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A System of Two Competitive Prey Species in Presence of Predator Under the Influence of Toxic Substances

Ashok Mondal[a], A. K. Pal[b], P. Dolai[a], G. P. Samanta[a]

[a] *Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, INDIA*
[b] *Department of Mathematics, S. A. Jaipuria College, Kolkata - 700005, INDIA*

Abstract. In this article, a two prey - one predator model has been studied where two prey species are competitive in nature and also uses toxic substances for own existence. Biologically well posedness of the model system has been shown through positivity and boundedness of solutions. Existence criterion and stability analysis of the non-negative equilibrium points have been discussed. The sufficient conditions for existence of Hopf bifurcation and stability switches induced by delay are investigated. The direction and the stability criteria of the bifurcating periodic solutions are determined with the help of the normal form theory and the center manifold theorem. Numerical simulations are performed to illustrate the theoretical analysis results.

1. Introduction

In the field of applied sciences, mathematics is used in population study for so many centuries. In 1798, English scholar Robert Malthus [16] initiated the study of population growth through mathematical modelling and after that several researchers are working in this field. Research on population dynamics is now dividing in multiple directions including prey-predator interaction. The dynamics of prey-predator interaction is one of the important topic in ecology as well as in mathematical ecology due to its ubiquitous entity and significance [1]. Lotka [12] and Volterra [25] initiated the study of prey-predator interaction.

The major environmental concern in the recent decades is the effects of toxicant on ecological communities. Hallam and Clark [6], Hallam et al.[7], Hallam and De Luna [8] started eco-toxicological modeling through mathematics. In this context it is mentioned that De Luna and Hallam [13], Freedman and Shukla [5], Samanta [23], Shukla and Dubey [24] and others [10, 17, 21] have made important contributions on this field. However, most of these models recognize with single species or two-species ecological communities. It became very difficult for the species to survive on the earth due to discharge of a large amount of toxicant. As a consequence many species are going to extinct. Therefore, a substantial amount of further research should be performed on this field. In recent time, mathematically modeling of eco-toxicological problems in aquatic environment are extensively studied by several researchers. They are paying attention

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Corresponding author: A. K. Pal
Email addresses: ashoke.2012@yahoo.com (Ashok Mondal), akpal_2002@yahoo.co.in (A. K. Pal), pralay_dolai@yahoo.in (P. Dolai), g_p_samanta@yahoo.co.uk, gpsamanta@math.iiests.ac.in (G. P. Samanta)

on the eco-toxicological effects of toxic substances released by the marine biological species themselves [10, 17, 20, 21, 23].

Dynamical complexities of predator-prey models have been extensively studied by several researchers to understand the long-time behavior of the species. Several studies in this area indicate that population modeling can be suitably extended by incorporating time delay. In nature, time delays occur in almost every biological situation [2, 3, 14, 15, 18, 19]. In population dynamics, a time delay is introduced when the rate of change of population biomass is not only a function of the present population biomass but also depends on the past population biomass and it (time delay) assumes to be one of the causes of regular fluctuations on population biomass.

In the present work, we have made an attempt to study a two-prey-one-predator model, each prey species obeys the logistic law of growth. It is assumed that the two prey species compete with each other for sharing a common food source and each species releases a substance toxic to the other species as a biological measure of deterring the competitor from sharing the food resource. The predator species is also affected by consuming the toxic through external toxic substances only. It is a very reasonable form of interaction between marine fish species competing for the use of a common food supply and a predator species depending on the both competing fish species. The remainder of this work is organized as follows: Section 2 deals with the construction and model assumptions. Positivity and boundedness of the solutions of the underlying model are discussed in Section 3. In Section 4, we have studied the existence, local stabilities of the equilibrium points and permanence of the model system. A detailed analysis of the effect of discrete time-delay has been performed in Section 5. Direction and stability of the Hopf-Bifurcation is discussed in Section 6. Numerical simulations have been made in Section 7 to validate the analytical findings. Finally, Section 8 deals with the general discussions and biological implications.

2. Mathematical formulation

Let us assume two prey species which compete with each other for sharing a common food source and also releases toxic substances to the other species as a biological measure of impeding the competitor from sharing the same food resource. It is assumed that in the absence of predator, both prey population grow according to logistic curve with carrying capacity k_1 and k_2 ($k_1,\ k_2 > 0$) and with an intrinsic growth rate r_1 and r_2 respectively. The predator species (feeding on both the prey species) is also affected by consuming the toxic released through external toxic substances only. It is also assumed that prey-predator interaction is governed by Holling type-II response function. Based on these assumptions the underlying model can be represented by the following system of ordinary differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= r_1 X\left(1-\frac{X}{k_1}\right) - \frac{\acute{\alpha}_{13}XZ}{\acute{a}_1+X} - \acute{\alpha}_{12}XY - \acute{\gamma}_1 X^2 Y \\
\frac{dY}{dT} &= r_2 Y\left(1-\frac{Y}{k_2}\right) - \frac{\acute{\alpha}_{23}YZ}{\acute{a}_2+Y} - \acute{\alpha}_{21}XY - \acute{\gamma}_2 XY^2 \\
\frac{dZ}{dT} &= \frac{b_1\acute{\alpha}_{13}XZ}{\acute{a}_1+X} + \frac{b_2\acute{\alpha}_{23}YZ}{\acute{a}_2+Y} - d_1 Z - \gamma_3 Z^2
\end{aligned} \tag{1}$$

with $X(0) > 0, Y(0) > 0, Z(0) > 0$. Here $X(T)$, $Y(T)$, $Z(T)$ denote the population biomasses of the two competing prey species and predator species at any time T respectively.

All the parameters involved in system (1) are assumed to be positive constants with the following interpretation:

r_i : represents the intrinsic (per capita) growth rate of the prey species i ($i = 1, 2$) in absence of other prey species and predator.

k_i : carrying capacity of the prey species i ($i = 1, 2$) in absence of other prey species and predator.

$\acute{\alpha}_{13}$, $\acute{\alpha}_{23}$: denotes the maximal consumption rate of the predator for the first and second prey respectively.

$\acute{\alpha}_{ij}$: measures the action of prey species j upon the growth rate of prey species ($i \neq j$, $i, j = 1, 2$).

$\acute{\gamma}_i$, γ_3 : toxic coefficients ($i = 1, 2$).

d_1 : per capita death rate of the predator.

$\acute{a}_1, \acute{a}_2$: half saturation constants.

b_i : conversion factors ($i = 1, 2$).

It is assumed that among the two prey species: (i) each species produces a substance toxic to the other (but only when the other is present), (ii) there is an accelerated growth in the production of the toxic substance since $\frac{d}{dX}(\acute{\gamma}_1 X^2) = 2\acute{\gamma}_1 X > 0$ and $\frac{d^2}{dX^2}(\acute{\gamma}_1 X^2) = 2\acute{\gamma}_1 > 0$ and similarly for the other species. The effect of toxicity on the predators being less because it comes through external toxic substances only.

To reduce the number of parameters, we rescale the variables as:

$$x = \frac{X}{k_1},\ y = \frac{Y}{k_2},\ z = Z \text{ and } t = T.$$

After some simplifications, system (1) takes the following form:

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x(1-x) - \frac{\alpha_{13} xz}{a_1 + x} - \alpha_{12} xy - \gamma_1 x^2 y \\
\frac{dy}{dt} &= r_2 y(1-y) - \frac{\alpha_{23} yz}{a_2 + y} - \alpha_{21} xy - \gamma_2 xy^2 \\
\frac{dz}{dt} &= \frac{\alpha_{31} xz}{a_1 + x} + \frac{\alpha_{32} yz}{a_2 + y} - d_1 z - \gamma_3 z^2
\end{aligned} \tag{2}$$

with $x(0) > 0, y(0) > 0, z(0) > 0$, where

$$\alpha_{13} = \frac{\acute{\alpha}_{13}}{k_1},\ a_1 = \frac{\acute{a}_1}{k_1},\ \alpha_{12} = \acute{\alpha}_{12} k_2,\ \gamma_1 = \acute{\gamma}_1 k_1 k_2,\ \alpha_{23} = \frac{\acute{\alpha}_{23}}{k_2},$$

$$a_2 = \frac{\acute{a}_2}{k_2},\ \alpha_{21} = \acute{\alpha}_{21} k_1,\ \gamma_2 = \acute{\gamma}_2 k_1 k_2,\ \alpha_{31} = b_1 \acute{\alpha}_{13},\ \alpha_{32} = b_2 \acute{\alpha}_{23}.$$

3. Positivity and Boundedness of system (2)

Positivity and boundedness together guarantees that the system is biologically well posed. The following theorem ensures the positivity and boundedness of system (2).

Theorem 3.1. *Every solution of system* (2) *with specified initial conditions exists and is unique in the interval* $[0, +\infty)$ *and* $x(t) > 0, y(t) > 0, z(t) > 0$, *for all* $t \geq 0$.

Proof. Since the right hand side of system (2) is absolutely continuous and locally Lipschitzian on $\mathbb{C}$, the solution $(x(t), y(t), z(t))$ of (2) with stated initial conditions exists and is unique on $[0, \kappa)$, where $0 < \kappa \leq +\infty$. From the first equation of system (2), we get

$$x(t) = x(0) \exp\left[\int_0^t \left\{r_1(1 - x(\phi)) - \frac{\alpha_{13} z(\phi)}{a_1 + x(\phi)} - \alpha_{12} y(\phi) - \gamma_1 x(\phi) y(\phi)\right\} d\phi\right] \qquad \Rightarrow x(t) > 0.$$

From the second equation of system (2), we get

$$y(t) = y(0) \exp\left[\int_0^t \left\{r_2(1 - y(\phi)) - \frac{\alpha_{23} z(\phi)}{a_2 + y(\phi)} - \alpha_{21} x(\phi) - \gamma_2 x(\phi) y(\phi)\right\} d\phi\right] \qquad \Rightarrow y(t) > 0.$$

From the third equation of system (2), we get

$$z(t) = z(0) \exp\left[\int_0^t \left\{\frac{\alpha_{31} x(\phi)}{a_1 + x(\phi)} + \frac{\alpha_{32} y(\phi)}{a_2 + y(\phi)} - d_1 - \gamma_3 z(\phi)\right\} d\phi\right] \qquad \Rightarrow z(t) > 0.$$

It completes the proof. □

Theorem 3.2. *All feasible solutions of system* (2) *that start in* $\mathbb{R}^3_+$ *are uniformly bounded.*

Proof. From the first equation of system (2):

$$\frac{dx}{dt} \leq r_1 x (1 - x).$$

It follows that

$$\limsup_{t \to \infty} x(t) \leq 1.$$

From the second equation of (2):

$$\frac{dy}{dt} \leq r_2 y (1 - y),$$

Therefore,

$$\limsup_{t \to \infty} y(t) \leq 1.$$

From the third equation of (2), for large time t:

$$\begin{aligned} \frac{dz}{dt} &\leq \tfrac{\alpha_{31} z}{a_1 + x} + \tfrac{\alpha_{32} z}{a_2 + y} - d_1 z - \gamma_3 z^2 \left[\because \limsup_{t \to \infty} x(t) \leq 1 \;\&\; \limsup_{t \to \infty} y(t) \leq 1, \;\; \forall t \geq 0\right] \\ &\leq \mu z - \gamma_3 z^2, \text{ where } \mu = \left(\tfrac{\alpha_{31}}{a_1} + \tfrac{\alpha_{32}}{a_2}\right) \\ &= \mu z \left(1 - \tfrac{z}{\frac{\mu}{\gamma_3}}\right). \end{aligned}$$

It follows that

$$\limsup_{t \to \infty} z(t) \leq \frac{\mu}{\gamma_3}.$$

Therefore, system (2) is ultimately bounded. □

4. Equilibrium points and their stability

The system (2) has six equilibrium points:

1. Trivial equilibrium: $(i)E_0(0,0,0)$.

2. Axial equilibrium: (i) $E_1(1,0,0)$, $(ii)E_2(0,1,0)$.

3. Planer equilibrium: (i) $E_3(0,\hat{y},\hat{z})$, where

$$\hat{y}^3 + A_1\hat{y}^2 + A_2\hat{y} + A_3 = 0 \text{ and } \hat{z} = \frac{r_2}{\alpha_{23}}(1-\hat{y})(a_2+\hat{y}),$$

$A_1 = \left(1 + 3a_2^2\right)$,
$A_2 = \left(-a_2 + \frac{\alpha_{32}\alpha_{23}}{r_2\gamma_3} - \frac{\alpha_{23}d_1}{r_2\gamma_3}\right)$,
$A_3 = -\left(a_2^2 + \frac{\alpha_{23}d_1a_2}{r_2\gamma_3}\right)$.

This equilibrium exists when $\hat{y} < 1$.

