Filomat 36:1 (2022), 89–98
https://doi.org/10.2298/FIL2201089F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Opial-Type Inequalities for Superquadratic Functions

Ghulam Farid[a], Ayesha Bibi[b], Waqas Nazeer[c]

[a]*Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan*
[b]*Govt. Girls High School Behboodi, Hazro, Pakistan*
[c]*Department of Mathematics, Govt. College University Lahore, Pakistan*

Abstract. In this paper we prove new Opial-type inequalities for arbitrary kernels using superquadratic functions, also their extensions are obtained. Furthermore, we find their fractional versions by applying different kinds of fractional integral and fractional derivative operators.

1. Introduction and preliminary results

In 1960 Opial established the following inequality:

Theorem 1.1. *[8] Let $\psi \in C^1[0,c]$ be such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t \in (0,c)$. Then*

$$\int_0^c |\psi(t)\psi'(t)|dt \leq \frac{c}{4}\int_0^c (\psi'(t))^2 dt. \tag{1}$$

Here $\frac{c}{4}$ is the best possible constant.

In literature the inequality (1) is well known as the Opial inequality, it has/had been studied extensively since its appearance. For different variants; extensions and generalizations of this inequality we refer the readers to [5, 7, 9, 10, 12–16, 20, 22, 25, 26]. Srivastava et al. in [20] studied the weighted versions of Opial type inequalities on time scale, where they also provided counter examples to the inequalities given in [24] and also suggested their corrections. In [11, p. 236-238], the Opial inequality is studied for convex functions by defining a class of functions for a nonnegative kernel. Motivating by these Opial type inequalities our aim in this paper is to give new versions of such inequalities for superquadratic functions. Also we will provide some fractional integral/derivative operator inequalities by fixing special cases of general kernels. For fractional integral inequalities we refer the readers to [2, 4, 5, 14, 21–23, 26].
The definition of superquadratic function is given as follows:

Definition 1.2. *[1] A function $\psi : [0,\infty) \to \mathbb{R}$ is called superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that*

$$\psi(y) \geq \psi(x) + C_x(y - x) + \psi(|y - x|) \tag{2}$$

for all $y \geq 0$.

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 31A10, 26A33
Keywords. Opial inequality, Superquadratic function, Fractional integral operators
Received: 13 January 2021; Revised: 02 June 2021; Accepted: 16 October 2021
Communicated by Miodrag Spalević
Email addresses: faridphdsms@hotmail.com; ghlmfarid@ciit-attock.edu.pk (Ghulam Farid), atk.ayesha1@gmail.com (Ayesha Bibi), nazeer.waqas@gmail.com (Waqas Nazeer)

Definition 1.3. *A function $\psi : [0,\infty) \to \mathbb{R}$ is called superaddative if for all $x, y \geq 0$ we have*

$$\psi(x+y) \geq \psi(x) + \psi(y). \tag{3}$$

In the next we give classes of functions needful to prove the results of this paper as follows: Let $U_1(u,k)$ denotes the class of functions $w : [a,b] \to \mathbb{R}$ having representation

$$w(x) = \int_a^x k(x,t)u(t)dt$$

where u is a continuous function and k is an arbitrary non negative kernel such that $k(x,t) = 0$ for $t > x$ and $u(x) > 0$ implies $w(x) > 0$ for every $x \in [a,b]$. Let $U_2(u,k)$ denotes the class of functions $w : [a,b] \to \mathbb{R}$ having representation

$$w(x) = \int_x^b k(x,t)u(t)dt$$

where u is a continuous function and k is an arbitrary non negative kernel such that $k(x,t) = 0$ for $t < x$ and $u(x) > 0$ implies $w(x) > 0$ for every $x \in [a,b]$. The Riemann-Liouville fractional integral and Riemann-Liouville, Liouville-Caputo fractional derivatives are defined as follows (see [6]):

Definition 1.4. *Let $h \in L_1[a,b]$. Then the left-sided and right sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are defined as follows:*

$$I_{a+}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}h(t)dt, \quad x > a,$$

$$I_{b-}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}h(t)dt, \quad x < b,$$

where $\Gamma(.)$ is the gamma function.

Definition 1.5. *Let $\alpha > 0$ and $\alpha \notin \{1,2,3,....,n\}$, $n = [\alpha]+1$, $h \in AC^m[a,b]$. Then the left-sided and right-sided Liouville-Caputo fractional derivatives of order α are defined as follows:*

$$({}^{LC}D_{a+}^{\alpha})h(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{h^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, \quad x > a,$$

$$({}^{LC}D_{b-}^{\alpha})h(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{h^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, \quad x < b.$$

In [4] Andrić et al. presented the following two composition identities for Liouville-Caputo fractional derivatives.

