Estimation of Almost Ricci-Yamabe Solitons on Static Spacetimes

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Abstract. This research work examines the standard static spacetime (*SSST*) in terms of almost Ricci-Yamabe soliton with conformal vector field. It is shown that almost Ricci-Yamabe soliton in standard static spacetime with function ψ satisfies Poisson-Laplace equation. Next, we consider the function ψ is harmonic and discuss the harmonic aspect of almost Ricci-Yamabe soliton on *SSST*. In addition, we investigate the nature of almost Ricci-Yamabe soliton on *SSST* with non-rotating Killing vector field. Also, we exhibit that non-steady non shrinking almost Ricci-Yamabe soliton i.e., $\lambda \geq 0$ on smooth, connected, and non-compact *SSST* with Killing vector field satisfies the Schrödinger equation for a smooth function ψ. Finally, we study almost Ricci-Yamabe soliton on static perfect fluid and vacuum static spacetime with conformal Killing vector field.

1. Introduction

Einstein's "Theory of General Relativity" (*GR*) is usually referred to as the geometric theory of gravitation. As one of the most successful theories of the physics in the 20th century, *GR* has revealed the fundamental interplay between physics and the geometry of spacetime. During the past 100 years, it has been one of the most active research fields in both physics and mathematics. Besides its essential role in the theoretical study, *GR* has also gained great success in engineering when applying to our daily life. After being proposed, seeking the various solution to Einstein's field equation become one of the most important problems.

The most obvious solution is the *Minikowski spacetime*, which is the four dimensional Euclidean space $\mathbb{R}^4$ equipped with Lorentzian metric. The other non-trivial solutions are like *Schwartzchild solution*, *de-Sitter*, *Kerr* etc. Static spacetime is a particular and relevant universal solution among all those solutions. One of the common features is that the time direction after a proper rescaling can be written as a Killing vector field, when evolving along the integral curve of which, the spatial slices remain invariant. This is the reason we call them "static" according to Newton's viewpoint on space and time.

In *GR*, Lorentzian twisted product manifolds were adapted in order to find a general solution to Einstein's field equations. Two well-known significant examples are generalized Robertson Walker spacetime (*GRW*) (for more details see [9], [16]) and standard static spacetime (*SSST*) [8]. Basically, a normal static spacetime

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can be treated as a Lorentzian warped product manifold where the warping function f is defined on Riemannian manifold and acting on the negative definite metric on an open interval I of real numbers .

The study of spacetime symmetries is an important task for solving the Einstein's field equations. Symmetries are a feature of geometry and thus expose the physics. In term of spacetime geometry and matter, there are various symmetries. As they simplify the solution to many problems, the metric equations are valuable. In *GR*, their primary utilization is that they classify solution of Einstein's field equations. Solitons are one such symmetry along with the geometric flow of geometrical spacetime. Ricci flow and Yamabe flow are useful because they can help to grasp the theory of energy and entropy. Ricci soliton and Yamabe soliton are the points at which Hamilton [12] applied the curvatures to a self-similar solution.

The symmetries of spacetime in terms of Ricci solitons were investigated by Ahsan and Ali [1]. In [3] Blaga characterized the geometrical features of a perfect fluid spacetime in form of Einstein solitons, Ricci solitons and their extensions. In [19] researchers have discussed certain geometrical significance of perfect fluid spacetime with conformal Ricci soliton. Furthermore, many geometers also illustrated the spacetime with various solitons, which are closely related to this paper (for more details see [4], [6], [18]).

Recently, in 2020, Siddiqi and Akyol [17] discussed about a new soliton named Ricci-Yamabe soliton. The scalar combination of Ricci and Yamabe flow is a new geometric flow [10]. This is also identified as the Ricci-Yamabe flow form (α, β). The Ricci-Yamabe flow is an evolution of the semi-Riemannian manifolds that are described as

$$\frac{\partial}{\partial t}g(t) = -2\alpha S(t) + \beta r(t)g(t), \quad g_0 = g(0). \tag{1}$$

The Ricc-Yamabe flow can also be a semi-Riemannian or singular Riemannian flow according to the indication of involved scalars α and β. In certain geometrical physical models, for example, relativistic theories, this kind of multiple choice may be efficient. Ricci-Yamabe soliton, therefore, inevitably occurs as the limit of the flow of Ricci-Yamabe soliton. This is valuable encouragement to lead the study of Ricci-Yamabe soliton. In [5] where the name Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow but relying on a single scalar, the interpolation solitons between Ricci and Yamabe soliton are interpreted.