(ii) $E_4(\bar{x},0,\bar{z})$, where

$$\bar{x}^3 + B_1\bar{x}^2 + B_2\bar{x} + B_3 = 0$$
$$\text{and } \bar{z} = \frac{r_1}{\alpha_{13}}(1-\bar{x})(a_1+\bar{x}),$$

$B_1 = \left(a_1^2 + 2a_1 + 1\right)$,
$B_2 = \left(\frac{\alpha_{13}}{\gamma_3 r_1}(\alpha_{31} - d_1) - 2a_1\right)$,
$B_4 = -\left(a_1^2 + \frac{\alpha_{13}d_1a_1}{r_1\gamma_3}\right)$.

This equilibrium point exist when $\bar{x} < 1$.

4. Interior equilibrium: $E^*(x^*,y^*,z^*)$ which is the positive solution of the following system:

$$r_1(1-x) - \frac{\alpha_{13}z}{a_1+x} - \alpha_{12}y - \gamma_1xy = 0 \tag{3}$$

$$r_2(1-y) - \frac{\alpha_{23}z}{a_2+y} - \alpha_{21}x - \gamma_2xy = 0 \tag{4}$$

$$\frac{\alpha_{31}x}{a_1+x} + \frac{\alpha_{32}y}{a_2+y} - d_1 - \gamma_3z = 0 \tag{5}$$

Solving (3) and (4), we get

$$f(x,y) = 0, \tag{6}$$

where

$$f(x,y) = r_1(1-x) - \alpha_{12}y - \gamma_1xy - \frac{\alpha_{13}(a_2+y)}{\alpha_{23}(a_1+x)}\{\alpha_{21}x + \gamma_2xy - r_2(1-y)\}.$$

Solving (4) and (5), we get

$$g(x,y) = 0, \tag{7}$$

where

$$g(x,y) = r_2(1-y) - \alpha_{21}x - \gamma_2 xy - \frac{\alpha_{23}}{\gamma_3(a_2+y)}\left\{-\frac{\alpha_{31}x}{a_1+x} - \frac{\alpha_{32}x}{a_2+y} + d_1\right\}.$$

From (7), it is noted that when $y \to 0$, then $x \to x_a$, where

$$x_a = r_1(1-x) - \frac{\alpha_{13}a_2}{\alpha_{23}a_1}(\alpha_{21}x - r_2). \tag{8}$$

Therefore, $x_a > 0$ if

$$r_1(1-x) + \frac{\alpha_{13}r_2a_2}{\alpha_{23}a_1} > \frac{\alpha_{13}a_2}{\alpha_{23}a_1}\alpha_{21}x. \tag{9}$$

Also from the equation (6), we have $\frac{dy}{dx} = \frac{\hat{A}}{\hat{B}}$, where

$$\hat{A} = r_1 + \gamma_1 y + \frac{\alpha_{13}(a_2+y)}{\alpha_{23}(a_1+x)^2}\{a_1\alpha_{21} + a_1\gamma_2 y + r_2(1-y)\},$$

and

$$\hat{B} = -\alpha_{12} - \gamma_1 x - \frac{\alpha_{13}}{\alpha_{23}(a_1+x)}\{\gamma_2 a_2 x + r_2 a_2 + \alpha_{21}x + 2\gamma_2 xy - r_2 + 2r_2 y\}.$$

It is clear that $\frac{dy}{dx} > 0$, if either

$$\begin{aligned} &(i)\ \hat{A} > 0 \text{ and } \hat{B} > 0, \text{ or} \\ &(ii)\ \hat{A} < 0 \text{ and } \hat{B} < 0. \end{aligned} \tag{10}$$

It is also noted that if $4a_2(a_1\gamma_2 - r_2)(a_1\alpha_{21} + r_2) > (a_1a_2\gamma_2 - a_2r_2 + a_2\alpha_{21} + r_2)^2$ then $\hat{A} > 0$. Then $\frac{dy}{dx} > 0$, if $4a_2(a_1\gamma_2 - r_2)(a_1\alpha_{21} + r_2) > (a_1a_2\gamma_2 - a_2r_2 + a_2\alpha_{21} + r_2)^2$ and $\hat{B} > 0$.

Again from (7): when $y \to 0$, then $x \to x_b$, where

$$x_b = r_2 - \alpha_{21}x + \frac{\alpha_{23}x}{\gamma_3 a_2}\left\{\frac{\alpha_{31}x}{a_1+x} + \frac{\alpha_{32}x}{a_2} - d_1\right\}.$$

We note that $x_b > 0$ if

$$r_2 + \frac{\alpha_{23}x}{\gamma_3 a_2}\left\{\frac{\alpha_{31}x}{a_1+x} + \frac{\alpha_{32}x}{a_2}\right\} > \frac{\alpha_{23}d_1x}{\gamma_3 a_2} + \alpha_{21}x. \tag{11}$$

We also have

$$\frac{dy}{dx} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}.$$

It is noted that $\frac{dy}{dx} < 0$ if either

$$\begin{aligned} &(i)\quad \tfrac{\partial g}{\partial x} > 0 \text{ and } \tfrac{\partial g}{\partial y} > 0, \text{ or} \\ &(ii)\quad \tfrac{\partial g}{\partial x} < 0 \text{ and } \tfrac{\partial g}{\partial y} < 0. \end{aligned} \tag{12}$$

From the previous analysis, it is noted that two isoclines (6) and (7) intersect at a unique (x^*, y^*) if in addition with conditions (9), (10) (11) and (12), the following inequality holds:

$$x_a < x_b. \tag{13}$$

After getting the value of x^* and y^*, we can computed the value of z^* from

$$z^* = \frac{(a_1 + x^*)}{\alpha_{13}} \{r_1(1 - x^*) - \alpha_{12}y^* - \gamma_1 x^* y^*\}. \tag{14}$$

It may be noted that for z^* to be positive, we must have

$$r_1(1 - x^*) > \alpha_{12}y^* + \gamma_1 x^* y^*. \tag{15}$$

This completes the existence of (x^*, y^*, z^*).

4.1. Local stability

Now we deal with local stability of the system (2) around each of the equilibrium points by computing the corresponding variational matrix V at any arbitrary point (x, y, z) is given by

$$V = \begin{bmatrix} r_1 - 2r_1x - \frac{a_1\alpha_{13}z}{(a_1+x)^2} - \alpha_{12}y - 2\gamma_1 xy & -\alpha_{12}x - \gamma_1 x^2 & -\frac{\alpha_{13}x}{a_1+x} \\ -\alpha_{21}y - \gamma_2 y^2 & r_2 - 2r_2y - \frac{\alpha_{23}a_2z}{(a_2+y)^2} - \alpha_{21}x - 2\gamma_2 xy & -\frac{\alpha_{23}y}{(a_2+y)} \\ \frac{a_1\alpha_{31}z}{(a_1+x)^2} & \frac{a_2\alpha_{32}z}{(a_2+y)^2} & \frac{\alpha_{31}x}{(a_1+x)} + \frac{\alpha_{32}y}{(a_2+y)} - d_1 - 2\gamma_3 z \end{bmatrix}$$

At E_0, the variational matrix $V(E_1)$ becomes

$$V(E_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -d_1 \end{bmatrix}$$

The corresponding eigenvalues are $r_1, r_2,$ and $-d_1$ and hence we have the following theorem:

Theorem 4.1. *E_0 is unstable.*

At E_1, the variational matrix $V(E_1)$ becomes

$$V(E_1) = \begin{bmatrix} -r_1 & -\alpha_{12} & -\frac{\alpha_{13}}{a_1+1} \\ 0 & r_2 - \alpha_{21} & 0 \\ 0 & 0 & \frac{\alpha_{31}}{a_1+1} - d_1 \end{bmatrix}$$

The corresponding eigenvalues are $-r_1, r_2 - \alpha_{21},$ and $\frac{\alpha_{32}}{a_2+1} - d_1$ and hence we have the following theorem:

Theorem 4.2. *E_1 is locally asymptotically stable if $r_2 < \alpha_{21}$ and $\frac{\alpha_{31}}{a_1+1} < d_1$.*

At E_2, the variational matrix $V(E_2)$ becomes

$$V(E_2) = \begin{bmatrix} r_1 - \alpha_{12} & 0 & 0 \\ -\alpha_{21} - \gamma_2 & -r_2 & -\frac{\alpha_{23}}{a_2+1} \\ 0 & 0 & \frac{\alpha_{32}}{a_2+1} - d_1 \end{bmatrix}$$

The corresponding eigenvalues are $r_1 - \alpha_{12}, -r_2,$ and $\frac{\alpha_{32}}{a_2+1} - d_1$ and hence we have the following theorem:

Theorem 4.3. *E_2 is locally asymptotically stable if $r_1 < \alpha_{12}$ and $\frac{\alpha_{32}}{a_1+1} < d_1$.*

At E_3, the variational matrix $V(E_3)$ is given by

$$V(E_3) = \begin{bmatrix} r_1 - \alpha_{12}\hat{y} - \frac{\alpha_{13}\hat{z}}{a_1} & 0 & 0 \\ -\alpha_{21}\hat{y} - \gamma_2\hat{y}^2 & r_2 - 2r_2\hat{y} - \frac{a_2\alpha_{23}\hat{z}}{(a_2+\hat{y})^2} & -\frac{\alpha_{23}\hat{y}}{a_2+\hat{y}} \\ \frac{a_1\alpha_{31}\hat{z}}{(a_1+\hat{x})^2} & \frac{a_2\alpha_{32}\hat{z}}{(a_2+\hat{y})^2} & \frac{\alpha_{32}\hat{y}}{a_2+\hat{y}} - d_1 - 2\gamma_3\hat{z} \end{bmatrix}.$$

The corresponding eigenvalues are λ_1, λ_2 and λ_3, where λ_1 and λ_2 are roots of the quadratic equation:

$$\lambda^2 + P_1\lambda + P_2 = 0$$

and

$$\lambda_3 = r_1 - \alpha_{12}\hat{y} - \frac{\alpha_{13}\hat{z}}{a_1},$$

where

$$P_1 = -r_2 + 2r_2\hat{y} + \frac{a_2\alpha_{23}\hat{z}}{(a_2+\hat{y})^2} - \frac{\alpha_{32}\hat{y}}{a_2+\hat{y}} + d_1 + 2\gamma_3\hat{z}$$

and

$$P_2 = \left(r_2 - 2r_2\hat{y} - \frac{a_2\alpha_{23}\hat{z}}{(a_2+\hat{y})^2}\right)\left(\frac{\alpha_{32}\hat{y}}{a_2+\hat{y}} - d_1 - 2\gamma_3\hat{z}\right) + \frac{\alpha_{23}\alpha_{32}a_2\hat{y}\hat{z}}{(a_2+\hat{y})^3}.$$

If $r_2 < 2r_2\hat{y} + \frac{a_2\alpha_{23}\hat{z}}{(a_2+\hat{y})^2}$ and $\frac{\alpha_{32}\hat{y}}{a_2+\hat{y}} < d_1 + 2\gamma_3\hat{z},$ then P_1 and P_2 both are positive. Then all roots of $\lambda^2 + P_1\lambda + P_2 = 0$ are negative or having negative real parts.

Theorem 4.4. *E_3 is locally asymptotically stable if*

$$\begin{array}{lll} & (i) & r_1 < \alpha_{12}\hat{y} + \frac{\alpha_{13}\hat{z}}{a_1}, \\ & (ii) & r_2 < 2r_2\hat{y} + \frac{a_2\alpha_{23}\hat{z}}{(a_2+\hat{y})^2}, \\ \textit{and} & (iii) & \frac{\alpha_{32}\hat{y}}{a_2+\hat{y}} < d_1 + 2\gamma_3\hat{z}. \end{array}$$

At E_4, the variational matrix $V(E_4)$ is given by

$$V(E_4) = \begin{bmatrix} r_1 - 2r_1\bar{x} - \frac{a_1\alpha_{13}\bar{z}}{(a_1+\bar{x})^2} & -\alpha_{12}\bar{x} - \gamma_1\bar{x}^2 & -\frac{\alpha_{13}\bar{x}}{a_1+\bar{x}} \\ 0 & r_2 - \frac{\alpha_{23}\bar{z}}{a_2} - \alpha_{21}\bar{x} & 0 \\ \frac{a_1\alpha_{31}\hat{x}\hat{z}}{(a_1+\bar{x})^2} & \frac{\alpha_{32}\bar{z}}{a_2} & \frac{\alpha_{31}\bar{x}}{a_1+\bar{x}} - d_1 - 2\gamma_3\bar{z} \end{bmatrix}.$$

The corresponding eigenvalues are λ_1, λ_2 and λ_3, where λ_1 and λ_2 are roots of the quadratic equation:

$$\lambda^2 + Q_1\lambda + Q_2 = 0$$

and

$$\lambda_3 = r_2 - \frac{\alpha_{23}\bar{z}}{a_2} - \alpha_{21}\bar{x},$$

where $Q_1 = -r_1 + 2r_1\bar{x} + \frac{a_1\alpha_{13}\bar{z}}{(a_1+\bar{x})^2} - \frac{\alpha_{31}\bar{x}}{a_1+\bar{x}} + d_1 + 2\gamma_3\bar{z}$ and $Q_2 = \left(r_1 - 2r_1\bar{x} - \frac{a_1\alpha_{13}\bar{z}}{(a_1+\bar{x})^2}\right)\left(\frac{\alpha_{31}\bar{x}}{a_1+\bar{x}} - d_1 - 2\gamma_3\bar{z}\right) + \frac{\alpha_{13}\alpha_{31}a_1\bar{x}^2\bar{z}}{(a_1+\bar{x})^3}$.