Lemma 1.6. *Let $\beta > \alpha \geq 0$, $m = [\beta]+1$ and $n = [\alpha]+1$ for $\alpha, \beta \notin N_0$. Let $h \in AC^m[a,b]$ be such that $h^i(a) = 0$ for $i = n, n+1, ..., m-1$. Let ${}^{LC}D_{a+}^{\beta}h, {}^{LC}D_{a+}^{\alpha}h \in L_1[a,b]$. Then*

$${}^{LC}D_{a+}^{\alpha}h(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1\,LC}D_{a+}^{\beta}h(t)dt, \quad x \in [a,b].$$

Lemma 1.7. *Let $\beta > \alpha \geq 0$, $m = [\beta]+1$ and $n = [\alpha]+1$ for $\alpha, \beta \notin N_0$. Let $f \in AC^m[a,b]$ be such that $h^i(b) = 0$ for $i = n, n+1, ..., m-1$. Let ${}^{LC}D_{b-}^{\beta}h, {}^{LC}D_{b-}^{\alpha}h \in L_1[a,b]$. Then*

$${}^{LC}D_{b-}^{\alpha}h(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_x^b (t-x)^{\beta-\alpha-1\,LC}D_{b-}^{\beta}h(t)dt, \quad x \in [a,b].$$

The Riemann-Liouville fractional derivative is defined in the forthcoming definition.

Definition 1.8. *For $f : [a,b] \to \mathbb{R}$, the left-sided Riemann-Liouville fractional derivative of order α is defined as follows:*

$$D^{\alpha}_{a+}h(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}h(t)dt = \frac{d^n}{dx^n}I^{n-\alpha}_{a+}h(x).$$

In [3] Andrić et al. presented a composition identity for the left-sided Riemann-Liouville fractional derivative, which is stated in the following lemma:

Lemma 1.9. *Let $\beta > \alpha \geq 0$, $m = [\beta]+1$, $n = [\alpha]+1$. The composition identity*

$$D^{\alpha}_{a+}h(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}D^{\beta}_{a+}h(t)dt, \quad x \in [a,b]$$

is valid if one of the following conditions holds:
(i) $h \in I^{\beta}_{+a}(L_1[a,b]) = \{h : h = I^{\beta}_{+a}\psi, \psi \in L_1[a,b]\}$.
(ii) $I^{m-\beta}_{a+}h \in AC^m[a,b]$ and $D^{\beta-k}_{a+}h(a) = 0$ for $k = 1, ..., m$.
(iii)$D^{\beta-1}_{a+}h \in AC[a,b]$, $D^{\beta-k}_{a+}h \in C[a,b]$ and $D^{\beta-k}_{a+}h(a) = 0$ for $k = 1, ..., m$.
(iv) $h \in AC^m[a,b]$, $D^{\beta}_{a+}h$, $D^{\alpha}_{a+}h \in L_1[a,b]$, $\beta-\alpha \notin N$, $D^{\beta-k}_{a+}h(a) = 0$ for $k = 1, ..., m$ and $D^{\alpha-k}_{a+}h(a) = 0$ $k = 1, ..., n$.
(v) $h \in AC^m[a,b]$, $D^{\beta}_{a+}h$, $D^{\alpha}_{a+}h \in L_1[a,b]$, $\beta-\alpha = l \in N$, $D^{\beta-k}_{a+}h(a) = 0$ for $k = 1, ..., l$.
(vii) $h \in AC^m[a,b]$, $D^{\beta}_{a+}h$, $D^{\alpha}_{a+}h \in L_1[a,b]$, $\beta \notin N$ and $D^{\beta-1}_{a+}h$ is bounded in a neighborhood of a.

For the detailed study of fractional integral operators we refer the readers to [6, 18, 19]. For a critical discussion on recently studied fractional integrals and derivatives we refer the readers to [17]. The rest of the paper is organized as follows: In Section 2, inequalities for superquadratic functions are established for an arbitrary kernel. In Section 3, the extensions of these inequalities are obtained by applying the Jensen inequality. In Section 4, results of Section 2 and Section 3 are applied for particular kernels and fractional integral inequalities of Opial-type are produced by using definitions of Riemann-Liouville, Liouville-Caputo fractional integral/derivative operators.

2. Opial-type inequalities for superquadratic function

In this section some new Opial-type inequalities for superquadratic functions are obtained.

Theorem 2.1. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Let $w \in U_1(u,k)$, $|k(x,t)| \leq K$ and $M < \int_a^x |u(t)|dt$. Then the following inequality holds for superquadratic functions:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}dx \leq \frac{1}{K^2M}\psi\left(K\int_a^b (|u(t)|dt)\right). \tag{4}$$