Ricci-Yamabe solitons are called a soliton in the Ricci-Yamabe flow if it moves diffeomorphism and scaling by one parameter group only. In fact a Ricci-Yamabe soliton on the semi-Riemannian manifold (M, g) is a data $(g, V, \lambda, \alpha, \beta)$ satisfying

$$\frac{1}{2}\mathcal{L}_V g + \alpha S + (\lambda - \frac{\beta r}{2})g = 0, \tag{2}$$

where $\mathcal{L}_V$ indicates the Lie derivative in the direction of vector field V, S and r is the Ricci tensor, and scalar curvature, respectively. In (M, g), Ricci-Yamabe soliton is called *expanding*, *shrinking*, or *steady*, corresponding to $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, respectively. Therefore, generalization of Ricci soliton and Yamabe soliton is termed as (α, β)-type Ricci-Yamabe soliton in (2). Moreover, Ricci-Yamabe soliton is called α-Ricci soliton and β-Yamabe soliton corresponding to $(\alpha, 0)$, and $(0, \beta)$, respectively.

Example 1.1. *Let us mention the situation of Einstein soliton, which generates self-similar solutions to Einstein flow in such a way that*

$$\frac{\partial}{\partial t}g(t) = -2\left(S - \frac{r}{2}\right)g.$$

Therefore, an Einstein soliton occurs as the limit of the Einstein flow solution, such that

$$\mathcal{L}_V g + 2S + (\lambda - \frac{r}{2})g = 0. \tag{3}$$

Comparing equation (3) with (2), we find $(1, 1)$ type-Ricci-Yamabe soliton.

According to Pigola et al. [23] if we swap the constant λ in (2) with a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we say that (M, g) is an almost Ricci-Yamabe soliton.

Therefore, encouraged by the earlier research work, in this research paper, we illustrate the geometry of a standard static spacetime ($SSST$) in terms of almost Ricci-Yamabe soliton.

2. Preliminaries

A standard static spacetime (briefly say *SSST*) is a Lorentzian warped product manifold $\bar{\mathbf{M}} = I_\psi \times \mathbf{M}$ with the metric $\bar{g} = -\psi^2 dt^2 \oplus g$, where $(\mathbf{M}, g)$ is a Riemannian manifold. $\psi : \mathbf{M} \longrightarrow (0, \infty)$ is smooth and $I = (t_1, t_2)$ with $-\infty \le t_1 < t_2 \le \infty$. We use the same notation for a vector field $X \in \chi(\mathbf{M})$ and for its lift to *SSST* $\bar{\mathbf{M}}$. Likewise, a function ω on $\mathbf{M}$ will be identified with $\omega \circ \pi$ on $\bar{\mathbf{M}}$, where $\pi : I_\psi \times \mathbf{M} \longrightarrow \mathbf{M}$ is the natural projection map of $I \times \mathbf{M}$ onto M. We use $grad\omega \in \chi(\mathbf{M})$ for the gradient vector field of ω on $\mathbf{M}$ and for its to $\chi(\mathbf{M})$ ([2], [22]). The submanifold $\{t\} \times \mathbf{M}$ and $\mathbf{M}$ are isomorphic for every $t \in \mathbf{M}$.

Let $\bar{\mathbf{M}} = I_\psi \times \mathbf{M}$ be a standard static spacetime with the metric tensor $\bar{g} = -\psi^2 dt^2 \oplus g$. Then the Levi-Civita connection $\bar{\nabla}$ on $\bar{\mathbf{M}}$ is given by [8]

$$\bar{\nabla}_{\partial_t}\partial_t = \psi\, grad\psi, \quad \bar{\nabla}_{\partial_t}E = \bar{\nabla}_E\partial_t = E(ln\psi)\partial_t, \quad \bar{\nabla}_E F = \nabla_E F \tag{4}$$

for any vector fields $E, F \in \chi(\mathbf{M})$, where ∇ is the Levi-Civita connection on $\mathbf{M}$. The Riemannian curvature tensor $\bar{R}$ is given by

$$\bar{R}(E, \partial_t)\partial_t = -\psi \nabla_E grad\psi, \tag{5}$$

$$\bar{R}(\partial_t, \partial_t)\partial_t = \bar{R}(\partial_t, \partial_t)E = \bar{R}(E, F)\partial_t = 0, \tag{6}$$