If $r_1 < 2r_1\bar{x} + \frac{a_1\alpha_{13}\bar{z}}{(a_1+\bar{x})^2}$ and $\frac{\alpha_{31}\bar{x}}{a_1+\bar{x}} < d_1 + 2\gamma_3\bar{z}$, then Q_1 and Q_2 are positive. Then all roots of $\lambda^2 + Q_1\lambda + Q_2 = 0$ are negative or having negative real parts.

Theorem 4.5. *E_4 is locally asymptotically stable if*

$$\begin{array}{lll} & (i) & r_2 < \frac{\alpha_{23}\bar{z}}{a_2} + \alpha_{21}\bar{x}, \\ & (ii) & r_1 < 2r_1\bar{x} + \frac{a_1\alpha_{13}\bar{z}}{(a_1+\bar{x})^2}, \\ \textit{and} & (iii) & \frac{\alpha_{31}\bar{x}}{a_1+\bar{x}} < d_1 + 2\gamma_3\bar{z}. \end{array}$$

At E^*, the variational matrix $V(E^*)$ is as follows:

$$V(E^*) = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix},$$

where

$$v_{11} = -r_1x^* + \frac{\alpha_{13}x^*z^*}{(a_1+x^*)^2} - \gamma_1x^*y^*, v_{12} = -\alpha_{12}x^* - \gamma_1x^{*2}, v_{13} = -\frac{\alpha_{13}x^*}{a_1+x^*},$$

$$v_{21} = -\alpha_{21}y^* - \gamma_2y^{*2}, v_{22} = -r_2y^* + \frac{\alpha_{23}y^*z^*}{(a_2+y^*)^2} - \gamma_2x^*y^*, v_{23} = -\frac{\alpha_{23}y^*}{a_2+y^*},$$

$$v_{31} = \frac{a_1\alpha_{31}z^*}{(a_1+x^2)^2}, v_{32} = \frac{a_2\alpha_{32}z^*}{(a_2+y^*)^2}, v_{33} = -\gamma_3z^*.$$

The corresponding characteristic equation is given by

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0,$$

where

$$C_1 = -(v_{11}+v_{22}+v_{33}), C_2 = (v_{11}v_{22} + v_{11}v_{33} + v_{22}v_{33} - v_{23}v_{32} - v_{12}v_{21} - v_{13}v_{31})$$
$$\text{and } C_3 = (v_{11}v_{23}v_{32} + v_{12}v_{21}v_{33} + v_{13}v_{31}v_{22} - v_{11}v_{22}v_{23} - v_{12}v_{23}v_{31} - v_{13}v_{23}v_{32}).$$

According to the Routh-Hurwitz criterion all three eigenvalues of $V(E^*)$ have negative real parts if

$$(i)\ C_1 > 0,\ (ii)\ C_3 > 0,\ \text{and } (iii)\ C_1C_2 - C_3 > 0.$$

Thus we have the following theorem:

Theorem 4.6. *E^* is locally asymptotically stable if*

$(i)\ C_1 > 0,$
$(ii)\ C_3 > 0$ *and*
$(iii)\ C_1C_2 - C_3 > 0.$

4.2. Permanence

Theorem 4.7. *Suppose the parameters of system* (2) *fulfill the following conditions:*

$$\begin{array}{l}(i)\ (r_2-\alpha_{21})>0 \text{ and/ or } \left(\frac{\alpha_{31}}{a_1+1}-d_1\right)>0;\\ (ii)\ (r_1-\alpha_{12})>0 \text{ and/ or } \left(\frac{\alpha_{32}}{a_2+1}-d_1\right)>0;\\ (iii)\ \left[r_1-\frac{\alpha_{13}\hat{z}}{a_1}-\alpha_{12}\hat{y}\right]>0;\\ (iv)\ \left[r_2-\frac{\alpha_{23}\bar{z}}{a_2}-\alpha_{21}\bar{x}\right]>0.\end{array} \tag{16}$$

then system (2) *is permanence.*

Proof. Here we assume the average Lyapunov function in the form $V_1(x,y,z) = x^{\theta_1}y^{\theta_2}z^{\theta_3}$ where each θ_i $(i=1,2,3)$ are positive constants. In the interior of $\mathbb{R}^3_+$, we have

$$\frac{\dot{V}_1}{V_1} = \varphi(x,y,z) = \theta_1\left[r_1(1-x)-\frac{\alpha_{13}z}{a_1+x}-\alpha_{12}y-\gamma_1xy\right]+\theta_2\left[r_2(1-y)-\frac{\alpha_{23}z}{a_2+y}-\alpha_{21}x-\gamma_2xy\right]$$
$$+\theta_3\left[\frac{\alpha_{31}x}{a_1+x}+\frac{\alpha_{32}y}{a_2+y}-d_1-\gamma_3z\right].$$

To verify permanence of system (2) we shall have to exhibit that $\varphi(x,y,z)>0$ for all boundary equilibria of (2). The values of $\varphi(x,y,z)$ at the boundary equilibria $E_0, E_1, E_2, E_3,$ and E_4 are as follows:

$$\begin{array}{lcl}E_0(0,0,0) & : & r_1\theta_1+r_2\theta_2-d_1\theta_3.\\ E_1(1,0,0) & : & \theta_2(r_2-\alpha_{21})+\theta_3\left(\frac{\alpha_{31}}{a_1+1}-d_1\right).\\ E_2(0,1,0) & : & \theta_1(r_1-\alpha_{12})+\theta_3\left(\frac{\alpha_{32}}{a_2+1}-d_1\right).\\ E_3(0,\hat{y},\hat{z}) & : & \theta_1\left[r_1-\frac{\alpha_{13}\hat{z}}{a_1}-\alpha_{12}\hat{y}\right].\\ E_4(\bar{x},0,\bar{z}) & : & \theta_2\left[r_2-\frac{\alpha_{23}\bar{z}}{a_2}-\alpha_{21}\bar{x}\right].\end{array}$$

Here $\varphi(0,0,0)>0$ for some $\theta_i>0$ $(i=1,2,3)$. Also, if the inequalities (16) satisfy, then φ is positive at E_0, E_1, E_2, E_3 and E_4 for some $\theta_i>0$ $(i=1,2,3)$. Therefore, system (2) is permanence if the inequalities stated in (16) hold ([4]). Hence the theorem. □

Remark: The conditions

$$\begin{array}{lcl}E_1 & : & (r_2-\alpha_{21})>0 \text{ and/ or } \left(\frac{\alpha_{31}}{a_1+1}-d_1\right)>0;\\ E_2 & : & (r_1-\alpha_{12})>0 \text{ and/ or } \left(\frac{\alpha_{32}}{a_2+1}-d_1\right)>0;\\ E_3 & : & \left[r_1-\frac{\alpha_{13}\hat{z}}{a_1}-\alpha_{12}\hat{y}\right]>0;\\ E_4 & : & \left[r_2-\frac{\alpha_{23}\bar{z}}{a_2}-\alpha_{21}\bar{x}\right]>0,\end{array}$$

assure that the respective planer equilibrium points E_1, E_2, E_3 and E_4 are unstable.

5. The delay model

It is already mentioned that time-delay is an important factor in the biological system. It is specified that animals must take time to digest their food before further exertion and responses take place and hence

any model of species dynamics without delays is an approximation at best [11]. For these reasons let us consider system (2) with a discrete time-delay as follows:

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x(1-x) - \frac{\alpha_{13} xz}{a_1 + x} - \alpha_{12} xy - \gamma_1 x^2 y \\
\frac{dy}{dt} &= r_2 y(1-y) - \frac{\alpha_{23} yz}{a_2 + y} - \alpha_{21} xy - \gamma_2 xy^2 \\
\frac{dz}{dt} &= \frac{\alpha_{31} x(t-\tau) z}{a_1 + x(t-\tau)} + \frac{\alpha_{32} y(t-\tau) z}{a_2 + y(t-\tau)} - d_1 z - \gamma_3 z^2.
\end{aligned} \tag{17}$$

Let us study the local stability analysis of the delay system (17) around the interior equilibrium $E^*(x^*, y^*, z^*)$, where $\tau \neq 0$. Using the transformations, we linearize the system (17):

$$x = x^* + u, y = y^* + v, z = z^* + w.$$

The linearize system is given by

$$\frac{dP}{dt} = D_1 P(t) + D_2 P(t-\tau) \tag{18}$$

with

$$P = [u, v, w]^T,$$

$$D_1 = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{312} & d_{322} & 0 \end{bmatrix}, \text{ where}$$

$$d_{11} = -r_1 x^* + \frac{\alpha_{13} x^* z^*}{(a_1 + x^*)^2} - \gamma_1 x^* y^*, d_{12} = -\alpha_{12} x^* - \gamma_1 x^{*2}, d_{13} = -\frac{\alpha_{13} x^*}{a_1 + x^*},$$

$$d_{21} = -\alpha_{21} y^* - \gamma_2 y^{*2}, d_{22} = -r_2 y^* + \frac{\alpha_{23} y^* z^*}{(a_2 + y^*)^2} - \gamma_2 x^* y^*, d_{23} = -\frac{\alpha_{23} y^*}{a_2 + y^*},$$

$$d_{33} = \frac{\alpha_{31} x^*}{a_1 + x^*} + \frac{\alpha_{32} y^*}{a_1 + y^*} - d_1 - 2\gamma_3 z^*, d_{312} = \frac{\alpha_{31} a_1 z^*}{(a_1 + x^*)^2}, d_{322} = \frac{\alpha_{32} a_2 z^*}{(a_2 + y^*)^2}.$$

The characteristic equation of (18) is

$$\lambda^3 + m_{11}\lambda^2 + m_{12}\lambda + e^{-\lambda\tau}(m_{13}\lambda + m_{14}) + m_{15} = 0, \tag{19}$$

where

$$\begin{aligned}
m_{11} &= -d_{11} - d_{22} - d_{331}, \\
m_{12} &= d_{11}d_{22} + d_{11}d_{331} + d_{22}d_{331} - d_{12}d_{21}, \\
m_{13} &= -d_{23}d_{322} - d_{13}d_{312},\ m_{14} = d_{11}d_{23}d_{322} + d_{13}d_{22}d_{312} - d_{12}d_{23}d_{312} - d_{13}d_{21}d_{322}, \\
m_{15} &= d_{12}d_{21}d_{331} - d_{11}d_{22}d_{331}.
\end{aligned}$$

It is well known that the signs of the real parts of the solutions of (19) characterize the stability behavior of E^*. Therefore substituting $\lambda = \eta + i\omega$ in (19) and separating real and imaginary parts, we obtain

$$\begin{aligned}
&\eta^3 - 3\eta\omega^2 + m_{11}(\eta^2 - \omega^2) + m_{12}\eta + m_{15} + [\{m_{13}\eta + m_{14}\}\cos\omega\tau]\, e^{-\eta\tau} \\
&+ [m_{13}\omega \sin\omega\tau]\, e^{-\eta\tau} = 0,
\end{aligned} \tag{20}$$

and

$$3\eta^2\omega - \omega^3 + 2\eta\omega m_{11} + m_{12}\omega + \left[\{m_{13}\omega\}\cos\omega\tau - \{m_{13}\eta + m_{14}\}\sin\omega\tau\right]e^{-\eta\tau} = 0. \tag{21}$$

Now, we check whether equation (19) has purely imaginary roots or not. So, we set $\eta = 0$, then (20) and (21) become:

$$-m_{11}\omega^2 + m_{15} + \{m_{14}\}\cos\omega\tau + \{m_{13}\omega\}\sin\omega\tau = 0, \tag{22}$$

and

$$-\omega^3 + m_{12}\omega + \{m_{13}\omega\}\cos\omega\tau - \{m_{14}\}\sin\omega\tau = 0. \tag{23}$$

For determining ω from (22) and (23), we eliminating τ and obtain the equation as:

$$\omega^6 + \left\{-2m_{12} + m_{11}^2\right\}\omega^4 + \left\{m_{12}^2 - 2m_{11}m_{15} - m_{13}^2\right\}\omega^2 + m_{15}^2 - m_{14}^2 = 0 \tag{24}$$

Substituting $\omega^2 = \mu$ in (24), we get a cubic equation given by

$$\mu^3 + P_{11}\mu^2 + P_{12}\mu + P_{13} = 0, \tag{25}$$

where

$$P_{11} = \left\{-2m_{12} + m_{11}^2\right\}, P_{12} = \left\{m_{12}^2 - 2m_{11}m_{15} - m_{13}^2\right\}, P_{13} = m_{15}^2 - m_{14}^2.$$

If $m_{15}^2 < m_{14}^2$, then $P_{13} < 0$, so equation (5.9) has at least one positive root.