Proof. The function $w \in U_1(u,k)$ has the representation $w(x) = \int_a^x k(x,t)u(t)dt$. By using $|k(x,t)| \leq K$, we find that $|w(x)| \leq K\int_a^x |u(t)|dt$. Let $z(x) = \int_a^x |u(t)|dt$. Then $z'(x) = |u(x)|$ and $|w(x)| \leq Kz(x)$. Since ψ is differentiable function and $\frac{\psi'(x)}{x}$ is increasing, we get $\frac{\psi'(|w(x)|)}{|w(x)|} \leq \frac{\psi'(Kz(x))}{Kz(x)}$. From which one can get the following inequality:

$$\frac{\psi'(|w(x)|)|u(x)|}{|w(x)|} \leq \frac{1}{K^2M}\psi'(Kz(x))(Kz'(x)).$$

By integrating over $[a,b]$ we get

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|}dx \leq \frac{1}{K^2M}\int_a^b \psi'(Kz(x))(Kz'(x))dx = \frac{1}{K^2M}\psi(K(z(b)).$$

By using $z(b) = \int_a^b |u(t)|dt$ we get the inequality (4). □

Next, we give another Opial-type inequality for superquadratic functions by using the Hölder inequality.

Theorem 2.2. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Let $w \in U_1(u,k)$, $\left(\int_a^x (k(x,t))^{p_1} dt\right)^{\frac{1}{p_1}} \leq K$, $M < \left(\int_a^x |u(t)|^{p_2} dt\right)^{\frac{2}{p_2}-1}$, $p_1 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then the following inequality holds for superquadratic functions:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|} dx \leq \frac{p_2}{K^2 M} \psi\left(K\left(\int_a^b |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}}\right). \tag{5}$$

Proof. The function $w \in U_1(u,k)$, therefore it has representation $w(x) = \int_a^x k(x,t)u(t)dt$.
Using Holder's inequality, we have

$$\begin{aligned} |w(x)| &\leq \left(\int_a^x (k(x,t))^{p_1} dt\right)^{\frac{1}{p_1}} \left(\int_a^x |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}} \\ &\leq K\left(\int_a^x |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}}. \end{aligned} \tag{6}$$

Let $z(x) = \int_a^x |u(t)|^{p_2} dt$. Then $z'(x) = |u(x)|^{p_2}$ and $|w(x)| \leq K(z(x))^{\frac{1}{p_2}}$. The function ψ is differentiable and $\frac{\psi'(x)}{x}$ is increasing, therefore we get $\frac{\psi'(|w(x)|)}{|w(x)|} \leq \frac{\psi'(K(z(x))^{\frac{1}{p_2}})}{K(z(x))^{\frac{1}{p_2}}}$. From which one can get the following inequality:

$$\begin{aligned} \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|} &\leq \frac{p_2}{K^2} \frac{\psi'\left(K(z(x))^{\frac{1}{p_2}}\right)\left(\frac{K}{p_2}(z(x))^{\frac{1}{p_2}-1}\right)z'(x)}{\left(\int_a^x |u(t)|^{p_2} dt\right)^{\frac{2}{p_2}-1}} \\ &\leq \frac{p_2}{K^2 M} \psi'\left(K(z(x))^{\frac{1}{p_2}}\right)\left(\frac{K}{p_2}(z(x))^{\frac{1}{p_2}-1}\right)z'(x). \end{aligned}$$

By integrating over $[a,b]$ we get

$$\begin{aligned} \int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|} dx &\leq \frac{p_2}{K^2 M} \int_a^b \psi'\left(K(z(x))^{\frac{1}{p_2}}\right)\left(\frac{K}{p_2}(z(x))^{\frac{1}{p_2}-1}\right)z'(x)dx \\ &= \frac{p_2}{K^2 M} \psi\left(K(z(b))^{\frac{1}{p_2}}\right). \end{aligned}$$

By using $z(b) = \int_a^b |u(t)|^{p_2} dt$ we will get the inequality (5). □

The inequalities (4) and (5) also hold for the class of functions $U_2(u,k)$. They are stated in the following theorems.

Theorem 2.3. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Let $w \in U_2(u,k)$, $|k(x,t)| \leq K'$ and $M' < \int_x^b |u(t)|dt$. Then the following inequality holds for superquadratic functions:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|} dx \leq \frac{1}{K'^2 M'} \psi\left(K' \int_a^b (|u(t)|dt)\right). \tag{7}$$

Proof. The proof can be followed from the proof of Theorem 2.1. □

Theorem 2.4. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Let $w \in U_2(u,k)$, $\left(\int_x^b (k(x,t))^{p_1} dt\right)^{\frac{1}{p_1}} \leq K'$, $M' < \left(\int_x^b |u(t)|^{p_2} dt\right)^{\frac{2}{p_2}-1}$, $p_1 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then the following inequality holds for superquadratic functions:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|} dx \leq \frac{p_2}{K'^2 M'} \psi\left(K'\left(\int_a^b |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}}\right). \tag{8}$$

Proof. The proof can be followed from the proof of Theorem 2.2. □

3. Extensions of Opial-type inequalities

In this section we present the extensions of of Theorems 2.1 and 2.2, by applying the well known Jensen integral inequality for convex functions.