$$\bar{R}(\partial_t, E)F = \frac{1}{\psi}H^\psi(E, F)\partial_t, \tag{7}$$

$$\bar{R}(E, F)G = R(E, F)G, \tag{8}$$

where R is the curvature tensor of $\mathbf{M}$ and $H^\psi(E, F) = g(\nabla_E grad\psi, F)$ is the *Hessian* of ψ. Now, the Ricci curvature tensor $\bar{S}$ and scalar curvature $\bar{r}$ of *SSST* $\bar{\mathbf{M}}$ are as follows

$$\bar{S}(\partial_t, \partial_t) = \psi\Delta\psi, \quad \bar{S}(E, \partial_t) = 0, \tag{9}$$

$$\bar{S}(E, F) = S(E, F) - \frac{1}{\psi}H^\psi(E, F), \tag{10}$$

$$\bar{r} = r - 2\frac{1}{\psi}\Delta\psi, \tag{11}$$

where $\Delta\psi$ denotes the *Laplacian* of ψ on $\mathbf{M}$. The Lie derivative $\mathcal{L}_\xi$ in the direction of ξ is given by

$$(\mathcal{L}_\xi g)(E, F) = g(\nabla_E \xi, F) + g(E, \nabla_F \xi) \tag{12}$$

where E, F are vector fields in $\chi(\mathbf{M})$.

3. Almost Ricci-Yamabe Soliton on standard static spacetime

In this section, first we recall the definition of conformal vector field [14].

Definition 3.1. *A vector field ξ is called conformal if $\mathcal{L}_\xi g = \rho g$ for some smooth function $\rho : M \longrightarrow \mathbb{R}$. If $\rho = 0$, then the vector field, is called Killing. Also ξ is a Killing vector field if and only if $g(\nabla_E \xi, E) = 0$ for any vector field $E \in \chi(M)$.*

Consider, that $h\partial_t, x\partial_t, y\partial_t \in \chi(I)$ and $\xi, E, F \in \chi(\mathbf{M})$, then we have

$$(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) = (\mathcal{L}_\xi g)(E, F) - 2xy\psi^2(h^. + \xi(ln\psi)), \tag{13}$$

where $\bar{\xi} = h\partial_t + \xi$, $\bar{E} = x\partial_t + E$ and $\bar{F} = y\partial_t + F$. The above expression (13) is a particular case of a notable one on warped product manifolds [9].

Now, let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton, where $\bar{M} = I_\psi \times M$ is a standard static spacetime and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$. It is clear that a potential field $\bar{\xi}$ is conformal on $(\bar{M}, \bar{g})$ if and only if $(\bar{M}, \bar{g})$ is an Einstein manifold.

Assume that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton, then from equation (2) we have

$$(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) + 2\alpha\bar{S}(\bar{E}, \bar{F}) + (2\lambda - \beta\bar{r})\bar{g}(\bar{E}, \bar{F}) = 0, \tag{14}$$

where $\bar{E} = xh\partial_t + E$ and $\bar{F} = yh\partial_t + F$ are vector fields on $\bar{M}$. Using (13), (10) and (11), we get

$$(\mathcal{L}_\xi g)(E, F) + 2\alpha S(E, F) - \frac{1}{\psi}H^\psi(E, F) + (2\lambda - \beta\left\{r - 2\frac{1}{\psi}\Delta\psi\right\})g(E, F) = 0. \tag{15}$$

Suppose now that $H^\psi = 0$ and $\Delta\psi = 0$, then we find

$$(\mathcal{L}_\xi g)(E, F) + 2\alpha S(E, F) + (2\lambda - \beta r)g(E, F) = 0. \tag{16}$$

Thus, we have the following results:

Theorem 3.2. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton where $\bar{M} = I_\psi \times M$ is a SSST and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ and assume that $H^\psi = 0$ and $\Delta\psi = 0$. Then $(M, g, \xi, \lambda, \alpha, \beta)$ is an almost Ricci-Yamabe soliton.*

Theorem 3.3. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, 0, 1)$ be an almost Ricci-Yamabe soliton of type $(0, 1)$, where $\bar{M} = I_\psi \times M$ is a SSST and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ and assume that $H^\psi = 0$ and $\Delta\psi = 0$. Then $(M, g, \xi, \lambda, \alpha, \beta)$ is an almost Yamabe soliton.*

Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton where $\bar{M} = I_\psi \times M$ is a standard static speacetime and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$. Then

$$(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) + 2\alpha\bar{S}(\bar{E}, \bar{F}) + (2\lambda - \beta\bar{r})\bar{g}(\bar{E}, \bar{F}) = 0,$$

