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Durrmeyer-Type Generalization of μ-Bernstein Operators

Arun Kajla[a], S. A. Mohiuddine[b,c], Abdullah Alotaibi[c]

[a]*Department of Mathematics, Central University of Haryana, Haryana 123031, India*
[b]*Department of General Required Courses, Mathematics, Faculty of Applied Studies, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[c]*Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589 Saudi Arabia*

Abstract. In the present manuscript, we consider μ-Bernstein-Durrmeyer operators involving a strictly positive continuous function. Firstly, we prove a Voronovskaja type, quantitative Voronovskaja type and Grüss-Voronovskaja type asymptotic formula, the rate of convergence by means of the modulus of continuity and for functions in a Lipschitz type space. Finally, we show that the numerical examples which describe the validity of the theoretical example and the effectiveness of the defined operators.

1. Introduction

For $f \in C(\hat{\mathcal{J}})$ with $\hat{\mathcal{J}} = [0,1]$. In 2017, Chen et al. [13] considered a generalization of the Bernstein operators involving a non-negative real parameter $\mu \in [0,1]$ as

$$T_m^{(\mu)}(q;z) = \sum_{\nu=0}^{m} \hat{r}_{m,\nu}^{(\mu)}(z) q\left(\frac{\nu}{m}\right), \quad z \in \hat{\mathcal{J}} \tag{1}$$

where $\hat{r}_{m,\nu}^{(\mu)}(z) = \left[\binom{m-2}{\nu}(1-\mu)z + \binom{m-2}{\nu-2}(1-\mu)(1-z) + \binom{m}{\nu}\mu z(1-z)\right] z^{\nu-1}(1-z)^{m-\nu-1}$ and $m \geq 2$.

Acar and Kajla [5] introduced a bivariate extension of these μ-Bernstein operators and studied the associated GBS operators and their order of approximation. The Durrmeyer variant of the operators (1) is considered by Kajla and Acar [28] and established direct results. Some amusing approximation properties can be seen in (cf. [1–4, 6, 7, 11, 12, 14, 18, 19, 21–23, 25, 28–30, 33–35, 37, 39]) and references therein.

Let us fix a strictly positive continuous function $\tau : C(\hat{\mathcal{J}}) \to C(\hat{\mathcal{J}})$. We construct a Durrmeyer type generalization of the operators (1) based on $\tau(z)$ $(0 < \tau(z) \leq 1)$ as

$$\mathcal{G}_{m,\tau,\varrho}^{(\mu)}(q;z) = \sum_{\nu=0}^{m} \hat{r}_{m,\nu}^{(\mu)}(z) \int_0^1 \left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right) q(t)dt, \tag{2}$$

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Email addresses: arunkajla@cuh.ac.in (Arun Kajla), mohiuddine@gmail.com (S. A. Mohiuddine), mathker11@hotmail.com (Abdullah Alotaibi)

where $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0$ and $\varrho > 0$.

The aim of the present manuscript is to compute some direct results for the operators given by (2). Throughout this manuscript, the positive constant $\mathscr{N}$ is not necessarily the same at each occurrence.

Lemma 1.1. *For $e_i = z^i, i = \overline{0,4}$ we conclude*

(i) $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_0; z) = 1;$

(ii) $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_1; z) = \dfrac{\tau + mz\varrho}{2\tau + m\varrho};$

(iii) $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_2; z) = \dfrac{z^2\left(-2 - m + m^2 + 2\mu\right)\varrho^2}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)} + \dfrac{z\varrho(-2(-1 + \mu)\varrho + m(1 + 2\tau + \varrho))}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)} + \dfrac{\tau(1 + \tau)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)};$

