Y\ \in TM.$$

We shall prove that the submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is invariant. Thus, we have to show $\phi X \notin T^{\perp}M$. To prove this, if possible, we suppose that the vector field ϕX has a component, say LX along the normal vector field of M. It is obvious that $A_{LX}Y \in TM,\ \forall\ X,\ Y \in TM$. Let $Z = A_{LX}Y \neq 0$. Then

$$g(Z, Z) = g(A_{LX}Y, Z) = g(h(Y, Z), LX) = 0. \tag{19}$$

Since Z is a non-null and non-zero vector field of TM implies $g(Z, Z) \neq 0$. Thus our hypothesis that ϕX has a component along $T^{\perp}M$ is inadmissible. Hence $\phi X \in TM$ and therefore the submanifold M of $\tilde{M}$ is invariant. □

In view of Theorem 4.6, we can state the following corollary as:

Corollary 4.7. *Every 3-dimensional geodesic submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is invariant.*

By considering the Theorem 4.5 and Corollary 4.7, we can state:

Theorem 4.8. *A 3-dimensional submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is totally geodesic if and only if it is invariant.*

Next, we have the following results:

Lemma 4.9. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then*

$$(\nabla_Y h)(Z,\xi) = -h(Z,\nabla_Y \xi) \tag{20}$$

for any $Y, Z \in TM$.

Proof. From Lemma 4.2 we turn up

$$(\nabla_Y h)(Z,\xi) = \nabla_Y h(Z,\xi) - h(\nabla_Y Z,\xi) - h(Z,\nabla_Y \xi),$$

which gives the statement of Lemma 4.9. □

Corollary 4.10. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then*

$$(\nabla_Y h)(Z,\xi) = h(Z,\varphi Y) \tag{21}$$

for any $Y, Z \in TM$.

Proof. By using equations (10) and (20), we get (21). □

Corollary 4.11. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then the following conditions are equivalent.*

1. *M is totally geodesic,*
2. *h is parallel,*
3. $(\nabla_Y \nabla_Z h)(\xi,\xi) = 0$,

where Y and Z are arbitrary vector fields on M.

Theorem 4.12. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$. Then*

$$\varphi(A_N Z) = A_{\varphi N} Z = -A_N \varphi Z \tag{22}$$

for all $Z \in TM$, $N \in TM^\perp$.

Proof. Adopting (9) and (16) for all $Z \in TM$, $N \in TM^\perp$ we turn up

$$g(\varphi(A_N Z), W) = -g(A_N Z, \varphi W) = -g(h(Z,\varphi Y), N)$$

$$= -g(h(\varphi Z, W), N) = -g(A_N \varphi Z, W).$$

Then, we have $\varphi(A_N Z) = -A_N \varphi Z$. Now using Lemma (4.2), we lead to

$$g(A_{\varphi N} Z, W) = g(h(Z,W),\varphi N) = -g(\varphi(h(Z,W)), N) = -g(h(Z,\varphi W), N)$$

$$= -g(A_N Z, \varphi W) = g(\varphi(A_N Z), W).$$

Thus, we have the result. □

5. η-Ricci-Bourguignon solitons on invariant submanifolds

Let us adopt $(g, \xi, \lambda, \omega, \rho)$ as an η-Ricci-Bourguignon soliton on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$. Then we have

$$(\mathcal{L}_\xi g)(X, Y) + 2Ric(X, Y) + 2(\lambda + \rho R)g(X, Y) + 2\omega\eta(X)\eta(Y) = 0. \tag{23}$$

From (10) and (14), we turn up

$$-\phi X = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{24}$$

If M is invariant in $\tilde{M}$, then $-\phi X, \xi \in TM$ and therefore equating the tangential and normal parts of (24) we find that

$$\nabla_X \xi = -\varphi X \quad and \quad h(X, \xi) = 0. \tag{25}$$

Again from (10) we lead to

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{26}$$

Adopting (23) and (25) we trun up

$$Ric(X, Y) = -(\lambda - \rho R)g(X, Y) - \omega\eta(X)\eta(Y), \tag{27}$$

which implies that M is η-Einstein. Also, from (17) and (25) we get $\eta(X)H = 0$, i.e., $H = 0$, since $\eta(X) \neq 0$ (in general) and therefore M is minimal in $\tilde{M}$. Thus, we turn up to the following:

Theorem 5.1. *If $(g, \xi, \lambda, \omega, \rho)$ is an η-Ricci-Bourguignon soliton on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$. Then we have the following:*