Thus

$$(\mathcal{L}_\xi g)(E, F) + 2\alpha S(E, F) - \frac{1}{\psi}H^\psi(E, F) + (2\lambda - \beta\left\{r - 2\frac{1}{\psi}\Delta\psi\right\})g(E, F) = 0. \tag{17}$$

$$h\,\dot{}\,\psi + \xi(\psi) - \alpha\Delta\psi = (\lambda - \frac{\beta r}{2})\psi. \tag{18}$$

Suppose that $grad\psi$ is a concircular vector field [14] with factor ρ, then

$$(\mathcal{L}_\xi g)(E, F) + 2\alpha S(E, F) + (2\lambda - \beta r + \frac{\rho}{\psi})g(E, F) = 0. \tag{19}$$

Hence we conclude that

Theorem 3.4. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton where $\bar{M} = I_\psi \times M$ is a SSST and assume that $grad\psi$ is a concircular vector field with factor ρ. Then $(M, g, \xi, \lambda, \alpha, \beta)$ is an almost Ricci-Yamabe soliton whenever $\frac{\rho}{\psi} - \beta r$ is constant.*

Theorem 3.5. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, 0, 1)$ be an almost Ricci-Yamabe soliton of type $(0, 1)$, where $\bar{M} = I_\psi \times M$ is a SSST and assume that $grad\psi$ is a concircular vector field with factor ρ. Then $(M, g, \xi, \lambda, \alpha, \beta)$ is an almost Yamabe soliton whenever $\frac{\rho}{\psi}$ is constant.*

Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton where $\bar{M} = I_\psi \times M$ is a standard static speacetime and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ be a conformal vector field on $\bar{M}$. Then

$$\alpha\bar{S}(\bar{E}, \bar{F}) = (\lambda - \frac{\beta\bar{r}}{2} - \rho)\bar{g}(\bar{E}, \bar{F}) \tag{20}$$

and hence from (9) we turn up

$$\alpha \Delta \psi = (\lambda - \frac{\beta \bar{r}}{2} - \rho)\psi. \tag{21}$$

Again using (11) in the above equation entails that

$$2\alpha S(E,F) - \frac{1}{\psi}H^\psi(E,F) + (2\lambda - \beta\left\{r - 2\frac{1}{\psi}\Delta\psi\right\})g(E,F) = 0. \tag{22}$$

Assuming that $H^\psi = 0$ and $\Delta\psi = 0$, we have $\alpha S(E,F) = 0$. Thus we have the following results.

Theorem 3.6. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST, $H^\psi = 0$ and $\Delta\psi = 0$ with $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ is a conformal vector field on $\bar{M}$ with factor $2\rho - \beta r$. Then (M, g) is Ricci flat and $\lambda = \rho + \frac{\beta r}{2}$.*

Theorem 3.7. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Yamabe soliton of type $(0, 1)$, where $\bar{M} = I_\psi \times M$ is a SSST, $H^\psi = 0$ and $\Delta\psi = 0$ with $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ is a conformal vector field on $\bar{M}$ with factor $2\rho - r$. Then (M, g) is Ricci flat and $\lambda = \rho + \frac{r}{2}$.*

Corollary 3.8. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST, $H^\psi = 0$ and $\Delta\psi = 0$ with $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ is a conformal Killing vector field on $\bar{M}$. Then (M, g) is an expanding almost Ricci-Yamabe soliton i.e., $\lambda = \frac{\beta r}{2}$.*

Corollary 3.9. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type $(0, 1)$, where $\bar{M} = I_\psi \times M$ is a SSST, $H^\psi = 0$ and $\Delta\psi = 0$ with $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ is a conformal Killing vector field on $\bar{M}$. Then (M, g) is an expanding almost Yamabe soliton i.e., $\lambda = \frac{r}{2}$.*

Corollary 3.10. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ and consider that $H^\psi = 0$ and $\Delta\psi = 0$ and (M, g) is Ricci flat. Then $\bar{\xi}$ is a conformal with factor $-2\lambda + \beta\bar{r}$.*