(iv)
$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_3; z) = \frac{(m-2)z^3\left(-6 - m + m^2 + 6\mu\right)\varrho^3}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)}$$
$$+ \frac{3z^2\varrho^2\left(m^2(1 + \tau + \varrho) + 2(-1 + \mu)(1 + \tau + 3\varrho) - m(1 + \tau - \varrho + 2\mu\varrho)\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)}$$
$$+ \frac{z\varrho\left(-6(-1 + \mu)\varrho(1 + \tau + \varrho) + m\left(2 + 3\tau^2 + 3\varrho + \varrho^2 + 3\tau(2 + \varrho)\right)\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)} + \frac{\tau(1 + \tau)(2 + \tau)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)};$$

(v)
$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_4; z) = \frac{(m-3)(m-2)z^4((m-1)m + 12(-1 + \mu))\varrho^4}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)(3 + 2\tau + m\varrho)}$$
$$+ \frac{2(m-2)z^3\varrho^3\left(\left(-6 - m + m^2 + 6\mu\right)(3 + 2\tau) + 3\left(-12 + m + m^2 - 2(-6 + m)\mu\right)\varrho\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)(3 + 2\tau + m\varrho)}$$
$$+ \frac{1}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)(3 + 2\tau + m\varrho)}\Big[z^2\varrho^2((-2 - m + m^2 + 2\mu)(11 + 6\tau(3 + \tau))$$
$$+ 6\left(-6 + m + m^2 + 6\mu - 2m\mu\right)(3 + 2\tau)\varrho + (m(29 + 7m - 36\mu) + 86(-1 + \mu))\varrho^2)\Big]$$
$$+ \frac{z\varrho\left(m(3 + 2\tau + \varrho)\left(2 + 2\tau(3 + \tau) + 3\varrho + 2\tau\varrho + \varrho^2\right) - 2(-1 + \mu)\varrho\left(11 + 6\tau(3 + \tau) + 18\varrho + 12\tau\varrho + 7\varrho^2\right)\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)(3 + 2\tau + m\varrho)}$$
$$+ \frac{\tau(1 + \tau)(2 + \tau)(3 + \tau)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)(2 + 2\tau + m\varrho)(3 + 2\tau + m\varrho)}.$$

Lemma 1.2. *By direct computation, we have*

(i) $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t - z); z) = \dfrac{\tau(1 - 2z)}{2\tau + m\varrho};$

(ii)
$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t - z)^2; z) = \frac{z^2\left(2\tau + 4\tau^2 - \varrho(m + (2 + m - 2\mu)\varrho)\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)} + \frac{z\left(-2\tau - 4\tau^2 + \varrho(m + (2 + m - 2\mu)\varrho)\right)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)}$$
$$+ \frac{\tau(1 + \tau)}{(2\tau + m\varrho)(1 + 2\tau + m\varrho)}.$$

2. Direct Results

Theorem 2.1. *For every $q \in C(\hat{\mathcal{J}})$,*

$$\lim_{m\to\infty} \mathscr{G}^{(\mu)}_{m,\tau,\varrho} = q(z), \textit{ uniformly on } \hat{\mathcal{J}},$$

i.e. $\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\right)_{m\in\mathbb{N}}$ is a positive convergence process on $C(\hat{\mathcal{J}})$.

Proof. We have $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_0;z)=1, \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_1;z)\to z, \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(e_2;z)\to z^2$ as $m\to\infty$, uniformly on $\hat{\mathcal{J}}$.

By application of Korovkin's Theorem,

$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\to q(z) \text{ as } m\to\infty, \text{ uniformly on } \hat{\mathcal{J}}.$$

□

Theorem 2.2. *If $q\in C^2(\hat{\mathcal{J}})$, then*

$$\lim_{m\to\infty} m\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right)=\frac{\tau(z)(1-2z)}{\varrho}q'(z)+\frac{(1-z)z(1+\varrho)}{2\varrho}q''(z).$$

Proof. By application of Taylor's series, we have

$$q(t)=q(z)+q'(z)(t-z)+\frac{1}{2}q''(z)(t-z)^2+\vartheta(t,z)(t-z)^2, \tag{3}$$

where $\lim_{t\to z}\vartheta(t,z)=0$.
Apply the operators $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}$ on both side of above equation (3), we obtain