Theorem 5.1. *Equation $\mu^3 + P_{11}\mu^2 + P_{12}\mu + P_{13} = 0$ has exactly three positive roots if $\vartheta_1^2 - 4\vartheta_0^3 \leq 0, P_{11} < 0, P_{12} > 0$ and $P_{13} < 0$, otherwise it has exactly one positive root, where $\vartheta_0 = P_{11}^2 - 3P_{12}$ and $\vartheta_1 = 3P_{11}\vartheta_0 - P_{11}^3 + 27P_{13}$.*

Proof. As we have $P_{13} < 0$ which implies that the equation (25) has at least one positive root. Also, since $\vartheta_1^2 - 4\vartheta_0^3 \leq 0$, so it has three real roots. Another two roots are real and positive or real and negative. Let μ_0 be a real positive root of equation (25). Then two other roots of the equation (25) are derived from

$$\mu^2 + (P_{11} + \mu_0)\mu + P_{12} + P_{11}\mu_0 + \mu_0^2 = 0. \tag{26}$$

Now we prove that equation (26) have two positive roots if $P_{11} < 0$. If not, we consider $P_{11} > 0$ then sum of two positive roots becomes $-(P_{11} + \mu_0) < 0$, which is a contradiction. So $P_{11} < 0$ and equation (5.10) has three real positive roots if $P_{12} > 0$ by Descartes' rule of sign . Hence the theorem. □

Theorem 5.2. *Let us consider μ_0 as a real positive root of equation* (25)*. Then* (25) *has*

(*i*) *exactly one root is real positive, two others are imaginary roots if $\rho(\mu_0) > P_{11}^2 - 3P_{12}$,*

(*ii*) *one positive, two negative real roots if $\rho(\mu_0) < P_{11}^2 - 3P_{12}$, $P_{12} + P_{11}\mu_0 + \mu_0^2 > 0$ and $P_{11} + \mu_0 > 0$,*

(*iii*) *three real positive roots if $\rho(\mu_0) < P_{11}^2 - 3P_{12}$, $P_{12} + P_{11}\mu_0 + \mu_0^2 > 0$ and $P_{11} + \mu_0 < 0$, where $\rho(\mu) = 3\mu^2 + 2\mu P_{11} + P_{12}$.*

Proof. Since $P_{13} < 0$, so (25) has at least one real positive root μ_0 (say). Another two roots of (25) are obtained from

$$\mu^2 + (P_{11} + \mu_0)\mu + P_{12} + P_{11}\mu_0 + \mu_0^2 = 0.$$

Then

$$\mu = \frac{-(P_{11} + \mu_0) \pm \sqrt{P_{11}^2 - 3P_{12} - \rho(\mu_0)}}{2}.$$

Thus if (*i*) holds, then equation (5.9) have one real positive root and two imaginary roots. If (*ii*) holds, then it has one positive and two negative roots. So finally, if (*iii*) holds, then (25) has three positive real roots. □

Now we present a lemma which was proved by Ruan and Wei [22].

Lemma 5.1. *Consider the exponential polynomial:*

$$\begin{aligned} G(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m}) &= \lambda^n + g_1^{(0)}\lambda^{n-1} + ... + g_{n-1}^{(0)}\lambda + g_n^{(0)} \\ &\quad + \left[g_1^{(1)}\lambda^{n-1} + ... + g_{n-1}^{(1)}\lambda + g_n^{(1)}\right]e^{-\lambda\tau_1} \\ &\quad + ... + \left[g_1^{(m)}\lambda^{n-1} + ... + g_{n-1}^{(m)}\lambda + g_n^{(m)}\right]e^{-\lambda\tau_m}, \end{aligned}$$

where $\tau_i \geq 0$ $(i = 1, 2, ..., m)$ *and* $g_j^{(i)}$ $(i = 0, 1, ..., m;\ j = 1, 2, ..., n)$ *are constants. As* $(\tau_1, \tau_2, ..., \tau_m)$ *vary, the sum of the orders of the zeros of* $G(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m})$ *on the open half plane can change only if a zero appears on or crosses the imaginary axis.*

Let us search the existence of Hopf-bifurcation around E^* by taking τ as bifurcation parameter.

Theorem 5.3. *Let* E^* *exists and let* $\mu_0 = \omega_0^2$ *be a positive root of* (25)*. Then there exists a* $\tau = \tau^*$ *such that* E^* *is locally asymptotically stable when* $0 \leq \tau < \tau^*$ *and unstable when* $\tau > \tau^*$*, where*

$$\tau_0^{(j)} = \frac{1}{\omega_0}\arccos\frac{\left(d_2\omega_0^4 + d_3\omega_0^2 + d_4\right)}{\left(d_5\omega_0^2 + d_6\right)} + \frac{2j\pi}{\omega_0}, j = 0, 1, 2\cdots \tag{27}$$

where,

$$d_2 = m_{13},\ d_3 = (m_{11}m_{14} - m_{12}m_{13}),$$
$$d_4 = -m_{14}m_{15}, d_5 = m_{13}^2, d_6 = m_{14}^2$$

and $\tau^* = \min_{j\geq 0}\tau_0^{(j)}$*. In other words, system* (17) *exhibits a Hopf-bifurcation near* E^* *for* $\tau = \tau^*$*.*

Proof. For $\tau = 0$, the characteristic equation (19) has negative real part of all the roots under some certain conditions. Now, the equation (19) has exactly one pair of purely imaginary roots when $\tau = \tau_0^{(j)}$.

It is easy to see that equation (19) has no root with zero real part when $\tau \neq \tau_0^{(j)}$, $j = 0, 1, 2, \cdots$, and it has exactly one pair of purely imaginary roots when $\tau = \tau_0^{(j)}$. Now, τ^* is the minimum value of $\tau_0^{(j)}$ for $j = 0, 1, 2, \cdots$ and so, we conclude that all roots of (19) have negative real parts when $0 \leq \tau < \tau^*$, by Lemma 5.1. That is, E^* is stable for $\tau < \tau^*$.

When $\tau = \tau^*$, the characteristic equation (19) has a pair of purely imaginary roots and the delayed system switches its stability. It is observed that

$$\left[\frac{d\eta}{d\tau}\right]_{\tau=\tau^*} = \frac{\mu_0\gamma(\mu_0)}{K_1^2 + K_2^2} = \frac{\omega_0^2\gamma(\omega_0^2)}{K_1^2 + K_2^2},$$

where $K_1 = -3\omega_0^2 + m_{12} + \{m_{13} - m_{14}\tau^*\}\cos\omega_0\tau^* - m_{13}\tau^*\omega_0\sin\omega_0\tau^*$ and $K_2 = -2m_{11}\omega_0 + m_{13}\omega_0\tau^*\cos\omega_0\tau^* + (m_{13} - m_{14}\tau^*)\sin\omega_0\tau^*$.

Since equation (25) has exactly only one positive root μ_0, therefore, other two roots of the characteristic equation are either negative or complex conjugates. Now we will show that, in both cases, $\gamma(\omega_0^2) > 0$.

First we assume that other two roots of (25) are negative, say $-\mu_4, -\mu_5$ (so that $\mu_4 > 0, \mu_5 > 0$). Then

$$f_1(\mu) \equiv \mu^3 + P_{11}\mu^2 + P_{12}\mu + P_{13} = (\mu - \mu_0)(\mu + \mu_4)(\mu + \mu_5),$$

$$\gamma(\omega_0^2) = f_1'(\mu_0) = 3\mu_0^2 + 2P_{11}\mu_0 + P_{12} = (\mu_0 + \mu_4)(\mu_0 + \mu_5) > 0. \tag{28}$$

Next we assume that other two roots of (25) are complex conjugates, say $\beta_6 \pm i\beta_7$. Then

$$f_1(\mu) = \mu^3 + P_{11}\mu^2 + P_{12}\mu + P_{13} = (\mu - \mu_0)\{(\mu - \mu_6)^2 + \mu_7^2\},$$

$$\gamma(\omega_0^2) = f_1'(\mu_0) = 3\mu_0^2 + 2P_{11}\mu_0 + P_{12} = (\mu_0 - \mu_6)^2 + \mu_7^2 > 0.$$

So, the characteristic equation (19) will have at least one root with positive real part when $\tau > \tau^*$, then the delayed system becomes unstable. That is, system (17) exhibits a Hopf-bifurcation near E^* for $\tau = \tau^*$. □

6. Direction and Stability of Hopf Bifurcation

In the previous section, we obtained the conditions under which the Hopf bifurcation occurs. In this section, we shall derive the direction of the Hopf bifurcation and sufficient conditions of the stability of bifurcating periodic solution from the positive equilibrium E^* of the system (17) at the critical value $\tau = \tau^*$. We will utilize the approach of the normal form method and center manifold theorem introduced by [9].

Let $x_1 = x - x^*, x_2 = y - y^*, x_3 = z - z^*, \tau = \tau^* + \mu$, where τ^* is defined by (27) and $\mu \in \mathbb{R}$. Dropping the bars for simplification of notation, system (17) can be written as functional differential equation (FDE) in $C = C([-1, 0], \mathbb{R}^3)$ as

$$\dot{x(t)} = L_\mu(x_t) + f(\mu, x_t), \tag{29}$$

where $x(t) = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, and $L_\mu : C \to \mathbb{R}, f : \mathbb{R} \times C \to \mathbb{R}$ are given, respectively, by

$$L_\mu(\psi) = (\tau^* + \mu)\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} \psi_1(0) \\ \psi_2(0) \\ \psi_3(0) \end{bmatrix} + (\tau^* + \mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{312} & d_{322} & 0 \end{bmatrix}\begin{bmatrix} \psi_1(-1) \\ \psi_2(-1) \\ \psi_3(-1) \end{bmatrix} \tag{30}$$

$$f(\mu, \psi) = (\tau^* + \mu)\begin{bmatrix} e_1\psi_1^2(0) + e_2\psi_1(0)\psi_2(0) + e_3\psi_1(0)\psi_3(0) + e_4\psi_1^2(0)\psi_2(0) \\ f_1\psi_1(0)\psi_2(0) + f_2\psi_2^2(0) + f_3\psi_2(0)\psi_3(0) + f_4\psi_1(0)\psi_2^2(0) \\ g_1\psi_1^2(-1) + g_2\psi_2^2(-1) + g_3\psi_3^2(0) + g_4\psi_1(-1)\psi_3(0) + g_5\psi_2(-1)\psi_3(0) \end{bmatrix}, \tag{31}$$

$\psi(\theta) = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta))^T \in \mathbb{C}^3$; the entries $e_i,\ f_i,\ g_i$ $(i = 1, 2, 3)$ are given as

$$e_1 = -r_1 + \frac{a_1\alpha_{13}z^*}{(a_1 + x^*)^3},\ e_2 = -\alpha_{12} - 2\gamma_1 x^*,\ e_3 = -\frac{a_1\alpha_{13}}{(a_1 + x^*)^2},\ e_4 = -\gamma_1,$$

$$f_1 = -r_2 + \frac{a_2\alpha_{23}z^*}{(a_2 + y^*)^3},\ f_2 = -\alpha_{21} - 2\gamma_2 y^*,\ f_3 = -\frac{a_2\alpha_{23}}{(a_2 + y^*)^2},\ f_4 = -\gamma_2,$$

$$g_1 = -\frac{a_1\alpha_{31}z^*}{(a_1 + x^*)^3},\ g_2 = \frac{a_2\alpha_{32}z^*}{(a_2 + y^*)^3},\ g_3 = -\gamma_3,\ g_4 = \frac{a_1\alpha_{31}}{(a_1 + x^*)^2},\ g_5 = \frac{a_2\alpha_{32}}{(a_2 + y^*)^2}.$$

By the Riesz representation theorem, there exists a function, $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu(\psi) = \int_{-1}^{0} d\eta(\theta, \mu)\psi(\theta), \text{ for } \psi \in C \tag{32}$$

In fact, we can choose

$$\omega(\theta, \mu) = (\tau^* + \mu)\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{bmatrix}\delta(\theta) - (\tau^* + \mu)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_{312} & d_{322} & 0 \end{bmatrix}\delta(\theta + 1), \tag{33}$$

where δ is a dirac delta function. For $\psi \in C^1([-1,0],\mathbb{R}^3)$, define

$$A(\mu)\psi(\theta) = \begin{cases} \frac{d\psi(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^{0} d\omega(\mu,s)\psi(s), & \theta = 0 \end{cases} \tag{34}$$

$$R(\mu)\psi(\theta) = \begin{cases} 0\,, & \text{for}\theta \in [-1,0) \\ f(\mu,\psi), & \text{for}\theta = 0 \end{cases} \tag{35}$$

Then, the system (29) is equivalent to

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \tag{36}$$

where $x_\theta = x_{t+\theta}$ for $\theta \in [-1,0]$. For $\phi \in C^1([0,1],(\mathbb{R}^*))$, define

$$A(\mu)\psi(\theta) = \begin{cases} \frac{-d\phi(s)}{ds}, & s \in [-1,0), \\ \int_{-1}^{0} d\omega^T(t,0)\phi(-t), & s = 0, \end{cases} \tag{37}$$

and a billinear product

$$< \phi(s), \psi(\theta) >= \bar{\phi}(0)\psi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\phi}(\xi-\theta)d\omega(\theta)\psi(\xi)d\xi, \tag{38}$$

where $\omega(\theta) = \omega(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. we know that $\pm i\tau^*$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvalues of $A(0)$ and A^* corresponding to $+i\tau^*\eta_0$ and $-i\tau^*\eta_0$ respectively.