Proof. Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST with $\bar{\xi} = h\partial_t + \xi \in \chi(M)$. Then

$$(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) + 2\alpha\bar{S}(\bar{E}, \bar{F}) + (2\lambda - \beta\bar{r})\bar{g}(\bar{E}, \bar{F}) = 0,$$

for any vector fields $\bar{E}, \bar{F} \in \chi(M)$. From (10) and (11), we obtain

$$(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) = -(2\lambda - \beta\bar{r})\bar{g}(\bar{E}, \bar{F}),$$

i.e., $\bar{\xi}$ is a conformal vector field with factor $-(2\lambda - \beta\bar{r})$. □

Theorem 3.11. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST and $\bar{\xi} = h\partial_t + \xi \in \chi(M)$ and consider that $H^\psi = 0$, $\Delta\psi = 0$ and (M, g) is Ricci flat. Moreover if $\bar{\xi}$ is a conformal vector field with factor $-2\lambda + \beta\bar{r}$, then $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ also admits an almost Yamabe soliton.*

4. Poisson-Laplace equation for standard static spacetime

Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST and $\bar{\xi} = h\partial_t + \xi$ is vector field on M and $\mu : I \longrightarrow \mathbb{R}$ is smooth function. We have

$$\frac{1}{2}(\bar{\mathcal{L}}_{\bar{\xi}}\bar{g})(\bar{E}, \bar{F}) + \alpha\bar{S}(\bar{E}, \bar{F}) = \left(\lambda - \frac{\beta\bar{r}}{2}\right)\bar{g}(\bar{E}, \bar{F}) \tag{23}$$

Now, computing above equation (23) at (∂_t, ∂_t), we find that

$$\Delta(\psi) = (\mu^{'} - \lambda - \frac{\beta r}{2})\psi + \xi(\psi) \tag{24}$$

It is clear that $\lambda = \mu^{'} + \frac{\beta r}{2}$ if ψ is a constant. Equation (24) at (E, F) where $E, F \in \chi(\mathbf{M})$ we get

$$\frac{1}{2}(\mathcal{L}_\xi g)(E, F) + \alpha S(E, F) = \left(\lambda - \frac{\beta r}{2} + \frac{1}{\psi}\Delta\psi\right)\bar{g}(\bar{E}, \bar{F}) + \frac{1}{\psi}H^\psi(E, F) \tag{25}$$

If ξ is a conformal vector field on $\mathbf{M}$, then we have

$$\alpha S(E, F) = \left(\lambda - \rho - \frac{\beta r}{2} + \frac{1}{\psi}\Delta\psi\right)\bar{g}(\bar{E}, \bar{F}) + \frac{1}{\psi}H^\psi(E, F) \tag{26}$$

Thus M is Einstein if $H^\psi = 0$. Taking trace on both sides we get

$$r = \left(\alpha + \frac{\beta n}{2}\right)\left[n(\lambda - \rho) + \frac{n}{\psi}\Delta\psi\right] \tag{27}$$

Theorem 4.1. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$. Consider that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β). Then the function ψ satisfies the Poisson-Laplace equation*

$$\Delta(\psi) = \frac{\psi}{n}\left(\alpha + \frac{\beta n}{2}\right) - \frac{1}{\psi}(\mu^{'} - \lambda). \tag{28}$$

Corollary 4.2. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$. Consider that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, 0)$ be an almost Ricci soliton of type $(1,0)$. Then the function ψ satisfies the Poisson-Laplace equation*

$$\Delta(\psi) = \frac{\psi}{n}\alpha - \frac{1}{\psi}(\mu^{'} - \lambda). \tag{29}$$

Corollary 4.3. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$. Consider that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, 0, \beta)$ be an almost Yamabe soliton of type $(0,1)$. Then the function ψ satisfies the Poisson-Laplace equation*

$$\Delta(\psi) = \frac{\psi}{n}\left(\frac{\beta n}{2}\right) - \frac{1}{\psi}(\mu^{'} - \lambda). \tag{30}$$

Theorem 4.4. *If $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β), where $\bar{M} = I_\psi \times M$ is a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ and ξ is a conformal vector field on $\mathbf{M}$, then the scalar curvature r of $\mathbf{M}$ is given by*

$$r = \left(\alpha + \frac{\beta n}{2}\right)\left[n(\lambda - \rho) + \frac{n}{\psi}\Delta\psi\right] \tag{31}$$