$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)=\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)q'(z)+\frac{1}{2}\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left((t-z)^2;z\right)q''(z)+\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\vartheta(t,z)\cdot(t-z)^2;z\right).$$

Therefore using Lemma 1.2, we may write

$$\begin{aligned}\lim_{m\to\infty} m\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right) &= \frac{\tau(z)(1-2z)}{\varrho}q'(z)+\frac{(1-z)z(1+\varrho)}{2\varrho}q''(z)\\ &\quad+\lim_{m\to\infty} m\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\vartheta(t,z)\cdot(t-z)^2;z\right)\right).\end{aligned} \tag{4}$$

By the Cauchy-Schwarz property,

$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\vartheta(t,z)\cdot(t-z)^2;z\right)\le\sqrt{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\vartheta^2(t,z);z\right)}\sqrt{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left((t-z)^4;z\right)}. \tag{5}$$

Because $\vartheta^2(z,z)=0$ and $\vartheta^2(\cdot,z)\in C[0,1]$, using Theorem 2.1, we easily obtain

$$\lim_{m\to\infty}\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\vartheta^2(t,z);z\right)=\vartheta^2(z,z)=0. \tag{6}$$

By Lemma 1.1, we have

$$\lim_{m\to\infty} m^2\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left((t-z)^4;z\right)=\left(\frac{3z^2(1-z)^2(1+\varrho)^2}{\varrho^2}\right). \tag{7}$$

Combining the (4-7), we obtain the desired result. □

Now, we compute a Grüss-Voronovskaja type theorem for $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}$.

Theorem 2.3. *Let $q,h\in C^2[0,1]$. Then, for each $z\in[0,1]$,*

$$\lim_{m\to\infty} m\left\{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((qh);z)-\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(h;z)\right\}=q'(z)h'(z)\frac{(1+\varrho)z(1-z)}{\varrho}.$$

Proof. The following relation holds
$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(qh;z)-\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(h;z)$

$$\begin{aligned} =\ & \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(qh;z)-q(z)h(z)-(qh)'(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)-\frac{1}{2}(qh)''(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\\ & -h(z)\left\{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)-q'(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)-\frac{1}{2}q''(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\right\}\\ & -\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\left\{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(h;z)-h(z)-h'(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)-\frac{1}{2}h''(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\right\}\\ & +\frac{1}{2}\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\left\{q(z)h''(z)+2q'(z)h'(z)-h''(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\right\}\\ & +\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)\left\{q(z)h'(z)-h'(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\right\}. \end{aligned}$$

From Theorems 2.1 and 2.2 and Lemma 1.2, we easily obtain

$$\begin{aligned} & \lim_{m\to\infty} m\left\{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(qh;z)-\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(h;z)\right\}\\ & =\lim_{m\to\infty} mq'(z)h'(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)+\lim_{m\to\infty}\frac{1}{2}m\,h''(z)\left\{q(z)-\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\right\}\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\\ & +\lim_{m\to\infty} m\,h'(z)\left\{q(z)-\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)\right\}\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z);z)=q'(z)h'(z)\frac{(1+\varrho)z(1-z)}{\varrho}. \end{aligned}$$

□

2.1. Local approximation

The K-functional is given by

$$K_2(q,\kappa)=\inf\{\|q-h\|+\kappa\|h''\|:h\in J^2\}\ (\kappa>0),$$

where $J^2=\{h:h''\in C(\hat{\mathcal{J}})\}$ and $\|.\|$ is the uniform norm on $C(\hat{\mathcal{J}})$. It is known from [15] that there exists a positive constant $M>0$ such that

$$K_2(q,\kappa)\le M\varpi_2(q,\sqrt{\kappa}). \tag{8}$$

For $C(\hat{\mathcal{J}})$ and $\kappa>0$ usual modulus of continuity and modulus of continuity of second order are given by the formulas

$$\varpi(q,\kappa)=\sup_{0<l\le\kappa}\ \sup_{z,z+l\in\hat{\mathcal{J}}}|q(z+l)-q(z)|.$$

and

$$\varpi_2(q,\sqrt{\kappa})=\sup_{0<l\le\sqrt{\kappa}}\ \sup_{z,z+2l\in\hat{\mathcal{J}}}|q(z+2l)-2q(z+l)+q(z)|.$$