1. *M is η-Einstein,*
2. *M is minimal and*
3. *ξ is a Killing vector field in $\tilde{M}$.*

Now, by choosing the different values of ρ in equation (27), we obtain the following table:

ρ	Solitons	Expression of Ricci tensor
$\frac{1}{2}$	η-Einstein solitons	$Ric(X, Y) = -(\lambda - \frac{R}{2})g(X, Y) - \omega\eta(X)\eta(Y)$
$\frac{1}{2(n-1)}$	η-Schouten solitons	$Ric(X, Y) = -(\lambda - \frac{R}{2(n-1)})g(X, Y) - \omega\eta(X)\eta(Y)$
0	η-Ricci solitons	$Ric(X, Y) = -\lambda g(X, Y) - \omega\eta(X)\eta(Y)$

(28)

From table (28) and Theorem 5.1, we state the following corollaries:

Corollary 5.2. *If $(g, \xi, \lambda, \omega, \rho)$ is an η-Einstein or η-Schouten or η-Ricci solitons on an invariant submanifold M of $\tilde{M}$, then we have*

1. *M is η-Einstein,*
2. *M is minimal and*
3. *ξ is Killing vector field in $\tilde{M}$.*

From Lemma (4.3), we have

$$Ric(X, \xi) = (n - 1)\eta(X),$$

and henceforth

$$Ric(\xi, \xi) = -(n - 1). \tag{29}$$

Also from (27) we trun up

$$Ric(\xi, \xi) = -(\lambda - \rho R) - \omega. \tag{30}$$

Thus from (29) and (30) we obtain $\lambda = \rho R + [(n - 1) - \omega]$. In particular, if we choose $\omega = 0$ then the η-Ricci-Bourguignon soliton becomes the Ricci-Bourguignon soliton and $\lambda = \rho R + (n - 1)$. This leads to the following:

Theorem 5.3. *Let $(g, \xi, \lambda, \omega, \rho)$ be an η-Ricci-Bourguignon soliton on an invariant submanifold M of $\tilde{M}$. Then $\lambda + \omega = \rho R + n - 1$. Also, the Ricci-Bourguignon soliton (g, ξ, λ, ρ) on M is steady, expanding or shrinking according as $\rho R > -(n-1)$, $\rho R = -(n-1)$ or $\rho R < -(n-1)$, respectively.*

Also, in consequence of Theorem 5.3 we can easily obtain the followings corollaries:

Corollary 5.4. *An Einstein soliton $(g, \xi, \lambda, \frac{1}{2})$ on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is steady, expanding or shrinking according as $\frac{R}{2} > -(n-1)$, $\frac{R}{2} = -(n-1)$ or $\frac{R}{2} < -(n-1)$, respectively.*

Corollary 5.5. *A Schouten soliton $(g, \xi, \lambda, \frac{1}{2(n-1)})$ on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is steady, expanding or shrinking according as $\frac{R}{2(n-1)} > -(n-1)$, $\frac{R}{2(n-1)} = -(n-1)$ or $\frac{R}{2(n-1)} < -(n-1)$, respectively.*

Corollary 5.6. *A Ricci soliton $(g, \xi, \lambda, 0)$ on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ is shrinking, expanding or steady if $Ric(\xi, \xi) > 0$, < 0 or $= 0$, respectively.*

6. η-Ricci-Bourguignon solitons with concircular vector field on invariant submanifolds

This section deals with the study of η-Ricci-Bourguignon solitons with concircular vector field on an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$.

In 1939, A. Fialkow [21] has been proposed the theory of concircular vector fields on a Riemannian manifold. A vector field v on a (semi-)Riemannian manifold M is said to be a concircular vector field if it satisfies

$$\nabla_U v = \mu U, \tag{31}$$

for any $U \in TM$, where ∇ denotes the Levi-Civita connection of the metric g and μ is a non-trivial smooth function on M. The concircular vector fields are also known as geodesic fields because their integral curves are geodesics [21]. Recently, Chen [16] studied the properties of Ricci solitons on submanifolds of a Riemannian manifold equipped with a concircular vector field. Particularly, if we choose $\mu = 1$ in equation (31), then the concircular vector field v is called *concurrent vector field*.