Corollary 4.5. *If $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, 0)$ be an almost Ricci soliton of type $(1, 0)$, where $\bar{M} = I_\psi \times M$ is a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ and ξ is a conformal vector field on $\mathbf{M}$, then the scalar curvature r of $\mathbf{M}$ is given by*

$$r = \alpha\left[n(\lambda - \rho) + \frac{n}{\psi}\Delta\psi\right] \tag{32}$$

Corollary 4.6. *If $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, 0, \beta)$ be an almost Yamabe soliton of type $(0, 1)$, where $\bar{M} = I_\psi \times M$ is a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ and ξ is a conformal vector field on M, then the scalar curvature r of $\mathbf{M}$ is given by*

$$r = \frac{\beta n^2}{2}\left[(\lambda - \rho) + \frac{\Delta\psi}{\psi}\right] \tag{33}$$

Now, from (26) we also have the following result

Theorem 4.7. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$. Consider that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type (α, β). Then M is an Einstein manifold if ξ is a conformal vector field and $H^\psi = 0$.*

Corollary 4.8. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$. Consider that $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, 0)$ be an almost Ricci soliton of type $(1, 0)$. Then M is an Einstein manifold if ξ is a conformal vector field and $H^\psi = 0$.*

4.1. Significance of Poisson-Laplace equation in Physics

The *GR* of solution of Poisson-Laplace equation is known as *potential theory* and the solution of Poisson-Laplace equation are harmonic functions, which are important in branches of physics, electrostatics, gravitation and fluid dynamics. In modern physics, there are two fundamental forces of the nature known at the time, namely, gravity and the electrostatics forces, could be modeled using functions called the gravitational potential and electrostatics potential both of which satisfy Poisson-Laplace equation. For example, consider the phenomena, if ψ be the gravitational filed, ρ the mass density and G the gravitational constant. The Gauss's law of gravitational in differential form is

$$\nabla \psi = -4\pi G \rho. \tag{34}$$

In case of gravitational field, ψ is conservative and can be expressed as the negative gradient of gravitational potential, i.e., $\psi = -grad\, f$ then by the Gauss's law of gravitational , we have

$$\nabla^2 f = 4\pi G \rho. \tag{35}$$

This physical phenomena is directly identical to the Theorem (4.1) and equation (28), which is a Poisson-Laplace equation with potential vector filed of gradient type.

4.2. Application of Poisson-Laplace equation in Cosmology

In the Newtonian cosmology the Poisson-Laplace equation for the gravitational field $\nabla^2 f = 4\pi G \rho$. This equation for the universe pre suppose that matter is continuously distributed with mass density ρ, while G stands for Newton's gravitational constant and f is the gravitational potential. Therefore, Newtonian gravitational potential also satisfy the Poisson-Laplace equation with Newtonian cosmological constant Λ such that

$$\nabla^2 f = 4\pi G \rho - \Lambda$$

Poisson-Laplace equation obey the principal of relativity, it describes gravitational field. The Azimuthally symmetric theory of gravitation (ASTG-model), Magneto-Hydro-Dynamic (MHD) modeling of molecular clouds are also based on the Poisson-Laplace equation.

5. Harmonic aspect of almost Ricci-Yamabe soliton on *SSST*

This section is based on the fact that a function $f : M \longrightarrow \mathbb{R}$ is said to be harmonic if $\Delta f = 0$, where Δ is the Poisson-Laplace operator on M [24], we have the following results:

Theorem 5.1. *Let $\bar{M} = I_\psi \times M$ be a SSST with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ and ψ is a harmonic function on M. Also $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ admits an almost Ricci-Yamabe soliton of type (α, β). Then the soliton is expanding, steady and shrinking according as*

1. $\frac{1}{\psi}\mu' > \frac{\psi}{n}\left(\alpha + \frac{\beta n}{2}\right),$

2. $\frac{1}{\psi}\mu' = \frac{\psi}{n}\left(\alpha + \frac{\beta n}{2}\right)$ *and*

3. $\frac{1}{\psi}\mu' < \frac{\psi}{n}\left(\alpha + \frac{\beta n}{2}\right)$ *respectively.*

Proof. Form equation (28) we can easily obtain the desired result. $\quad\square$

Corollary 5.2. *Let* $\bar{M} = I_\psi \times M$ *be a SSST with metric* $\bar{g} = -\psi^2 dt^2 \oplus g$ *and* ψ *is a harmonic function on* **M**. *Also* $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, 0)$ *admits an almost Ricci soliton of type* $(\alpha, 0)$. *Then soliton is expanding, steady and shrinking according as*