Suppose that $q(\theta) = (1, q_1, q_2)^T e^{i\theta\eta_0\tau^*}$, is the eigenvector of $A(0)$ corresponding to $i\tau^*\eta_0$. Then $A(0)q(\theta) = i\tau^*\eta_0 q(\theta)$. It follows from the definition of $A(0)$ and (30), (32), (33) that

$$\tau^* \begin{bmatrix} i\eta_0 - d_{11} & -d_{12} & -d_{13} \\ -d_{21} & i\eta_0 - d_{22} & -d_{23} \\ -d_{312}e^{-i\eta_0\tau^*} & -d_{322}e^{-i\eta_0\tau^*} & i\eta_0 - d_{33} \end{bmatrix} q(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{39}$$

Thus, we can easily obtain

$$q(0) = (1, q_1, q_2)^T, \tag{40}$$

where

$$q_1 = \frac{(i\eta_0 - d_{11})(i\eta_0 - d_{33}) - d_{13}d_{312}e^{-i\eta_0\tau^*}}{d_{12}(i\eta_0 - d_{33}) + d_{13}d_{322}e^{-i\eta_0\tau^*}},\ q_2 = \frac{(i\eta_0 - d_{11})d_{322} + d_{12}d_{312}}{d_{12}(i\eta_0 - d_{33})e^{i\eta_0\tau^*} + d_{13}d_{322}}$$

Similarly, let $q^*(s) = D(1, q_1^*, q_2^*)^T e^{is\eta_0\tau^*}$ be the eigenvector of A^* corresponding to $-i\eta_0\tau^*$. By the definition of A^*, we can compute

$$q^*(s) = D(1, q_1^*, q_2^*)^T e^{is\eta_0\tau^*} = D\left(1, \frac{d_{12}d_{312} - (i\eta_0 + d_{11})d_{322}}{d_{21}d_{322} - d_{312}d_{21}}, \frac{(i\eta_0 + d_{11})(i\eta_0 + d_{22}) - d_{12}d_{21}}{\{d_{21}d_{322} - d_{312}(i\eta_0 + d_{22})\}e^{-i\eta_0\tau^*}}\right) \tag{41}$$

In order to assure $< q^*(s), q(\theta) >= 1$, we need to determine the value of D. From (38), we have

$$\begin{aligned} < q^*(s), q(\theta) > &= \bar{D}(1, \bar{q}_1^*, \bar{q}_2^*)(1, q_1, q_2)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{q}_1^*, \bar{q}_2^*)e^{i\eta_0\tau^*(\xi-\theta)}d\omega(\theta)(1, q_1, q_2)^T e^{i\eta_0\xi\tau^*}d\xi \\ &= \bar{D}\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* - \int_{-1}^{0}(1, \bar{q}_1^*, \bar{q}_2^*)\theta e^{i\eta_0\theta\tau^*}d\omega(\theta)(1, q_1, q_2)^T\} \\ &= \bar{D}\{1 + q_1\bar{q}_1^* + q_2\bar{q}_2^* + \tau^*(\bar{q_1}^* q_2 d_{312} + \bar{q_2}^* q_1 d_{322})e^{-i\eta_0\tau^*}\}. \end{aligned} \tag{42}$$

Thus we can choose $\bar{D}$ as

$$\bar{D} = \frac{1}{1 + q_1\bar{q_1}^* + q_2\bar{q_2}^* + \tau^*(\bar{q_1}^* q_2 d_{312} + \bar{q_2}^* q_1 d_{322})e^{-i\eta_0\tau^*}},$$

$$D = \frac{1}{1 + q_1\bar{q_1}^* + q_2\bar{q_2}^* + \tau^*(\bar{q_1}^* q_2 d_{312} + \bar{q_2}^* q_1 d_{322})e^{i\eta_0\tau^*}} \tag{43}$$

In the remainder of this section, we use the theory of Hassard et al. [9] to compute the conditions describing center manifold C_0 at $\mu = 0$. Let x_t be the solution of (6.8) when $\mu = 0$.

Define

$$z(t) =< q^*, x_t >,\ W(t,\theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{44}$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{45}$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{46}$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We only consider real solutions. For solution $x_t \in C_0$ of (36). Since $\mu = 0$, we have,

$$\dot{z}(t) = i\eta_0\tau^* z + \bar{q}^*(0)f(0, W(z,\bar{z},\theta)) + 2\mathrm{Re}zq(\theta)\overset{def}{=} i\eta_0\tau^* z + \bar{q}^*(0)f(z,\bar{z}). \tag{47}$$

We rewrite this equation as

$$\dot{z}(t) = i\eta_0\tau^* z(t) + g(z,\bar{z}) \tag{48}$$

where

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) \\ &= g_{20}\tfrac{z^2}{2} + g_{11}z\bar{z} + g_{02}\tfrac{\bar{z}^2}{2} + g_{21}\tfrac{z^2\bar{z}}{2} + \cdots \end{aligned} \tag{49}$$

We have $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))$ and $q(\theta) = (1, q_1, q_2)^T e^{i\theta\omega_0\tau^*}$, so from (44) and (46) it follows that

$$\begin{aligned} x_t(\theta) &= W(t,\theta) + 2Rez(t)q(t) \\ &= W_{20}\tfrac{z^2}{2} + W_{11}z\bar{z} + W_{02}\tfrac{\bar{z}^2}{2} + (1, q_1, q_2)^T e^{i\eta_0\tau^* z} + (1, \bar{q}_1, \bar{q}_2)^T e^{-i\eta_0\tau^*}\bar{z} + \cdots \end{aligned}$$

and then, we have

$$\begin{aligned} x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}\tfrac{z^2}{2} + W_{11}^{(1)}z\bar{z} + W_{02}^{(1)}\tfrac{\bar{z}^2}{2} + \cdots, \\ x_{2t}(0) &= q_1 z + \bar{q}_1\bar{z} + W_{20}^{(2)}\tfrac{z^2}{2} + W_{11}^{(2)}z\bar{z} + W_{02}^{(2)}\tfrac{\bar{z}^2}{2} + \cdots, \\ x_{3t}(0) &= q_2 z + \bar{q}_2\bar{z} + W_{20}^{(3)}\tfrac{z^2}{2} + W_{11}^{(3)}z\bar{z} + W_{02}^{(3)}\tfrac{\bar{z}^2}{2} + \cdots, \\ x_{1t}(-1) &= ze^{-i\omega_0\tau^*} + \bar{z}e^{i\omega_0\tau^*} + W_{20}^{(1)}\tfrac{z^2}{2} + W_{11}^{(1)}z\bar{z} + W_{02}^{(1)}\tfrac{\bar{z}^2}{2} + \cdots, \\ x_{2t}(-1) &= q_1 ze^{-i\omega_0\tau^*} + \bar{q}_1\bar{z}e^{i\omega_0\tau^*} + W_{20}^{(2)}\tfrac{z^2}{2} + W_{11}^{(2)}z\bar{z} + W_{02}^{(2)}\tfrac{\bar{z}^2}{2} + \cdots, \\ x_{3t}(-1) &= q_2 ze^{-i\omega_0\tau^*} + \bar{q}_2\bar{z}e^{i\omega_0\tau^*} + W_{20}^{(3)}\tfrac{z^2}{2} + W_{11}^{(3)}z\bar{z} + W_{02}^{(3)}\tfrac{\bar{z}^2}{2} + \cdots \end{aligned} \tag{50}$$

It follows from together with (31) that

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = \bar{q}^*(0)f(0,x_t)$$

$$= \tau^*\bar{D}(1, \bar{q_1}^*, \bar{q_2}^*)\begin{pmatrix} e_1x_{1t}^2(0) + e_2x_{1t}(0)x_{2t}(0) + e_3x_{1t}(0)x_{3t}(0) + e_4x_{1t}^2(0)x_{2t}(0) \\ f_1x_{1t}(0)x_{2t}(0) + f_2x_{2t}^2(0) + f_3x_{2t}(0)x_{3t}(0) + f_4x_{1t}(0)x_{2t}^2(0) \\ g_1x_{1t}^2(-1) + g_2x_{2t}^2(-1) + g_3x_{3t}^2(0) + g_4x_{1t}(-1)x_{3t}(0) + g_5x_{2t}(-1)x_{3t}(0) \end{pmatrix}$$

$$= \tau^*\bar{D}[\{e_1 + e_2q_1 + e_3q_2 + f_1|q_1|^2 + f_2q_1|q_1|^2 + f_3|q_1|^2q_2 + g_1\bar{q_2}^*e^{-2i\eta_0\tau^*} + g_2q_1^2\bar{q_2}^*e^{-2i\eta_0\tau^*} + g_3q_2|q_2|^2 + g_4|q_2|^2e^{-i\eta_0\tau^*} + g_5q_1|q_2|^2e^{-i\eta_0\tau^*}\}z^2 + \{2e_1 + 2e_2\mathrm{Re}(q_1) + 2e_3\mathrm{Re}(q_2) + 2f_1\bar{q_1}^*\mathrm{Re}(q_1) + 2f_2|q_1|^2\bar{q_1} + 2f_3\bar{q_1}^*\mathrm{Re}(q_1\bar{q_2}) + 2g_1 + 2g_2|q_1|^2 + 2g_3|q_2|^2\bar{q_2} + g_4(|q_2|^2e^{i\eta_0\tau^*} + \bar{q_2}\bar{q_2}^*e^{-i\eta_0\tau^*}) + g_5(\bar{q_1}|q_2|^2e^{i\eta_0\tau^*} + q_1|q_2|^2e^{-i\eta_0\tau^*})\}z\bar{z} + \{e_1 + e_2\bar{q_1} + e_3\bar{q_2} + f_1\bar{q_1}\bar{q_1}^* + f_2\bar{q_1}^2\bar{q_1}^* + f_3\bar{q_1}\bar{q_1}^*\bar{q_2} + g_1\bar{q_2}^*e^{2i\eta_0\tau^*} + g_2\bar{q_1}^2\bar{q_2}^*e^{2i\eta_0\tau^*} + g_3\bar{q_2}^*\bar{q_2}^2 + g_4\bar{q_2}\bar{q_2}^*e^{i\eta_0\tau^*} + g_5\bar{q_1}\bar{q_2}\bar{q_2}^*e^{i\eta_0\tau^*}\}\bar{z}^2 + \{e_1(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + e_2(W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(0) + q_1W_{11}^{(1)}(0)) + e_3(W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0) + \frac{\bar{q_2}}{2}W_{20}^{(1)}(0) + q_2W_{11}^{(1)}(0)) + e_4(\mathrm{Re}(q_1) + q_1) + f_1(\bar{q_1}^*W_{11}^{(2)}(0) + \frac{\bar{q_1}^*}{2}W_{20}^{(2)}(0) + \frac{\bar{q_1}^*}{2}W_{20}^{(1)}(0) + |q_1|^2W_{11}^{(1)}(0)) + f_2(2\bar{q_1}^*W_{11}^{(2)}(0) + \bar{q_1}\bar{q_1}^*W_{20}^{(2)}(0)) + f_3(\frac{\bar{q_1}\bar{q_1}^*}{2}W_{30}^{(3)} + \frac{\bar{q_1}^*\bar{q_2}}{2}W_{20}^{(2)}(0) + |q_1|^2W_{11}^{(3)}(0) + \bar{q_1}^*q_2W_{11}^{(2)}(0)) + f_4(q_1|q_1|^2 + 2|q_1|^2\bar{q_1}) + g_1\bar{q_2}^*(W_{20}^{(1)}(-1)e^{i\eta_0\tau^*} + 2W_{11}^{(1)}(-1)e^{-i\eta_0\tau^*}) + g_2\bar{q_2}^*(\bar{q_1}W_{20}^{(2)}(-1)e^{i\eta_0\tau^*} + 2q_1W_{11}^{(2)}(-1)e^{-i\eta_0\tau^*}) + g_3(\bar{q_2}\bar{q_2}^*W_{20}^{(3)}(0) + 2|q_2|^2W_{11}^{(3)}(0)) + g_4(|q_2|^2W_{11}^{(-1)}(-1) + \frac{\bar{q_2}\bar{q_2}^*}{2}W_{20}^{(1)}(-1) + \bar{q_2}^*W_{11}^{(3)}(0)e^{-i\eta_0\tau^*} + \frac{1}{2}W_{20}^{(3)}(0)e^{i\eta_0\tau^*}) + g_5(|q_2|^2W_{11}^{(2)}(-1) + \frac{\bar{q_2}\bar{q_2}^*}{2}W_{20}^{(2)}(-1) + \frac{\bar{q_1}\bar{q_2}^*}{2}W_{20}^{(3)}(0)e^{i\eta_0\tau^*} + \bar{q_1}\bar{q_2}^*W_{11}^{(3)}(0)e^{-i\eta_0\tau^*})\}z^2\bar{z}]$$