For an invariant submanifold, from Lemma (4.1), we can write

$$v = v^T + v^{\perp}, \tag{32}$$

where $v \in TM$, $v^T \in D$ and $v^{\perp} \in D^{\perp}$.
Since v is a concircular vector field on $\tilde{M}$ and from (32), we get

$$\mu U = \tilde{\nabla}_U v^T + \tilde{\nabla}_U v^{\perp} \tag{33}$$

for any $U \in D$. Also, from (14) and (15), we turn up

$$\mu U = \nabla_U v^T + h(U, v^T) - A_{v^{\perp}} U + \nabla^{\perp}_U v^{\perp}. \tag{34}$$

By comparing the tangential and normal components of equation (34), we conclude that

$$h(U, v^T) = -\nabla^{\perp}_U v^{\perp}, \qquad \nabla_U v^T = \mu U - A_{v^{\perp}} U. \tag{35}$$

Now, we prove the following theorem as:

Theorem 6.1. *Let M be an invariant submanifold of $\tilde{M}$ admitting an η-Ricci-Bourguignon soliton with a concircular vector field v. Then the Ricci tensor Ric_D on the invariant distribution D is given by*

$$Ric_D(U, W) = -\left\{\left(\lambda - \frac{\rho R}{2} + \mu\right) g(U, W) + g(h(U, W), v^{\perp}) + \omega \eta(U)\eta(W)\right\} \tag{36}$$

for any $U, W \in D$.

Proof. By the definition of Lie-derivative, we have

$$(\mathcal{L}_{v^T} g)(U, W) = g(\nabla_U v^T, W) + g(U, \nabla_W v^T). \tag{37}$$

Adopting (35) together with (37), the above equation infers that

$$(\mathcal{L}_{v^T} g)(U, W) = 2\mu g(U, W) - 2g(h(U, W), v^\perp). \tag{38}$$

Again, since the invariant submanifold M admits an η-Ricci-Bourguignon soliton, therefore we observe that

$$(\mathcal{L}_{v^T} g)(U, W) + 2Ric_D(U, W) + 2(\lambda - \rho R)g(U, W) + 2\omega\eta(U)\eta(W) = 0. \tag{39}$$

From equations (38) and (39), we obtain the statement of Theorem 6.1. □

In particular, if we take $\mu = 1$ in equation (31) then we will get concurrent vector field. Let us suppose that v is a concurrent vector field and $(g, v, \lambda, \omega, \rho)$ is an η-Ricci-Bourguignon soliton in an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$. Then by following the similar process of Theorem 6.1, we can state the following:

Theorem 6.2. *If an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ admits an η-Ricci-Bourguignon soliton with the concircular vector field v, then the invariant distribution D of M is η-Einstein, provided the invariant distribution D of M is D-geodesic.*

In view of Theorem 6.1, we can write the following corollaries as:

Corollary 6.3. *Let M be an invariant submanifold of $\tilde{M}$ admitting an η-Ricci-Bourguignon soliton with a concurrent vector field v. Then the Ricci tensor Ric_D on the invariant distribution D is given by*

$$Ric_D(U, W) = -\left\{\left(\lambda - \frac{\rho R}{2} + 1\right) g(U, W) + g(h(U, W), v^\perp) + \omega\eta(U)\eta(W)\right\} \tag{40}$$

for any $U, W \in D$.

Corollary 6.4. *Suppose that an invariant submanifold M of $\tilde{M}$ admits an η-Ricci-Bourguignon soliton with a concurrent vector field v. If the invariant distribution D of M is D-geodesic, then the invariant distribution D is η-Einstein.*

Remark 6.5. *It is noticed that if we choose $D^\perp$ distribution and $D^\perp$-geodesic condition in the above theorems and corollaries, then after following the similar steps we obtain the results as stated in Theorem 6.1, Theorem 6.2 and Corollary 6.3, Corollary 6.4.*

Now, with the help of Table (28) and Theorem 6.1, we can easily obtain following corollaries as:

Corollary 6.6. *If an invariant submanifold M of a hyperbolic Sasakian manifold $\tilde{M}$ admits an η-Einstein soliton with a concircular vector field v, then the Ricci tensor Ric_D on the invariant distribution D is given by*

$$Ric_D(U, W) = -\left\{\left(\lambda - \frac{R}{2} + \mu\right) g(U, W) + g(h(U, W), v^\perp) + \omega\eta(U)\eta(W)\right\} \tag{41}$$

for any $U, W \in D$.

Corollary 6.7. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$ admitting an η-Schouten soliton with a concircular vector field v. Then the Ricci tensor Ric_D of the invariant distribution D is given by*

$$Ric_D(U, W) = -\left\{\left(\lambda - \frac{R}{2(n-1)} + \mu\right) g(U, W) + g(h(U, W), v^\perp) + \omega\eta(U)\eta(W)\right\} \tag{42}$$

for any $U, W \in D$.