1. $\frac{1}{\psi}\mu' > \frac{\psi}{n}\alpha$,

2. $\frac{1}{\psi}\mu' = \frac{\psi}{n}\alpha$ *and*

3. $\frac{1}{\psi}\mu' < \frac{\psi}{n}\alpha$ *respectively.*

Corollary 5.3. *Let* $\bar{M} = I_\psi \times M$ *be a SSST with metric* $\bar{g} = -\psi^2 dt^2 \oplus g$ *and* ψ *is a harmonic function on* **M**. *Also* $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, 0, \beta)$ *admits an almost Yamabe soliton of type* $(0, \beta)$. *Then soliton is expanding, steady and shrinking according as*

1. $\frac{1}{\psi}\mu' > \frac{\beta\psi}{2}$,

2. $\frac{1}{\psi}\mu' = \frac{\beta\psi}{2}$ *and*

3. $\frac{1}{\psi}\mu' < \frac{\beta\psi}{2}$ *respectively.*

6. Almost Ricci-Yamabe soliton on *SSST* with non-rotating Killing vector field

First, we remind the concept of a *curl* operator on a semi-Riemannian manifold of an arbitrary finite dimension, that is , if V is a vector field on a semi-Riemannian manifold $(\bar{M}, \bar{g})$, then *curlV* is an anti-symmetric 2-covariant tensor field described by [15]

$$curlV(E, F) = g(\nabla_E V, F) - g(\nabla_F V, E), \tag{36}$$

where $E, F \in \chi(\mathbf{M})$.

Remark 6.1. *If* V *a Killing vector field on a semi-Riemannian manifold* $(\mathbf{M}, g)$, *then* V *is non-rotating if and only if it is parallel. Indeed, for any* $E, F \in \chi(\mathbf{M})$

$$(\mathcal{L}_V g)(E, F) = curlV(E, F) + 2g(\nabla_F V, E) = 0. \tag{37}$$

Thus $CurlV(E, F) = 0$ *for any* $E, F \in \chi(\mathbf{M})$ *and for any* $F \in \chi(\mathbf{M})$ *then* $g(\nabla_F V, E) = 0$ *for any* $E \in \chi(\mathbf{M})$.

Now, from above remark and using Corollaries (3.8) and (3.9), we obtain the following results

Theorem 6.2. *Let* $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ *be an almost Ricci-Yamabe soliton of type* (α, β), *where* $\bar{M} = I_\psi \times M$ *is a SSST,* $H^\psi = 0$ *and* $\Delta\psi = 0$ *with* $\bar{\xi} = h\partial_t + \xi \in \chi(\mathbf{M})$ *is a non-rotating Killing vector field on* $\bar{M}$. *Then* $(\mathbf{M}, g)$ *is an expanding almost Ricci-Yamabe soliton i.e.,* $\lambda = \frac{\beta r}{2}$.

Corollary 6.3. *Let* $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ *be an almost Ricci-Yamabe soliton of type* $(0, 1)$, *where* $\bar{M} = I_\psi \times M$ *is a SSST,* $H^\psi = 0$ *and* $\Delta\psi = 0$ *with* $\bar{\xi} = h\partial_t + \xi \in \chi(\mathbf{M})$ *is a non-rotating Killing vector on* $\bar{M}$. *Then* $(\mathbf{M}, g)$ *is an expanding almost Yamabe soliton i.e.,* $\lambda = \frac{r}{2}$.

7. Schrödinger equation for connected and non-compact standard static spacetime

In [11] Grigoyan and Losev considering the Schrodinger equation if **M** be a smooth, connected and non-compact Riemannian manifold, Δ the Laplace-Beltrami operator on **M**, and f is a smooth non negative function on **M** not identically zero, then considering the Schrödinger equation on **M** such that

$$\Delta u - f(x)u = 0, \tag{38}$$

Therefore, we adopt the equation (21) for $\alpha = 1$, $\beta = 0$ and assume the condition for Killing vector [8] filed i.e., $\rho = 0$

$$\Delta \psi - \lambda \psi = 0. \tag{39}$$

Thus, we can state the following theorem

Theorem 7.1. *Let $(\bar{M}, \bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ be an almost Ricci-Yamabe soliton of type $(1, 0)$, where $\bar{M} = I_\psi \times M$ is a connected and non-compact standard static spacetime with a Killing vector field on $\bar{M}$ and (M, g) is a non-shrinking and non-steady almost Ricci-Yamabe soliton i.e., $\lambda > 0$. Then the smooth function $\psi : M \longrightarrow \mathbb{R}$ satisfies the Schrödinger equation*

$$(\Delta - \lambda)\psi = 0. \tag{40}$$

8. Almost Ricci-Yamabe solitons in Static perfect fluid and Vacuum static spacetime

Static spacetimes for Einstein's field equations are significant universal solutions. Therefore, we are concerned in this part with static spacetime bearing perfect fluid matter and admitting an almost Ricci-Yamabe solitons with conformal Killing vector field [14].