Comparing the coefficients with (49) that, we get

$$\begin{aligned}
g_{20} &= 2\tau^*\bar{D}[e_1 + e_2q_1 + e_3q_2 + f_1|q_1|^2 + f_2q_1|q_1|^2 + f_3|q_1|^2q_2 + g_1\bar{q_2}^*e^{-2i\eta_0\tau^*} \\
&+ g_2q_1^2\bar{q_2}^*e^{-2i\eta_0\tau^*} + g_3q_2|q_2|^2 + g_4|q_2|^2e^{-i\eta_0\tau^*} + g_5q_1|q_2|^2e^{-i\eta_0\tau^*}] \\
g_{11} &= \tau^*\bar{D}[2e_1 + 2e_2\mathrm{Re}(q_1) + 2e_3\mathrm{Re}(q_2) + 2f_1\bar{q_1}^*\mathrm{Re}(q_1) + 2f_2|q_1|^2\bar{q_1} \\
&+ 2f_3\bar{q_1}^*\mathrm{Re}(q_1\bar{q_2}) + 2g_1 + 2g_2|q_1|^2 + 2g_3|q_2|^2\bar{q_2} \\
&+ g_4(|q_2|^2e^{i\eta_0\tau^*} + \bar{q_2}\bar{q_2}^*e^{-i\eta_0\tau^*}) + g_5(\bar{q_1}|q_2|^2e^{i\eta_0\tau^*} + q_1|q_2|^2e^{-i\eta_0\tau^*})] \\
g_{02} &= 2\tau^*\bar{D}[e_1 + e_2\bar{q_1} + e_3\bar{q_2} + f_1\bar{q_1}\bar{q_1}^* + f_2\bar{q_1}^2\bar{q_1}^* + f_3\bar{q_1}\bar{q_1}^*\bar{q_2} + g_1\bar{q_2}^*e^{2i\eta_0\tau^*} \\
&+ g_2\bar{q_1}^2\bar{q_2}^*e^{2i\eta_0\tau^*} + g_3\bar{q_2}^*\bar{q_2}^2 + g_4\bar{q_2}\bar{q_2}^*e^{i\eta_0\tau^*} + g_5\bar{q_1}\bar{q_2}\bar{q_2}^*e^{i\eta_0\tau^*}]
\end{aligned} \tag{51}$$

$$\begin{aligned}
g_{21} &= e_1(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + e_2(W_{11}^{(2)}(0) + \tfrac{1}{2}W_{20}^{(1)}(0) + \tfrac{1}{2}W_{20}^{(1)}(0) + q_1W_{11}^{(1)}(0)) \\
&+ e_3(W_{11}^{(3)}(0) + \tfrac{1}{2}W_{20}^{(3)}(0) + \tfrac{\bar{q_2}}{2}W_{20}^{(1)}(0) + q_2W_{11}^{(1)}(0)) + e_4(\mathrm{Re}(q_1) + q_1) \\
&+ f_1(\bar{q_1}^*W_{11}^{(2)}(0) + \tfrac{\bar{q_1}^*}{2}W_{20}^{(2)}(0) + \tfrac{\bar{q_1}^*}{2}W_{20}^{(1)}(0) + |q_1|^2W_{11}^{(1)}(0)) + f_2(2\bar{q_1}^*W_{11}^{(2)}(0) + \bar{q_1}\bar{q_1}^*W_{20}^{(2)}(0)) \\
&+ f_3(\tfrac{\bar{q_1}\bar{q_1}^*}{2}W_{30}^{(3)} + \tfrac{\bar{q_1}^*\bar{q_2}}{2}W_{20}^{(2)}(0) + |q_1|^2W_{11}^{(3)}(0) + \bar{q_1}^*q_2W_{11}^{(2)}(0)) + f_4(q_1|q_1|^2 + 2|q_1|^2\bar{q_1}) \\
&+ g_1\bar{q_2}^*(W_{20}^{(1)}(-1)e^{i\eta_0\tau^*} + 2W_{11}^{(1)}(-1)e^{-i\eta_0\tau^*}) + g_2\bar{q_2}^*(\bar{q_1}W_{20}^{(2)}(-1)e^{i\eta_0\tau^*} + 2q_1W_{11}^{(2)}(-1)e^{-i\eta_0\tau^*}) \\
&+ g_3(\bar{q_2}\bar{q_2}^*W_{20}^{(3)}(0) + 2|q_2|^2W_{11}^{(3)}(0)) + g_4(|q_2|^2W_{11}^{(-1)}(-1) + \tfrac{\bar{q_2}\bar{q_2}^*}{2}W_{20}^{(1)}(-1) + \bar{q_2}^*W_{11}^{(3)}(0)e^{-i\eta_0\tau^*} \\
&+ \tfrac{1}{2}W_{20}^{(3)}(0)e^{i\eta_0\tau^*}) + g_5(|q_2|^2W_{11}^{(2)}(-1) + \tfrac{\bar{q_1}\bar{q_2}^*}{2}W_{20}^{(2)}(-1) + \tfrac{\bar{q_1}\bar{q_2}^*}{2}W_{20}^{(3)}(0)e^{i\eta_0\tau^*} + \bar{q_1}\bar{q_2}^*W_{11}^{(3)}(0)e^{-i\eta_0\tau^*})
\end{aligned}$$

Since these are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (36) and (44),we have

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\mathrm{Re}\bar{q}^*f_0q(\theta), & \theta \in [-1, 0) \\ AW - 2\mathrm{Re}\bar{q}^*f_0q(\theta) + f_0, & \theta = 0. \end{cases} \tag{52}$$

By definition, is equal to $AW + H(z, \bar{z}, \theta)$, where

$$H(z, \bar{z}, \theta) = H_{10}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}\frac{\bar{z}}{2} + \cdots \tag{53}$$

Substituting the corresponding series into (52) and comparing the coefficients, we obtain

$$(A - 2i\eta_0\tau^*)W_{20} = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta). \tag{54}$$

From (52), we know that for $\theta \in [-1, \theta)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(\theta) = -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \bar{q}(\theta). \tag{55}$$

Comparing the coefficients with (53), we get

$$H_{20}(\theta) = -g_{20}(\theta) q(\theta) - \bar{g_{02}}(\theta) \bar{q}(\theta) \tag{56}$$

$$H_{11}(\theta) = -g_{11}(\theta) q(\theta) - \bar{g_{11}}(\theta) \bar{q}(\theta). \tag{57}$$

From (54) and (57) and the definition of A, it follows that

$$\dot{W_{20}}(\theta) = 2i\eta_0\tau^* W_{20}(\theta) + g_{20}(\theta) q(\theta) + \bar{g_{02}}(\theta) \bar{q}(\theta). \tag{58}$$

Notice that $q(\theta) = (1, q_1, q_2)^T e^{i\eta_0\tau^*\theta}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\eta_0\tau^*} q(0) e^{i\eta_0\tau^*\theta} + \frac{i\bar{g_{20}}}{3\eta_0\tau^*} \bar{q_0} e^{-i\eta_0\tau^*\theta} + P_1 e^{2i\eta_0\tau^*\theta} \tag{59}$$

where $P_1 = (P_1^1, P_1^2, P_1^3) \in \mathbb{R}^3$ is a constant vector. Similarly, from (54) and (58), we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\eta_0\tau^*} q(0) e^{i\eta_0\tau^*\theta} + \frac{i\bar{g_{11}}}{3\eta_0\tau^*} \bar{q_0} e^{-i\eta_0\tau^*\theta} + P_2 \tag{60}$$

where $P_2 = (P_2^1, P_2^2, P_2^3) \in \mathbb{R}^3$ is also a constant vector.

In what follows, are will seek appropriate P_1 and P_2 .From the definition of a A and (54), we obtain

$$\int_{-1}^{0} d\omega(\theta) W_{20}(\theta) = 2i\eta_0\tau^* W_{20}(0) - H_{20}(0), \tag{61}$$

$$\int_{-1}^{0} d\omega(\theta) W_{11}(\theta) = -H_{11}(0), \tag{62}$$

where $\omega(\theta) = \omega(0, \theta)$. By (52) we have

$$H_{20}(0) = -g_{20} q(0) - \bar{g_{02}} \bar{q_0} + 2\tau^* \begin{bmatrix} e_1 + e_2 q_1 + e_3 q_2 \\ f_1 q_1 + f_2 q_1^2 + f_3 q_1 q_2 \\ g_1 e^{-2i\eta_0\tau^*} + g_2 q_1^2 e^{-2i\eta_0\tau^*} + g_3 q_2^2 + g_4 q_2 e^{-i\eta_0\tau^*} + g_5 q_1 q_2 e^{-i\eta_0\tau^*} \end{bmatrix}, \tag{63}$$

$$H_{11}(0) = -g_{11} q(0) - \bar{g_{11}} \bar{q_0} + 2\tau^* \begin{bmatrix} \frac{e_1}{2} + e_2 \mathrm{Re}(q_1) + e_3 \mathrm{Re}(q_2) \\ f_1 \mathrm{Re}(q_1) + f_2 q_1^2 + f_3 \mathrm{Re}(q_1 \bar{q_2}) \\ g_1 + g_2 q_1^2 + g_3 q_2^2 + \frac{g_4}{2}(q_2 e^{i\eta_0\tau^*} + \bar{q_2} e^{-i\eta_0\tau^*}) + \frac{g_5}{2}(\bar{q_1} q_2 e^{i\eta_0\tau^*} + q_1 \bar{q_2} e^{-i\eta_0\tau^*}) \end{bmatrix}, \tag{64}$$

Substituting (59) and (63) into (61) and noting that

$$(i\eta_0\tau^* I - \int_{-1}^{0} e^{i\eta_0\tau^*\theta} d\omega(0)) q(0) = 0, \tag{65}$$

$$(-i\eta_0\tau^* I - \int_{-1}^{0} e^{-i\eta_0\tau^*\theta} d\omega(0)) q(\bar{0}) = 0,$$

We obtain

$$\left(i\eta_0\tau^* I - \int_{-1}^{0} e^{i\eta_0\tau^*\theta} d\omega(0)\right) P_1 = 2\tau^* \begin{pmatrix} Q_1^{(1)} \\ Q_1^{(2)} \\ Q_1^{(3)} \end{pmatrix} \tag{66}$$

where

$$Q_1^{(1)} = e_1 + e_2 q_1 + e_3 q_2, \;\; Q_1^{(2)} = f_1 q_1 + f_2 q_1^2 + f_3 q_1 q_2,$$

$$Q_1^{(3)} = g_1 e^{-2i\eta_0\tau^*} + g_2 q_1^2 e^{-2i\eta_0\tau^*} + g_3 q_2^2 + g_4 q_2 e^{-i\eta_0\tau^*} + g_5 q_1 q_2 e^{-i\eta_0\tau^*}.$$