Theorem 3.1. *Under the conditions of Theorem 2.1, in addition if ψ is nonnegative, then we have*

$$\begin{aligned} \int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|} dx &\leq \frac{1}{K^2 M} \psi\left(K \int_a^b (|u(t)|dt)\right) \\ &\leq \frac{1}{(b-a)MK^2} \int_a^b \psi\left(K(b-a)|u(t)|\right) dt. \end{aligned} \tag{9}$$

Proof. Since ψ is nonnegative, by [1, Lemma 2.2] we have that ψ is convex. By Jensen inequality we have

$$\psi\left(\frac{1}{(b-a)} \int_a^b (b-a)K|u(t)|dt\right) \leq \frac{1}{(b-a)} \int_a^b \psi\left(K(b-a)|u(t)|\right) dt. \tag{10}$$

Using inequality (10) in (4) one can get the required inequality (9).
□

Theorem 3.2. *Under the conditions of Theorem 2.2 in addition if ψ is nonnegative, then we have*

$$\begin{aligned} \int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|} dx &\leq \frac{p_2}{K^2 M} \psi\left(K\left(\int_a^b |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}}\right) \\ &\leq \frac{p_2}{K^2 M(b-a)} \int_a^b \psi\left((b-a)^{\frac{1}{p_2}} K|u(t|\right) dt. \end{aligned} \tag{11}$$

Proof. Since $\psi(x^{\frac{1}{p_2}})$ is convex, the following Jensen's inequality holds:

$$\psi\left(\left(\frac{1}{(b-a)} \int_a^b h(t)dt\right)^{\frac{1}{p_2}}\right) \leq \frac{1}{b-a} \int_a^b \psi\left(h^{\frac{1}{p_2}}(t)\right) dt. \tag{12}$$

Therefore inequality (11) can be obtained from the inequalities (12) and (5). □

The inequalities (9) and (11) also hold for the class of functions $U_2(u,k)$. They are stated in the following theorems.

Theorem 3.3. *Under the conditions of Theorem 2.3, in addition if ψ is nonnegative, then the following inequalities hold:*

$$\begin{aligned} \int_a^b \frac{\psi'(|w(x)|)|u(x)|}{|w(x)|} dx &\leq \frac{1}{K'^2 M'} \psi\left(K' \int_a^b (|u(t)|dt)\right) \\ &\leq \frac{1}{(b-a)M'K'^2} \int_a^b \psi\left(K'(b-a)|u(t)|\right) dt. \end{aligned} \tag{13}$$

Proof. The proof can be followed from the proof of Theorem 3.1. □

Theorem 3.4. *Under the conditions of Theorem 2.4 in addition if ψ is nonnegative, then the following inequalities hold:*

$$\int_a^b \frac{\psi'(|w(x)|)|u(x)|^{p_2}}{|w(x)|}dx \leq \frac{p_2}{K'^2M'}\psi\left(K'\left(\int_a^b |u(t)|^{p_2}dt\right)^{\frac{1}{p_2}}\right) \leq \frac{p_2}{K'^2M'(b-a)}\int_a^b \psi\left((b-a)^{\frac{1}{p_2}}K'|u(t|\right)dt. \tag{14}$$

Proof. The proof can be followed from the proof of Theorem 3.2. □

4. Fractional Opial-type inequalities for superquadratic functions

In this section, we apply results of Sections 2 & 3 for special kernels to get fractional Opial-type inequalities for Riemann-Liouville fractional integral, Liouville-Caputo fractional derivative. First we give the following results for the left-sided Riemann-Liouville fractional integrals.

Theorem 4.1. *Under the conditions of Theorem 3.1, for $\alpha \geq 1$ the following fractional inequalities hold:*

$$\int_a^b \frac{\psi'(|I^\alpha_{+a}u(x)|)|u(x)|}{|I^\alpha_{+a}u(x)|}dx \leq \frac{\Gamma^2(\alpha)}{(b-a)^{2(\alpha-1)}M}\psi\left(\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_a^b |u(t)|dt\right)\right) \leq \frac{\Gamma^2(\alpha)}{(b-a)^{2\alpha-1}M}\int_a^b \psi\left(\frac{(b-a)^\alpha}{\Gamma(\alpha)}|u(t)|dt\right). \tag{15}$$

Proof. Let us define the kernel $k(x,t)$ as follows:

$$k(x,t) = \begin{cases} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}, & t \in [a,x], \\ 0\ , & t \in (x,b]. \end{cases} \tag{16}$$

Further, we take function w as follows:

$$w(x) := I^\alpha_{a+}u(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}u(t)dt. \tag{17}$$

It is clear that for $\alpha \geq 1$, $0 \leq k(x,t) \leq \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \leq \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$, $t \in [a,x]$, $x \in [a,b]$.