Corollary 6.8. *Let M be an invariant submanifold of a hyperbolic Sasakian manifold $\tilde{M}$ admitting an η-Ricci soliton with a concircular vector field v. Then the Ricci tensor Ric_D on the invariant distribution D is given by*

$$Ric_D(U, W) = -\{(\lambda + \mu)g(U, W) + g(h(U, W), v^\perp) + \omega\eta(U)\eta(W)\}$$

for any $U, W \in D$.

7. Example

Let $\Re^n$ be an n-dimensional real number space. Define $M^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in \Re,\ i = 1, 2, ..., 5\}$. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be a set of linearly independent vector fields of M^5 given by

$$e_1 = e^{2x_1}\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3}, \qquad e_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2}, \qquad e_3 = \frac{\partial}{\partial x_3}, \qquad e_4 = x_4\frac{\partial}{\partial x_4} - 2x_5\frac{\partial}{\partial x_3}, \qquad e_5 = \frac{\partial}{\partial x_5}.$$

We define the $(1, 1)$-tensor field ϕ of M^5 as

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = e_4.$$

Also, we define the associated metric g of M^5 by the following relation.

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

By the linearity property of ϕ and g, we can show that the relations

$$\phi^2 e_i = e_i + \eta(e_i)\xi, \quad \eta(\xi) = -1$$

hold for $i = 1, 2, 3, 4, 5$ and $\xi = e_3$. Again, for $\xi = e_3$, M^5 satisfies $g(e_i, e_3) = \eta(e_i)$, $g(\phi e_i, e_j) = -g(e_i, \phi e_j)$ and $g(\phi e_i, \phi e_j) = -g(e_i, e_j) - \eta(e_i)\eta(e_j)$, where $i, j = 1, 2, 3, 4, 5$. It can be easily obtained that

$$[e_i, e_j] = \begin{cases} 2e_3, & i = 1, \quad j = 2 \\ 2e_3, & i = 4, \quad j = 5 \\ 0, & \text{otherwise}, \end{cases}$$

where $i,\ j = 1, 2, 3, 4, 5$. Koszul's formula for the Levi-Civita connection ∇ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

for all vector fields X, Y and Z on M^5. In view of this formula and the above results, we have

$$\begin{array}{lllll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = e_3, & \nabla_{e_1} e_3 = -e_2, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = 0, \\ \nabla_{e_2} e_1 = -e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = -e_1, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = 0, \\ \nabla_{e_3} e_1 = -e_2, & \nabla_{e_3} e_2 = -e_1, & \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_4 = -e_5, & \nabla_{e_3} e_5 = -e_4, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = -e_5, & \nabla_{e_4} e_4 = 0, & \nabla_{e_4} e_5 = -e_3, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = -e_4, & \nabla_{e_5} e_4 = -e_3, & \nabla_{e_5} e_5 = 0. \end{array}$$

From the above equations, it is obvious that $\nabla_X \xi = -\phi X$ for all $X \in TM^5$ and $\xi = e_3$. Thus, the structure (ϕ, ξ, η, g) is an almost hyperbolic Sasakian structure and M^5 equipped with the structure (ϕ, ξ, η, g) is an almost hyperbolic Sasakian manifold of dimension 5.

We suppose that $\mathfrak{f}$ is an isometric immersion from M^3 to M^5 defined by $\mathfrak{f}(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, 0, 0)$.

Let the triplet (x_1, x_2, x_3) be the standard coordinates in $\Re^3$. We define $M^3 = \{(x_1, x_2, x_3) \in \Re^3$ such that $(x_1, x_2, x_3) \neq 0\}$. If we consider the vector fields

$$e_1 = e^{2x_1}\frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3}, \qquad e_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2}, \qquad e_3 = \frac{\partial}{\partial x_3}$$

on M^3, then they form a basis for M^3. By considering these vectors, we can easily find the components of Lie bracket for e_1, e_2 and e_3 as:

$$[e_1,\ e_2] = 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0 \tag{43}$$

and other components can be obtained by using the skew-symmetric property of Lie bracket. Let the associated metric g of M^3 is defined by

$$g(e_i, e_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Suppose that the 1-form η with respect to the metric g is defined as $\eta(X) = g(X, \xi)$, $\forall\ X \in TM^3$. The $(1,1)$-type tensor ϕ of M^3 is defined by

$$\phi e_1 = e_2, \qquad \phi e_2 = e_1, \qquad \phi e_3 = 0.$$

By the linearity of ϕ and g, we can easily see that for $\xi = e_3$

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1$$

and

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \qquad \forall\ X,\ Y \in TM^3.$$