A static spacetime $\bar{M} = I_\psi \times M$ with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ satisfies the Einstein equation

$$\bar{S} - \frac{\bar{r}}{2}\bar{g} = (\mu + p)\eta \otimes \eta + p\bar{g}, \tag{41}$$

for the energy-momentum stress tensor of a perfect fluid

$$T = -\mu\psi^2 dt^2 - pg \tag{42}$$

where $\bar{S}$ and $\bar{r}$ indicates the Ricci tensor and scalar curvature, , respectively for metric $\bar{g}$ in (41). In addition, η is a 1-form with $\bar{g}(\eta, \eta) = -1$ whose related vector field reflects the flux of the fluid, μ and p are smooth functions, namely time independent mass-energy density and pressure of the perfect fluid respectively in equation (42), also some solutions of (41) provide models for galaxies, stars and black holes (for more details see [15, 20]).

From (42) we know that the *static perfect fluid spacetime* equation is equivalent to [21]

$$\left(S - \frac{r}{n}g\right)\psi = \nabla^2\psi - \frac{\Delta\psi}{n}g, \tag{43}$$

and

$$r = 2\mu, \quad \psi p = \frac{n-1}{n}\left[\Delta\psi - \frac{n-2}{2(n-1)}r\psi\right].$$

In particular $\bar{M} = I_\psi \times M$ with metric $\bar{g} = -\psi^2 dt^2 \oplus g$ is said to be a *vacuum static spacetime* if (43) reduces to [21]

$$\nabla^2\psi - \left(S - \frac{r}{n}g\right)\psi = 0, \tag{44}$$

where S, r, and $\nabla^2\psi$ are, respectively, the Ricci tensor, scalar curvature, and Hessian traceless tensor, Δ is the Laplacian for g.

Now consider the fact that vector field $\bar{\xi}$ is Killing and using (16), (43), and (44), we find

$$\lambda = \left(\frac{\alpha}{\psi}\nabla^2\psi + \alpha\frac{\Delta\psi}{n\psi}\right) - 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right). \tag{45}$$

Thus, we have followings results:

Theorem 8.1. *If a static perfect fluid spacetime $(\bar{M}, \bar{g})$ satisfying the Einstein equation with a conformal Killing vector field admits an almost Ricci-Yamabe soliton $(\bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ of type (α, β), then almost Ricci-Yamabe soliton is expanding, steady and shrinking according as*

1. $) \left(\frac{\alpha}{\psi}\nabla^2\psi + \alpha\frac{\Delta\psi}{n\psi}\right) > 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$

2. $\left(\frac{\alpha}{\psi}\nabla^2\psi + \alpha\frac{\Delta\psi}{n\psi}\right) = 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$ *and*

3. $\left(\frac{\alpha}{\psi}\nabla^2\psi + \alpha\frac{\Delta\psi}{n\psi}\right) < 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$ *respectively.*

Theorem 8.2. *If a Vacuum static spacetime $(\bar{M}, \bar{g})$ satisfying the Einstein equation with a conformal Killing vector field admits an almost Ricci-Yamabe soliton $(\bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ of type (α, β), then almost Ricci-Yamabe soliton is expanding, steady and shrinking according as*

1. $\frac{\alpha}{\psi}\nabla^2\psi > 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$

2. $\frac{\alpha}{\psi}\nabla^2\psi = 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$ *and*

3. $\frac{\alpha}{\psi}\nabla^2\psi < 2\mu\left(\frac{\alpha}{n} - \frac{\beta}{2}\right),$ *respectively.*

Corollary 8.3. *If a static perfect fluid spacetime $(\bar{M}, \bar{g})$ satisfying the Einstein equation with a conformal Killing vector field admits an almost Yamabe soliton $(\bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ of type $(0, \beta)$, then almost Yamabe soliton is expanding i.e $\lambda = \beta\mu$.*

Corollary 8.4. *If a Vacuum static spacetime $(\bar{M}, \bar{g})$ satisfying the Einstein equation with a conformal Killing vector field admits an almost Yamabe soliton $(\bar{g}, \bar{\xi}, \lambda, \alpha, \beta)$ of type $(0, \beta)$, then almost Yamabe soliton is also expanding.*

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