This leads to

$$\begin{bmatrix} 2i\eta_0 - d_{12} & -d_{12} & -d_{13} \\ -d_{21} & 2i\eta_0 - d_{22} & -d_{23} \\ -d_{312}e^{i\eta_0\tau^*} & -d_{322}e^{i\eta_0\tau^*} & 2i\eta_0 - d_{33} \end{bmatrix} P_1 = 2 \begin{pmatrix} Q_1^{(1)} \\ Q_1^{(2)} \\ Q_1^{(3)} \end{pmatrix}. \tag{67}$$

$$P_1^1 = \frac{2}{\triangle_1} \begin{vmatrix} Q_1^{(1)} & -d_{12} & -d_{13} \\ Q_1^{(2)} & 2i\eta_0 - d_{22} & -d_{23} \\ Q_1^{(3)} & -d_{322}e^{2i\eta_0\tau^*} & 2i\eta_0 - d_{33} \end{vmatrix}, \;\; P_1^2 = \frac{2}{\triangle_1} \begin{vmatrix} 2i\eta_0 - d_{11} & Q_1^{(1)} & -d_{13} \\ -d_{21} & Q_1^{(2)} & -d_{23} \\ -d_{312}e^{2i\eta_0\tau^*} & Q_1^{(3)} & 2i\eta_0 - d_{33} \end{vmatrix}$$

$$P_1^3 = \frac{2}{\triangle_1} \begin{vmatrix} 2i\eta_0 - d_{11} & -d_{12} & Q_1^{(1)} \\ -d_{21} & 2i\eta_0 - d_{22} & Q_1^{(2)} \\ -d_{312}e^{2i\eta_0\tau^*} & -d_{322}e^{2i\omega_0\tau^*} & Q_1^{(3)} \end{vmatrix} \tag{68}$$

where

$$\triangle_1 = \begin{vmatrix} 2i\eta_0 - d_{11} & -d_{12} & -d_{13} \\ -d_{21} & 2i\eta_0 - d_{22} & -d_{23} \\ -d_{312}e^{i\eta_0\tau^*} & -d_{322}e^{i\eta_0\tau^*} & 2i\eta_0 - d_{33} \end{vmatrix}. \tag{69}$$

Similarly, substituting (57) and (64) into (62), we get

$$\begin{pmatrix} 2i\eta_0 - d_{12} & -d_{12} & -d_{13} \\ -d_{21} & 2i\eta_0 - d_{22} & -d_{23} \\ -d_{312}e^{i\eta_0\tau^*} & -d_{322}e^{i\eta_0\tau^*} & 2i\eta_0 - d_{33} \end{pmatrix} P_2 = 2 \begin{pmatrix} Q_2^{(1)} \\ Q_2^{(2)} \\ Q_2^{(3)} \end{pmatrix}. \tag{70}$$

where

$$Q_2^{(1)} = \frac{e_1}{2} + e_2 \mathrm{Re}(q_1) + e_3 \mathrm{Re}(q_2), \;\; Q_1^{(2)} = f_1 \mathrm{Re}(q_1) + f_2 q_1^2 + f_3 \mathrm{Re}(q_1 \bar{q_2}),$$

$$Q_1^{(3)} = g_1 + g_2 q_1^2 + g_3 q_2^2 + \frac{g_4}{2}(q_2 e^{i\eta_0\tau^*} + \bar{q_2} e^{-i\eta_0\tau^*}) + \frac{g_5}{2}(\bar{q_1} q_2 e^{i\eta_0\tau^*} + q_1 \bar{q_2} e^{-i\eta_0\tau^*}).$$

$$P_2^1 = \frac{2}{\triangle_2} \begin{vmatrix} Q_2^{(1)} & -d_{12} & -d_{13} \\ Q_2^{(2)} & 2i\eta_0 - d_{22} & -d_{23} \\ Q_2^{(3)} & -d_{322} & -d_{33} \end{vmatrix}, \;\; P_2^2 = \frac{2}{\triangle_2} \begin{vmatrix} 2i\eta_0 - d_{11} & Q_2^{(1)} & -d_{13} \\ -d_{21} & Q_2^{(2)} & -d_{23} \\ -d_{312} & Q_1^{(3)} & -d_{33} \end{vmatrix},$$

$$P_2^3 = \frac{2}{\triangle_2} \begin{vmatrix} 2i\eta_0 - d_{11} & -d_{12} & Q_2^{(1)} \\ -d_{21} & 2i\eta_0 - d_{22} & Q_2^{(2)} \\ -d_{312} & -d_{322} & Q_2^{(3)} \end{vmatrix} \tag{71}$$

$$\triangle_2 = \begin{vmatrix} 2i\eta_0 - d_{11} & -d_{12} & -d_{13} \\ d_{21} & 2i\eta_0 - d_{22} & -d_{23} \\ -d_{312} & -d_{322} & 2i\eta_0 - d_{33} \end{vmatrix}. \tag{72}$$

Thus we determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (60) and (64) into (62). Furthermore, g_{21} in (60) can be expressed by the parameters and delay. Thus, we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\eta_0\tau^*}(g_{20}g_{11} - 2 \mid g_{11} \mid^2 - \frac{2|g_{12}|^2}{3}) + \frac{g_{11}}{2}, \\ \gamma_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\acute{\xi}(\tau^*)\}}, \\ \beta_2 &= 2\mathrm{Re}\{c_1(0)\}, \quad T_2 = -\frac{\mathrm{Im}c_1(0) + \gamma_2 \mathrm{Im}\acute{\xi}(\tau^*)}{\eta_0\tau^*}. \end{aligned} \tag{73}$$

which determines the qualities of bifurcating periodic solution in the centre manifold at the critical value τ^*.

7. Numerical Simulations

In this section, Extensive numerical simulations have been performed for various values of parameters to determine the the dynamics of the system (2) and (17). This study provides stability analysis of each of the equilibrium points and occurrence of Hopf-bifurcation of the system (17).

7.1. I. Non-delayed system (2)

Let us take the values of the parameters as $r_1 = 0.3, \alpha_{13} = 0.3, a_1 = 0.6, \alpha_{12} = 0.1, \gamma_1 = 0.2, r_2 = 0.2, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 0.1, \alpha_{32} = 0.4, d_1 = 0.1, \gamma_3 = 0.2$. Then it is observed that $E_1(1, 0, 0)$ is locally asymptotically stable (LAS) (see Fig. 1. Now we take the parameters as $r_1 = 0.05, \alpha_{13} = 0.3, a_1 = 0.6, \alpha_{12} = 0.1, \gamma_1 = 0.2, r_2 = 0.8, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 0.1, \alpha_{32} = 0.1, d_1 = 0.1, \gamma_3 = 0.2$. Then $E_2(0, 1, 0)$ is LAS (see Fig. 1).

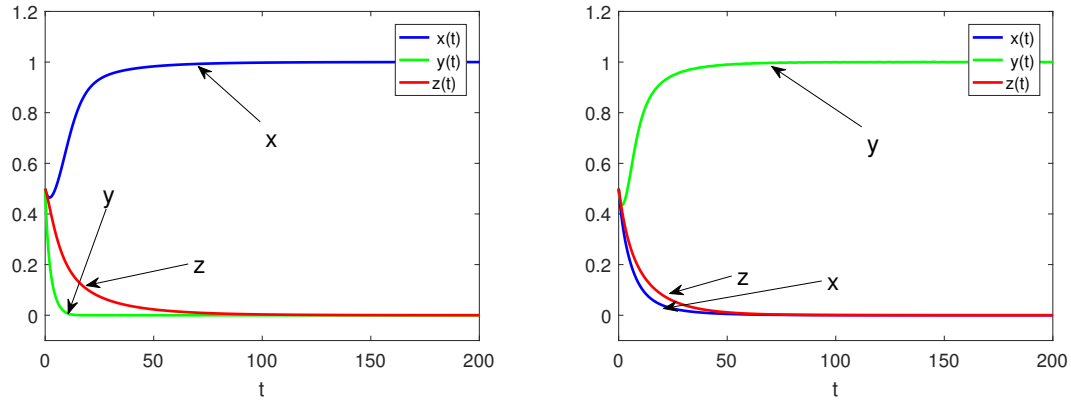

Figure 1: *Local asymptotic stability of* $E_1(1, 0, 0)$ *and local asymptotic stability of* $E_2(0, 1, 0)$.

Next, the parameter values are taken as $r_1 = 0.05, \alpha_{13} = 0.3, a_1 = 0.6, \alpha_{12} = 0.1, \gamma_1 = 0.2, r_2 = 0.8, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 0.1, \alpha_{32} = 0.4, d_1 = 0.1, \gamma_3 = 0.2$. Then the conditions of existence are satisfied and consequently $E_3(0, \hat{y}, \hat{z})$ is locally asymptotically stable (see Fig. 2). Let us take the parameter values as $r_1 = 0.3, \alpha_{13} = 0.3, a_1 = 0.6, \alpha_{12} = 0.1, \gamma_1 = 0.2, r_2 = 0.2, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 0.8, \alpha_{32} = 0.4, d_1 = 0.1, \gamma_3 = 0.2$. Then the conditions of existence are fulfilled and consequently $E_4(\bar{x}, 0, \bar{z})$ is LAS (see Fig. 2).

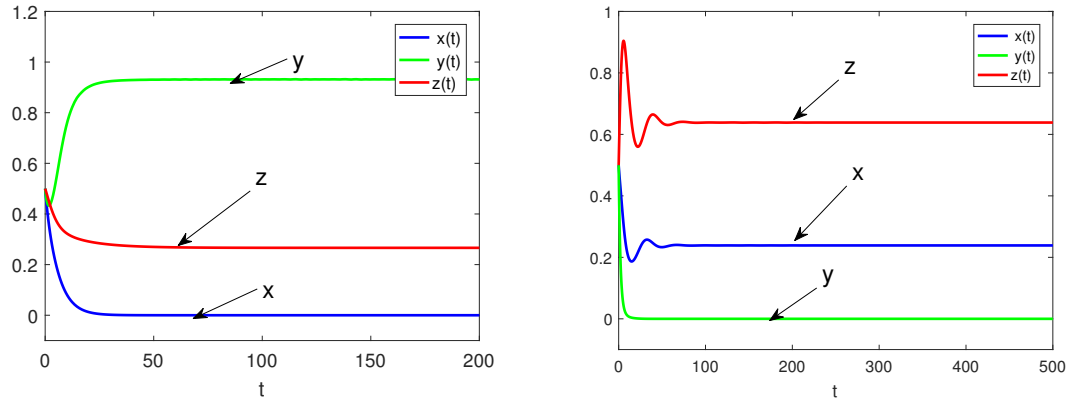

Figure 2: *Local asymptotic stability of* $E_3(0, \hat{y}, \hat{z})$ *and local asymptotic stability of* $E_4(\bar{x}, 0, \bar{z})$.

Now, take $r_1 = 0.12, \alpha_{13} = 0.2, a_1 = 0.6, \alpha_{12} = 0.05, \gamma_1 = 0.01, r_2 = 0.8, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 1.1, \alpha_{32} = 0.6, d_1 = 0.3, \gamma_3 = 0.2$ then the conditions of Theorem 4.6 is fulfilled and hence E^* exists. Also, E^* is locally asymptotically stable. The phase portrait and stable behaviour of (x, y, z) is depicted in Fig. 3.

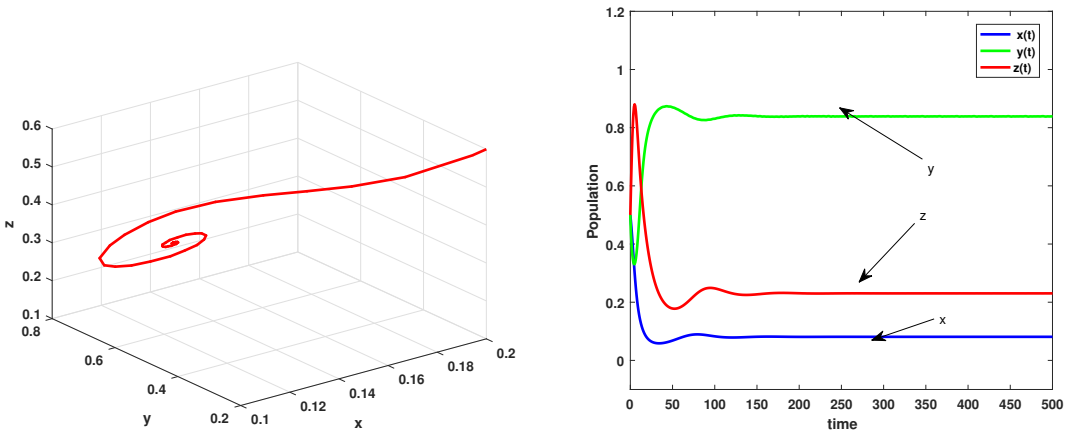

Figure 3: *Phase portrait and Local asymptotic stability of* E^*.