Therefore by applying Theorem 3.1 for $K = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$ we get the fractional inequalities required in (15). □

Theorem 4.2. *Under the conditions of Theorem 3.2, in addition if $\alpha > \frac{1}{p_2}$, then the following inequalities hold:*

$$\int_a^b \frac{\psi'(|I^\alpha_{a+}u(x)|)|I^\alpha_{a+}u(x)|^{p_2}}{|I^\alpha_{a+}u(x)|}dx \leq \frac{p_2\left(\Gamma(\alpha)\left(p_1(\alpha-1)+1\right)^{\frac{1}{p_1}}\right)^2}{\left((b-a)^{\alpha-\frac{1}{p_2}}\right)^2 M} \tag{18}$$
$$\times\psi\left(\frac{(b-a)^{\alpha-\frac{1}{p_2}}}{\Gamma(\alpha)\left(p_1(\alpha-1)+1\right)^{\frac{1}{p_1}}}\left(\int_a^x |u(t)|^{p_2}dt\right)^{\frac{1}{p_2}}\right)$$
$$\leq \frac{p_2\left(\Gamma(\alpha)\left(p_1(\alpha-1)+1\right)^{\frac{1}{p_1}}\right)^2}{(b-a)^{2\left(\alpha-\frac{1}{p_2}\right)+1}M}\int_a^b \psi\left(\frac{(b-a)^\alpha|u(t)|}{\Gamma(\alpha)\left(p_1(\alpha-1)+1\right)^{\frac{1}{p_1}}}\right).$$

Proof. Let us consider the kernel $k(x,t)$ as follows:

$$k(x,t) = \begin{cases} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}, & t \in [a,x], \\ 0 \ , & t \in (x,b]. \end{cases}$$

Further, we take w same as defined in (17). Let $Q(x) := \left(\int_a^x (k(x,t))^{p_1} dt\right)^{\frac{1}{p_1}}$. Then it is easy to see that $Q(x) \leq \frac{(b-a)^{\alpha-\frac{1}{p_2}}}{\Gamma(\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}}$ for $\alpha > \frac{1}{p_2}$. By setting $K = \frac{(b-a)^{\alpha-\frac{1}{p_2}}}{\Gamma(\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}}$ and applying Theorem 3.2, we get inequalities required in (18). □

The next results are obtained by using the right-sided Riemann-Liouville fractional integral which is stated as follows:

Theorem 4.3. *Under the conditions of Theorem 3.3, in addition if $\alpha \geq 1$, then the following fractional inequalities hold:*

$$\begin{aligned} \int_a^b \frac{\psi'(|I_{b-}^{\alpha} u(x)|)|u(x)|}{|I_{b-}^{\alpha} u(x)|} dx &\leq \frac{\Gamma^2(\alpha)}{(b-a)^{2(\alpha-1)} M'} \psi\left(\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_a^b |u(t)| dt\right)\right) \\ &\leq \frac{\Gamma^2(\alpha)}{(b-a)^{2\alpha-1} M'} \int_a^b \psi\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha)} |u(t)| dt\right). \end{aligned} \tag{19}$$

Proof. The proof is similar to the proof of Theorem 4.1. □

Theorem 4.4. *Under the conditions of Theorem 3.4 in addition if $\alpha > \frac{1}{p_2}$, then the following inequalities hold:*

$$\begin{aligned} &\int_a^b \frac{\psi'(|I_{b-}^{\alpha} u(x)|)|I_{b-}^{\alpha} u(x)|^{p_2}}{|I_{b-}^{\alpha} u(x)|} dx \leq \frac{p_2 \left(\Gamma(\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}\right)^2}{((b-a)^{\alpha-\frac{1}{p_2}})^2 M'} \\ &\times \psi\left(\frac{(b-a)^{\alpha-\frac{1}{p_2}}}{\Gamma(\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}} \left(\int_a^x |u(t)|^{p_2} dt\right)^{\frac{1}{p_2}}\right) \\ &\leq \frac{p_2 \left(\Gamma(\alpha)\,(p_1(\alpha-1)+1)^{\frac{1}{p_1}}\right)^2}{(b-a)^{2\left(\alpha-\frac{1}{p_2}\right)+1} M'} \int_a^b \psi\left(\frac{(b-a)^{\alpha}|u(t)|}{\Gamma(\alpha)\,(p_1(\alpha-1)+1)^{\frac{1}{p_1}}}\right). \end{aligned} \tag{20}$$

Proof. The proof is similar to the proof of Theorem 4.2. □

By using the composition identity for the left-sided Liouville-Caputo fractional derivative given in Lemma 1.6, the following results are given.