The Steklov mean is considered as

$$q_l(z)=\frac{4}{l^2}\int_0^{\frac{l}{2}}\int_0^{\frac{l}{2}}\left[2q(z+u+v)-q(z+2(u+v))\right]du\,dv. \tag{9}$$

By direct computation, we have

(a) $\|q_l-q\|_{C(\hat{\mathcal{J}})}\le\varpi_2(q,l)$.

(b) $q_l',q_l''\in C(\hat{\mathcal{J}})$ and $\|q_l'\|_{C(\hat{\mathcal{J}})}\le\frac{5}{l}\varpi(q,l)$, $\quad\|q_l''\|_{C(\hat{\mathcal{J}})}\le\frac{9}{l^2}\varpi_2(q,l)$,

Theorem 2.4. *Suppose that $q \in C(\hat{\mathcal{J}})$ and $z \in \hat{\mathcal{J}}$. Then, we have*

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q;z) - q(z)\right| \leq 5\varpi\left(q, \sqrt{\gamma_{m,\tau,\varrho}(z)}\right) + \frac{13}{2}\varpi_2\left(q, \sqrt{\gamma_{m,\tau,\varrho}(z)}\right),$$

where $\gamma_{m,\tau,\varrho}(z) = \mathscr{G}_{m,\tau,\varrho}^{(\mu)}((t-z)^2;z)$.

Proof. For $z \in \hat{\mathcal{J}}$, and using (9), we obtain

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q;z) - q(z)\right| \leq \mathscr{G}_{m,\tau,\varrho}^{(\mu)}(|q-q_l|;z) + |\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q_l - q_l(z);z)| + |q_l(z) - q(z)|. \tag{10}$$

From (2), for every $q \in C(\hat{\mathcal{J}})$ we get

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q;z)\right| \leq \|q\|. \tag{11}$$

Using assumption (a) of Steklov mean and (11), we have

$$\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(|q-q_l|;z) \leq \|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q-q_l)\| \leq \|q-q_l\| \leq \varpi_2(q,l).$$

By Cauchy-Schwarz inequality and Taylor's formula, we may write

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q_l - q_l(z);z)\right| \leq \|q_l'\| \sqrt{\mathscr{G}_{m,\tau,\varrho}^{(\mu)}((t-z)^2;z)} + \frac{1}{2}\|q_l''\| \mathscr{G}_{m,\tau,\varrho}^{(\mu)}\left((t-z)^2;z\right).$$

Using Lemma 1.2 and inequality (b) of Steklov mean, we get

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q_l - q_l(z);z)\right| \leq \frac{5}{l}\varpi(q,l)\sqrt{\gamma_{m,\tau,\varrho}(z)} + \frac{9}{2l^2}\varpi_2(q,l)\gamma_{m,\tau,\varrho}(z).$$

Choosing $l = \sqrt{\gamma_{m,\tau,\varrho}(z)}$, and substituting the values in (10), we get the desired result. □

Next, we investigate the approximation of functions in a Lipschitz-type space [36] involving two parameters $\varsigma_1 \geq 0, \varsigma_2 > 0$, defined as

$$Lip_M^{(\varsigma_1,\varsigma_2)}(\beta) := \left\{ q \in C(\hat{\mathcal{J}}) : |q(y) - q(z)| \leq M \frac{|y-z|^\beta}{(y + \varsigma_1 z^2 + \varsigma_2 z)^{\frac{\beta}{2}}}; y \in \hat{\mathcal{J}}, z \in (0,1] \text{ and } 0 < \beta \leq 1 \right\}.$$