7.2. *II. Delayed system* (17)

It has already been mentioned that the stability criteria in absence of delay ($\tau = 0$) will not necessarily guarantee the stability of system (17) in presence of delay ($\tau \neq 0$). If we choose the values of the parameters of system (17) as $r_1 = 0.12, \alpha_{13} = 0.2, a_1 = 0.6, \alpha_{12} = 0.05, \gamma_1 = 0.01, r_2 = 0.8, \alpha_{23} = 0.5, a_2 = 1.5, \alpha_{21} = 0.9, \gamma_2 = 0.1, \alpha_{31} = 1.1, \alpha_{32} = 0.6, d_1 = 0.3, \gamma_3 = 0.2$ then equation (25) has unique positive root and Hopf-bifurcation occurs at $\tau = \tau_0^* = 3.8846$. For $\tau < \tau_0^*$, it is seen that E^* is stable (Figs. 4 and 5). Clearly the phase portrait is a stable spiral converging to E^*. If we gradually increase the value of τ (keeping other parameters fixed), it is observed that E^* loses its stability at $\tau = \tau_0^* = 3.8846$. For $\tau > \tau_0^*, E^*$ is unstable and there is a bifurcating periodic solution near E^* which is shown in Figs. 6 and 7.

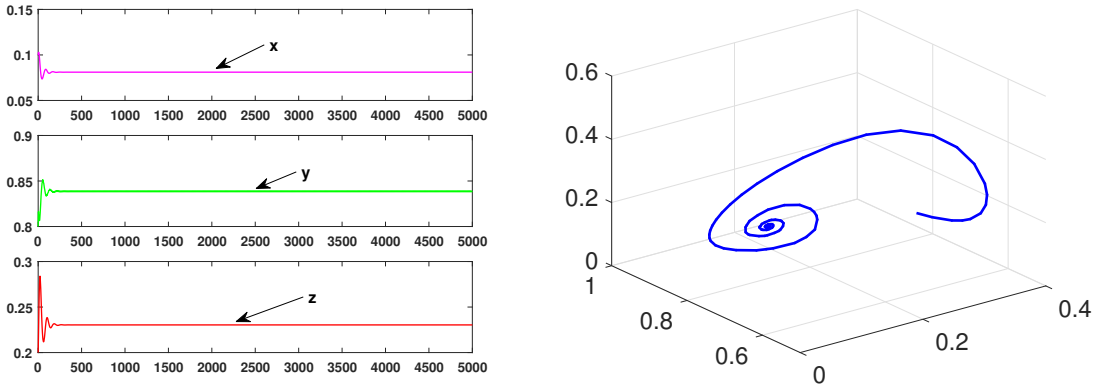

Figure 4: *Keeping other parameters fixed, if we take* $\tau = 2.0 < \tau_0^*$, *it shows that* $E^*(x^*, y^*, z^*)$ *is stable and the phase portrait of the solution being a stable spiral.*

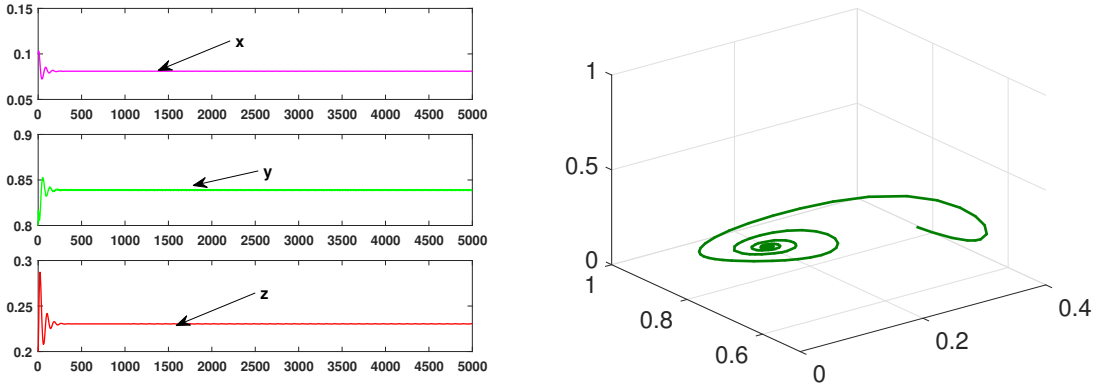

Figure 5: *Local asymptotic stability and phase portrait of* E^*, *keeping other parameters fixed, if we take* $\tau = 3.0 < \tau_0^*$, *it shows that* $E^*(x^*, y^*, z^*)$ *is stable and the phase portrait of the solution being a stable spiral.*

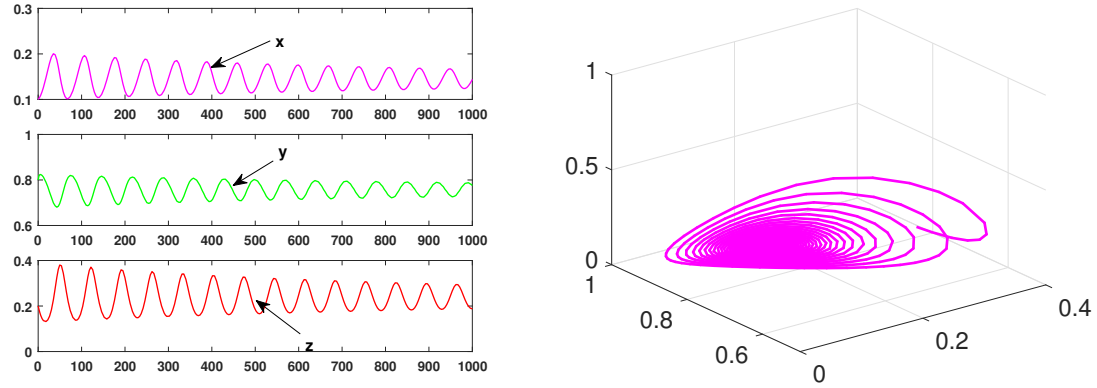

Figure 6: *Keeping other parameters fixed, if we take* $\tau = 5.0 > \tau_0^*$, *it shows that* $E^*(x^*, y^*, z^*)$ *is unstable and there is a bifurcating periodic solution near* E^*.

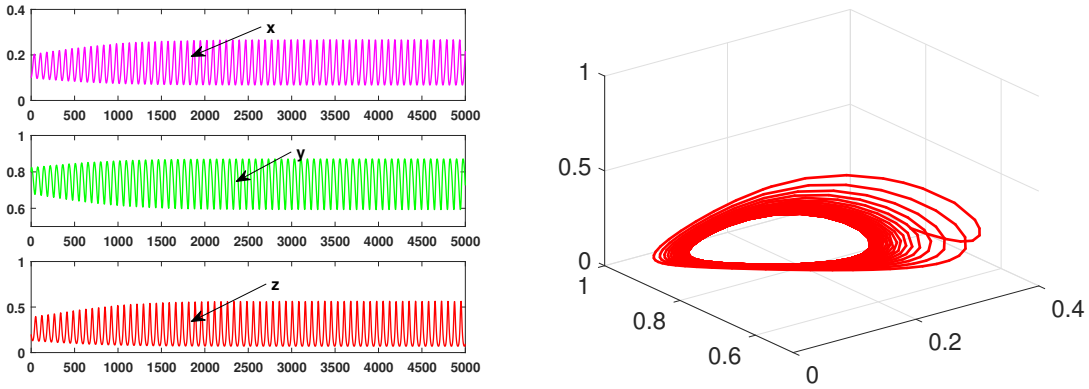

Figure 7: *Keeping other parameters fixed, if we take* $\tau = 6.0 > \tau_0^*$, *it shows that* $E^*(x^*, y^*, z^*)$ *is unstable and there is a bifurcating periodic solution near* E^*.

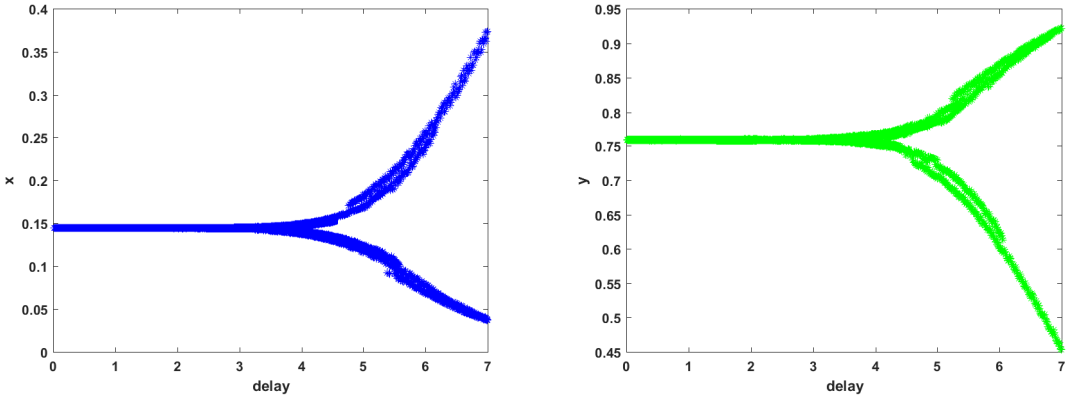

Figure 8: *Bifurcation diagram for the delay* (τ) *with respect to x and y having* $\tau = \tau_0^* = 3.8846$.

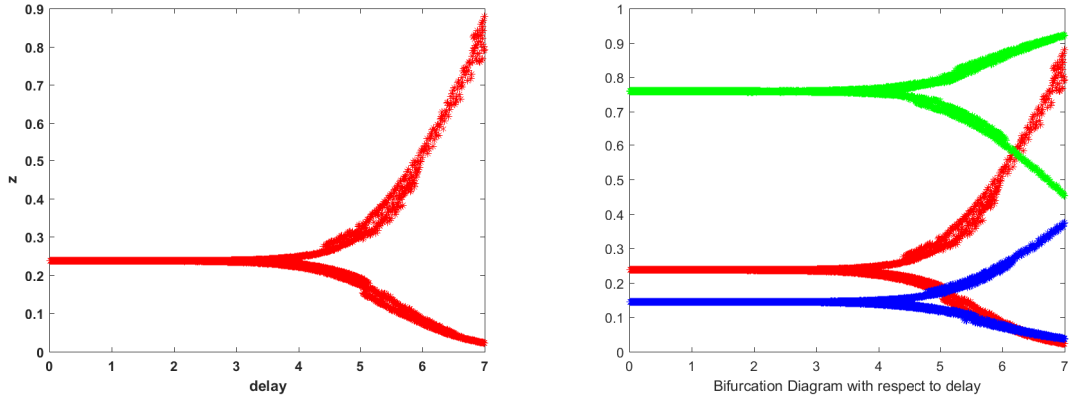

Figure 9: *Bifurcation diagram for the delay* (τ) *with respect to z having* $\tau = \tau_0^* = 3.8846$ *and bifurcation diagram for the delay* (τ) *and keeping other parameters fixed as given in Figure 3 having* $\tau = \tau_0^* = 03.8846$.

8. Conclusion

In this work, we have introduced a mathematical model with two prey species and one predator, each prey species obeys the logistic law of growth in absence of predator and toxicity. It is also assumed that the two prey species compete with each other for using a common source of food and each species releases a substance toxic to the other species as a biological measure of deterring the competitor from sharing the food resource. The predator species is also affected by consuming the toxic released through external toxic substances only. The underlying models can be treated as very reasonable form of interaction between marine fish species competing for the use of a common food supply and a predator species depending on the both competing fish species.

Here we have considered a Holling type -II functional response for predator on prey populations. The number of parameters of the model have been reduced by suitable scalings. The dynamical behaviours of the resulting model are analyzed. It is shown that the solutions of the system remain positive forever, and they are uniformly bounded. Then we have studied the behaviour of the underlying system at various equilibrium points. It is observed that the interior equilibrium points is locally asymptotically stable under certain conditions. The conditions for permanence of the system have been discussed by means of average Lyapunov function.

We have also investigated the effect of discrete time delay on the underlying model where the delay can be regarded as a gestation period or reaction time of the predator population. A rigorous analysis on the stability and bifurcation of the coexistence (interior) equilibrium point has been performed. Our analysis indicates that the value of delay in certain specified range could guarantee the stable coexistence of the species. On the other hand, the delay could drive the system to an unstable state. Thus the time-delay has a regulatory impact on the whole system. We have derived the explicit formulae to determine the stability, direction and other properties of bifurcating periodic solutions by using the normal form theory and center manifold reduction.

The theoretical investigations carried out in this work will definitely help the ecologists to do some experimental studies and as a result the theoretical ecology may be developed to some extent. Analytical studies can never be completed without numerical verifications. Our analytical findings are numerically verified using MATLAB. Numerical simulations depict the dynamical behaviour of the system at boundary and interior equilibria, which are in good agreement with analytical findings. Our model is not a case study and so it is difficult to choose parameter values from quantitative estimation. The hypothetical sets of parameter values are used to verify the analytical findings obtained in this work.

There is still some work to do in the proposed model (2) such as we would consider maturity of both the species to release toxic substances within the surrounding environment. These modifications make the model more interesting and realistic. We leave this for future consideration.

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