From the Koszul's formula, we have

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \qquad \nabla_{e_1} e_2 = e_3, \qquad \nabla_{e_1} e_3 = -e_2, \quad \nabla_{e_2} e_1 = -e_3, \quad \nabla_{e_2} e_2 = 0, \\ &\nabla_{e_2} e_3 = -e_1, \quad \nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_3} e_2 = -e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned} \tag{44}$$

It is obvious from these equations that $\nabla_X \xi = -\phi X$ for $\xi = e_3$ and $X \in TM^3$ holds on M^3. Thus, (M^3, g) is a 3-dimensional hyperbolic Sasakian manifold. It is obvious that (M^3, g) is a submanifold of $(\tilde{M}^5, g)$. To achieve our goal, we have to prove that M^3 is an invariant as well as totally geodesic.

If possible, we suppose that $\langle e_1 \rangle = D$ and $\langle e_2 \rangle = D^\perp$, then the tangent space TM of M^3 takes the form $TM = D \oplus D^\perp \oplus \langle \xi \rangle$. Let $U \in D$ and $W \in D^\perp$, then we can write $U = \alpha e_1$ and $W = \beta e_2$, where α and β are the smooth functions. We have

$$\phi U = \phi(\alpha\, e_1) = \alpha\, \phi e_1 = \alpha\, e_2 \ \in D^\perp \ \in TM$$

and

$$\phi W = \phi(\beta\, e_2) = \beta\, \phi e_2 = \beta\, e_1 \ \in D \ \in TM.$$

Hence, we can say that $M^3 = M$ under consideration is an invariant submanifold of $M^5 = \tilde{M}$.

From equation (14), we have

$$h(e_i,\ e_j) = \tilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j.$$

It is evident from the above results that

$$h(e_i,\ e_j) = 0, \quad \forall\ \ i,\ j = 1, 2, 3.$$

Let $U,\ W \in TM$. Then we can write $U = \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3$ and $W = \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3$, where α_i, β_i and γ_i, for $i = 1, 2$, are scalars. We have

$$\begin{aligned} h(U, W) &= h(\alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_3, \alpha_2 e_1 + \beta_2 e_2 + \gamma_2 e_3) \\ &= \alpha_1\, \alpha_2\, h(e_1, e_1) + \alpha_1\, \beta_2\, h(e_1, e_2) + \alpha_1\, \gamma_2\, h(e_1, e_3) \\ &\quad + \beta_1\, \alpha_2\, h(e_2, e_1) + \beta_1\, \beta_2\, h(e_2, e_2) + \beta_1\, \gamma_2\, h(e_2, e_3) \\ &\quad + \gamma_1\, \alpha_2\, h(e_3, e_1) + \gamma_1\, \beta_2\, h(e_3, e_2) + \gamma_1\, \gamma_2\, h(e_3, e_3). \end{aligned}$$

The last two equations give

$$h(U, W) = 0, \quad \forall\ U,\ W \in TM.$$

This shows that the 3-dimensional hyperbolic Sasakian submanifold M^3 of the 5-dimensional hyperbolic Sasakian manifold M^5 is totally geodesic. Hence, the statement of Theorem 4.8 is verified.

Also, by using equations (43), (44) and the metric of M^3 in $\mathcal{R}(U,Z)W = \nabla_U\nabla_Z W - \nabla_Z\nabla_U W - \nabla_{[U,Z]}W$, we find the non-zero components of curvature tensor $\mathcal{R}$ as:

$$\mathcal{R}(e_1,e_2)e_1 = 3e_2,\ \ \mathcal{R}(e_1,e_3)e_1 = -e_3\ \ \mathcal{R}(e_1,e_2)e_2 = 3e_1,\ \ \mathcal{R}(e_2,e_3)e_2 = e_3,\ \ \mathcal{R}(e_1,e_3)e_3 = -e_1,\ \ \mathcal{R}(e_2,e_3)e_3 = -e_2.$$

The other components of the curvature tensor can be obtained by the symmetric properties. The above results together with equations (5) and (6) reveal that

$$Ric(e_1,e_1) = Ric(e_3,e_3) = -2,\quad Ric(e_2,e_2) = 2\ \ and\ R = -2.$$

By the straight forward calculations, we can show that the Lemma 4.2, Lemma 4.3, Lemma 4.4, Theorem 4.5 and Corollary 4.7 hold on M^3.

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