Theorem 4.5. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Also let $m = [\beta] + 1$ and $n = [\alpha] + 1$ for $\alpha, \beta \notin N_0$, $u \in AC^m[a,b]$ such that $h^i(a) = 0$ for $i = n, n+1, ..., m-1$. Let ${}^{LC}D_{+a}^{\alpha} u, {}^{LC}D_{+a}^{\beta} u \in L_1[a,b]$. Then for $\alpha < \beta - 1$, the following fractional inequalities hold:*

$$\begin{aligned} &\int_a^b \frac{\psi'(|{}^{LC}D_{+a}^{\alpha} u(x)|)|{}^{LC}D_{+a}^{\beta} u(x)|}{|{}^{LC}D_{+a}^{\alpha} u(x)|} dx \\ &\leq \frac{(\Gamma(\beta-\alpha))^2}{((b-a)^{\beta-\alpha-1})^2 M} \psi\left(\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \left(\int_a^b |{}^{LC}D_{+a}^{\beta} u(t)| dt\right)\right) \\ &\leq \frac{(\Gamma(\beta-\alpha))^2}{(b-a)^{2\beta-2\alpha-1} M} \int_a^b \psi\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)} |{}^{LC}D_{+a}^{\beta} u(t)| dt\right). \end{aligned} \tag{21}$$

Proof. Let us consider the kernel $k(x,t)$ as follows:

$$k(x,t)=\begin{cases}\frac{(x-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & t\in[a,x],\\ 0, & t\in(x,b].\end{cases} \tag{22}$$

Further, we take w as follows:

$$w(x):=^{LC}D^{\alpha}_{+a}u(x)=\frac{1}{\Gamma(\beta-\alpha)}\int_a^x(x-t)^{\beta-\alpha-1LC}D^{\beta}_{a+}u(t)dt. \tag{23}$$

It is clear that for $\alpha<\beta-1$, $0\le k(x,t)\le\frac{(x-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\le\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$, $t\in[a,x]$, $x\in[a,b]$. We let $K=\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$ and apply Theorem 3.1 to get the fractional inequalities required in (21). □

Theorem 4.6. *Let $\psi:[0,\infty)\to\mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0)=0$. Also let $m=[\beta]+1$ and $n=[\alpha]+1$ for $\alpha,\beta\notin N_0$, $u\in AC^m[a,b]$ such that $h^i(a)=0$ for $i=n,n+1,...,m-1$. Let ${}^{LC}D^{\beta}_{+a}u,{}^{LC}D^{\beta}_{+a}u\in L_1[a,b]$. Then for $\alpha<\beta-\frac{1}{p_2}$, the following fractional inequalities hold:*

$$\begin{aligned}&\int_a^b\frac{\psi'(|^{LC}D^{\alpha}_{+a}w(x)|)|^{LC}D^{\beta}_{+a}u(x)|^{p_2}}{|^{LC}D^{\alpha}_{+a}u(x)|}dx\le\frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{((b-a)^{\beta-\alpha-\frac{1}{p_2}})^2M}\\&\psi\left(\frac{(b-a)^{\beta-\alpha-\frac{1}{p_2}}}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\left(\int_a^b|^{LC}D^{\beta}_{+a}u(t)|^{p_2}dt\right)^{\frac{1}{p_2}}\right)\\&\le\frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{(b-a)^{2\left(\beta-\alpha-\frac{1}{p_2}\right)+1}M}\int_a^b\psi\left(\frac{(b-a)^{\beta-\alpha}|^{LC}D^{\beta}_{+a}u(t)|}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\right).\end{aligned} \tag{24}$$

Proof. Let us consider the kernel $k(x,t)$ as follows:

$$k(x,t)=\begin{cases}\frac{(x-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & t\in[a,x],\\ 0, & t\in(x,b].\end{cases}$$

Further, we take w same as defined in (23). Let $P(x):=\left(\int_a^x(k(x,t))^{p_1}dt\right)^{\frac{1}{p_1}}$. Then it is easy to see that $P(x)\le\frac{(b-a)^{\beta-\alpha-\frac{1}{p_2}}}{\Gamma(\beta-\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}}$ for $\alpha<\beta-\frac{1}{p_2}$. By setting $K=\frac{(b-a)^{\beta-\alpha-\frac{1}{p_2}}}{\Gamma(\beta-\alpha)(p_1(\alpha-1)+1)^{\frac{1}{p_1}}}$ and applying Theorem 3.2, we get inequalities required in (24). □

The next results are obtained by using the composition identity for the right-sided Liouville-Caputo fractional derivative given in Lemma 1.7.

Theorem 4.7. *Let $\psi:[0,\infty)\to\mathbb{R}$ be a differentiable such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0)=0$. Also let $m=[\beta]+1$ and $n=[\alpha]+1$ for $\alpha,\beta\notin N_0$; $u\in AC^m[a,b]$ such that $h^i(a)=0$ for $i=n,n+1,...,m-1$. Let ${}^{LC}D^{\beta}_{b-}v,{}^{LC}D^{\alpha}_{b-}u\in L_1[a,b]$. Then for $\alpha<\beta-1$, the following fractional inequalities hold:*

$$\begin{aligned}&\int_a^b\frac{\psi'(|^{LC}D^{\alpha}_{b-}u(x)|)|^{LC}D^{\beta}_{b-}u(x)|}{|^{LC}D^{\alpha}_{b-}u(x)|}dx\\&\le\frac{(\Gamma(\beta-\alpha))^2}{((b-a)^{\beta-\alpha-1})^2M}\psi\left(\frac{(b-a)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}\left(\int_a^b|^{LC}D^{\beta}_{b-}u(t)|dt\right)\right)\\&\le\frac{(\Gamma(\beta-\alpha))^2}{(b-a)^{2\beta-2\alpha-1}M}\int_a^b\psi\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)}|^{LC}D^{\beta}_{b-}u(t)|dt\right).\end{aligned} \tag{25}$$