Theorem 2.5. *Suppose that $q \in Lip_M^{(\varsigma_1,\varsigma_2)}(\beta)$, we have*

$$\left|\mathscr{G}_{m,\tau,\varrho}^{(\mu)}(q;z) - q(z)\right| \leq M\left(\frac{\gamma_{m,\tau,\varrho}(z)}{\varsigma_1 z^2 + \varsigma_2 z}\right)^{\beta/2} \quad \forall z \in (0,1].$$

Proof. Using the application of Holder's inequality, we have

$$\begin{aligned}
\left|\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right| &\leq \sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1\left|q(t)-q(z)\right|\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\\
&\leq \sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\left(\int_0^1\left|q(t)-q(z)\right|^{\frac{2}{\beta}}\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right)^{\frac{\beta}{2}}\\
&\leq \left\{\sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1\left|q(t)-q(z)\right|^{\frac{2}{\beta}}\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right\}^{\frac{\beta}{2}}\\
&\quad\times\left(\sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right)^{\frac{2-\beta}{2}}\\
&= \left(\sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1\left|q(t)-q(z)\right|^{\frac{2}{\beta}}\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right)^{\frac{\beta}{2}}\\
&\leq M\left(\sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1\frac{(t-z)^2}{(t+\varsigma_1 z^2+\varsigma_2 z)}\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right)^{\frac{\beta}{2}}\\
&\leq \frac{M}{(\varsigma_1 z^2+\varsigma_2 z)^{\frac{\beta}{2}}}\left(\sum_{\nu=0}^{m}\hat{r}^{(\mu)}_{m,\nu}(z)\int_0^1(t-z)^2\left(\frac{t^{\nu\varrho+\tau(z)-1}(1-t)^{(m-\nu)\varrho+\tau(z)-1}}{B\left(\nu\varrho+\tau(z),(m-\nu)\varrho+\tau(z)\right)}\right)dt\right)^{\frac{\beta}{2}}\\
&= \frac{M}{(\varsigma_1 z^2+\varsigma_2 z)^{\frac{\beta}{2}}}\left(\mathcal{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\right)^{\frac{\beta}{2}} = M\left(\frac{\gamma_{m,\tau,\varrho}(z)}{\varsigma_1 z^2+\varsigma_2 z}\right)^{\frac{\beta}{2}}.
\end{aligned}$$

□

Theorem 2.6. *For $q \in C^1(\hat{\mathcal{J}})$ and $z \in \hat{\mathcal{J}}$, we have*

$$\left|\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right| \leq \left|\frac{\tau(z)(1-2z)}{(m\varrho+2\tau(z))}\right||q'(z)| + 2\sqrt{\gamma_{m,\tau,\varrho}(z)}\,\varpi\left(q',\sqrt{\gamma_{m,\tau,\varrho}(z)}\right). \tag{12}$$

Proof. Let $q \in C^1(\hat{\mathcal{J}})$. For any $t, z \in \hat{\mathcal{J}}$, we have

$$q(t)-q(z) = q'(z)(t-z) + \int_z^t (q'(v)-q'(z))\,dv.$$

Using $\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(\cdot;z)$ on both sides of the above relation, we have

$$\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(q(t)-q(z);z) = q'(z)\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(t-z;z) + \mathcal{G}^{(\mu)}_{m,\tau,\varrho}\left(\int_z^t (q'(v)-q'(z))\,dv;z\right).$$

Using $|q(t)-q(z)| \leq \varpi(q,\kappa)\left(\frac{|t-z|}{\kappa}+1\right)$, $\kappa > 0$, we may write

$$\left|\int_z^t (q'(v)-q'(z))\,dv\right| \leq \varpi(q',\kappa)\left(\frac{(t-z)^2}{\kappa}+|t-z|\right),$$

it follows that

$$\left|\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right| \leq |q'(z)| \quad |\mathcal{G}^{(\mu)}_{m,\tau,\varrho}(t-z;z)| + \varpi(q',\kappa)\left\{\frac{1}{\kappa}\mathcal{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z) + \mathcal{G}^{(\mu)}_{m,\tau,\varrho}(|t-z|;z)\right\}.$$