Proof. The proof is similar to the proof of Theorem 4.5. □

Theorem 4.8. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Also let $m = [\beta] + 1$ and $n = [\alpha] + 1$ for $\alpha, \beta \notin N_0$, $u \in AC^m[a,b]$ such that $h^i(a) = 0$ for $i = n, n+1, ...m-1$. Let ${}^{LC}D^{\beta}_{b-}u, {}^{LC}D^{\alpha}_{b-}u \in L_1[a,b]$. Then for $\alpha < \beta - \frac{1}{p_2}$, the following fractional inequalities hold:*

$$\int_a^b \frac{\psi'(|{}^{LC}D^{\alpha}_{b-}w(x)|)|{}^{LC}D^{\beta}_{b-}u(x)|^{p_2}}{|{}^{LC}D^{\alpha}_{b-}u(x)|}dx \le \frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{((b-a)^{\beta-\alpha-\frac{1}{p_2}})^2 M} \tag{26}$$

$$\psi\left(\frac{(b-a)^{\beta-\alpha-\frac{1}{p_2}}}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\left(\int_a^b |{}^{LC}D^{\beta}_{b-}u(t)|^{p_2}dt\right)^{\frac{1}{p_2}}\right)$$

$$\le \frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{(b-a)^{2\left(\beta-\alpha-\frac{1}{p_2}\right)+1}M}\int_a^b \psi\left(\frac{(b-a)^{\beta-\alpha}|{}^{LC}D^{\beta}_{b-}u(t)|}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\right).$$

Proof. The proof is similar to the proof of Theorem 4.6. □

For the composition identity of Riemann-Liouville fractional derivative given in Lemma 1.9, the following results hold:

Theorem 4.9. *let $\psi : [0,\infty) \to \mathbb{R}$ be a differential function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Also let $\alpha \ge 0$, $m = [\beta] + 1$ and $n = [\alpha] + 1$. Suppose that one of the following conditions (i)-(vii) in Lemma 1.9 holds for $\{\beta, \alpha, u\}$ and let $D^{\beta}_{a+}u \in L_q[a,b]$. Then, for $\alpha < \beta - 1$, the following fractional inequalities holds:*

$$\int_a^b \frac{\psi'(|D^{\alpha}_{+a}u(x)|)|D^{\beta}_{+a}u(x)|}{|D^{\alpha}_{+a}u(x)|}dx \tag{27}$$

$$\le \frac{(\Gamma(\beta-\alpha))^2}{((b-a)^{\beta-\alpha-1})^2 M}\psi\left(\frac{(b-a)^{\beta-\alpha-1}}{(\Gamma(\beta-\alpha)}\left(\int_a^b |D^{\beta}_{a+}u(t)|dt\right)\right)$$

$$\le \frac{(\Gamma(\beta-\alpha))^2}{(b-a)^{2\beta-2\alpha-1}M}\int_a^b \psi\left(\frac{(b-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha)}|D^{\beta}_{+a}u(t)|dt\right).$$

Proof. The proof is similar to the proof of Theorem 4.5. □

Theorem 4.10. *let $\psi : [0,\infty) \to \mathbb{R}$ be a differential function such that $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = 0$. Also let $\alpha \ge 0$, $m = [\beta] + 1$ and $n = [\alpha] + 1$. Suppose that one of the following conditions (i)-(vii) in Lemma 1.9 holds for $\{\beta, \alpha, u\}$ and let $D^{\beta}_{a+}u \in L_q[a,b]$. Then, for $\alpha < \beta - \frac{1}{p_2}$, the following fractional inequalities holds:*

$$\int_a^b \frac{\psi'(|D^{\alpha}_{+a}w(x)|)|D^{\beta}_{+a}u(x)|^{p_2}}{|D^{\alpha}_{+a}u(x)|}dx \le \frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{((b-a)^{\beta-\alpha-\frac{1}{p_2}}))^2 M} \tag{28}$$

$$\psi\left(\frac{(b-a)^{\beta-\alpha-\frac{1}{p_2}}}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\left(\int_a^b |D^{\beta}_{+a}u(t)|^{p_2}dt\right)^{\frac{1}{p_2}}\right)$$

$$\le \frac{p_2(\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}})^2}{(b-a)^{2\left(\beta-\alpha-\frac{1}{p_2}\right)+1}M}\int_a^b \psi\left(\frac{(b-a)^{\beta-\alpha}|D^{\beta}_{+a}u(t)|}{\Gamma(\beta-\alpha)(p_1(\beta-\alpha-1)+1)^{\frac{1}{p_1}}}\right).$$

5. Concluding remarks

New inequalities of Opial-type for superquadratic functions are established in this research article. Their extensions are established for convex functions. These inequalities are further discussed for fractional calculus operators named Riemann-Liouville and Liouville-Caputo. The results of this paper can be further estimated by mean value theorems, which are under considerations.

References

[1] S. Abramovich, J. Barić, J. Pečarić, Fejér and Hermite-Hadamard type inequalities for superquadratic functions, Journal of Mathematical Analysis and Applications 344 (2008) 1048–1056.
[2] M. Andrić, A. Barbir, G. Farid, J. Pečarić, Opial-type inequality due to Agarwal-Pang and fractional difffferential inequalities, Integral Transforms and Special Functions 25 (2014) 324–335.
[3] M. Andrić, J. Pečarić, I. Perić, A multiple Opial type inequality for the Riemann–Liouville fractional derivatives, Journal of Mathematical Inequalities 7 (2013) 139–150.
[4] M. Andrić, J. Pečarić, I. Perić, Composition identities for the Caputo fractional derivatives and applications to Opial-type inequalities, Mathematical Inequalities and Applications 16 (2013) 657–670.
[5] G. Farid, A. U. Rehman, S. Ullah, A. Nosheen, M. Waseem, Y. Mehboob, Opial-type inequalities for convex function and associated results in fractional calculus, Advances in Difference Equations 2019 (2019) 2019:152.
[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Application of Fractional Differential Equations, North-Holland Mathematics studies, 204, Elsevier, New York-London, 2006.
[7] D. S. Mitrinovic, J. E. Pečarić, Generalization of two inequalities of Godunova and Levin, Bulletin of the Polish Academy of Sciences 36 (1988) 645–648.
[8] Z. Opial, Sur une inégalité, Annales Polonici Mathematici 8 (1960) 29–32.
[9] B. G. Pachpatte, A note on generalization Opial type inequalities, Tamkang Journal of Mathematics 24 (1993) 229–235.
[10] B. G. Pachpatte, On Opial-type integral inequalities, Journal of Mathematical Analysis and Applications 120 (1986) 547–556.
[11] J. Pečarić, F. Proschan, Y. C. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Inc., 1992.
[12] Z. Qi, Further generalization of Opial's inequality, Acta Mathematica Sinica 1 (1985) 196–200.
[13] R. Redheffffer, Inequalities with three functions, Journal of Mathematical Analysis and Applications 16 (1966) 219–242.
[14] A. U. Rehman, G. Farid, Y. Mehboob, Mean value theorems associated to the differences of Opial-type and their fractional versions, Fractional Differential Calculus 10 (2020) 213–224.
[15] D. T. Shum, A general and sharpened form of Opial's inequality, Canadian Mathematical Bulletin 17 (1974) 385–389.
[16] G. J. Sinnamon, Weighted Hardy and Opial-type inequalities, Journal of Mathematical Analysis and Applications 160 (1991) 434–445.
[17] H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, Journal of Nonlinear Convex Analysis 22 (2021) 1501–1520.
[18] H. M. Srivastava, Fractional-order derivatives and integrals: introductory overview and recent developments, Kyungpook Mathematical Journal 60 (2020) 73–116.
[19] H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iranian Journal of Science and Technology, Transactions A: Science 44 (2020) 327–344.
[20] H. M. Srivastava, K-L. Tseng, S-J. Tseng and J-C. Lo, Some weighted Opial-type inequalities on time scales, Taiwanese Journal of Mathematics 14 (2010) 107–122.
[21] H. M. Srivastava, Z-H Zhang and Y-D. Wu, Some further refinements and extensions of the Hermite-Hadamard and Jensen inequalities in several variables, Mathematics and Computer Modeling 54 (2011) 2709–2717.
[22] Z. Tomovski, J. Pecaric, and G. Farid, Weighted Opial inequalities for fractional integral and differential operators involving generalized Mittag-Leffler function, European Journal of Pure and Applied Mathematics 10 (2017) 419–439.
[23] T. Tunç, M. Z. Sarikaya and H. M. Srivastava, Some generalized Steffensen's inequalities via a new identity for local fractional integrals, International Journal of Analysis and Applications 13 (2017) 98–107.
[24] F.-H. Wong, W.-C. Lian, S.-L. Yu and C.-C. Yeh, Some generalizations of Opial's inequalities on time scales, Taiwanese Journal of Mathematics 12 (2008) 463–471.
[25] G. S. Yang, On a certain result of Z. Opial, Proceedings of the Japan Academy 42 (1966) 78–83.
[26] G. S. Yang, A note on some integro-difffferential inequalities, Soochow Journal of Mathematics 9 (1983) 231–236.