Hence using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\right| &\le |q'(z)|\ \ |\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(t-z;z)| \\ &\quad +\varpi(q',\kappa)\left\{\frac{1}{\kappa}\sqrt{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)}+1\right\}\sqrt{\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)}.\end{aligned}$$

Now, taking $\kappa=\sqrt{\gamma_{m,\tau,\varrho}(z)}$, the required result follows. □

Suppose that $\theta(z)=\sqrt{z(1-z)}$ and $q\in C(\hat{\mathcal{J}})$. The Ditzian-Totik first order modulus of smoothness [16] is defined by

$$\varpi_\theta(q,\lambda)=\sup_{0<h\le\lambda}\left\{\left|q\Big(z+\frac{h\theta(z)}{2}\Big)-q\Big(z-\frac{h\theta(z)}{2}\Big)\right|, z\pm\frac{h\theta(z)}{2}\in\hat{\mathcal{J}}\right\},$$

and an appropriate K-functional is defined by

$$\overline{K}_\theta(q,\lambda)=\inf_{g\in J_\theta}\{\|q-g\|+\lambda\|\theta g'\|+\lambda^2\|g'\|\}\ (\lambda>0),$$

where $J_\theta=\{g: g\in AC_{loc}, \|\theta g'\|<\infty, \|g'\|<\infty\}$.
From [16], there exists a constant $\mathscr{N}>0$, such that

$$\mathscr{N}^{-1}\varpi_\theta(q,\lambda)\le\overline{K}_\theta(q,\lambda)\le\mathscr{N}\varpi_\theta(q,\lambda). \tag{13}$$

Now, we compute the order of convergence theorem for the operators $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}$.

Theorem 2.7. *Suppose that $q\in C(\hat{\mathcal{J}})$ and $z\in[0,1)$*

$$|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)| \le \mathscr{N}\varpi_\theta\left(q,2\sqrt{\frac{1+\varrho}{m\varrho}}\right),$$

where $\mathscr{N}>0$ is a constant.

Proof. Using the relation $g(t)=g(z)+\int_z^t g'(s)ds$, we can write

$$\left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(g;z)-g(z)\right| = \left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\Big(\int_z^t g'(s)ds;z\Big)\right|. \tag{14}$$

For any $z,t\in(0,1)$, we have

$$\left|\int_z^t g'(s)ds\right|\le\|\theta g'\|\left|\int_z^t\frac{1}{\theta(s)}ds\right|. \tag{15}$$

Now,

$$\begin{aligned}\left|\int_z^t\frac{1}{\theta(s)}ds\right| &= \left|\int_z^t\frac{1}{\sqrt{s(1-s)}}ds\right|\le\left|\int_z^t\Big(\frac{1}{\sqrt{s}}+\frac{1}{\sqrt{1-s}}\Big)ds\right| \\ &\le 2\Big(|\sqrt{t}-\sqrt{z}|+|\sqrt{1-t}-\sqrt{1-z}|\Big) \\ &< 2|t-z|\Big(\frac{1}{\sqrt{z}}+\frac{1}{\sqrt{1-z}}\Big)\le\frac{2\sqrt{2}\,|t-z|}{\theta(z)}.\end{aligned} \tag{16}$$

Collecting (14)-(16) and operating Cauchy-Schwarz property, we can write

$$\begin{aligned} |\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(g;z)-g(z)| &< 2\sqrt{2}\|\theta g'\|\theta^{-1}(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(|t-z|;z)\\ &\leq 2\sqrt{2}\|\theta g'\|\theta^{-1}(z)\Big(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\Big)^{1/2}\\ &< 4\|\theta g'\|\sqrt{\frac{1+\varrho}{m\varrho}}. \end{aligned} \tag{17}$$

Using (11) and (17), we get

$$\begin{aligned} |\ \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)\ | &\leq |\ \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q-g;z)\ | + |q-g| + |\ \mathscr{G}^{(\mu)}_{m,\tau,\varrho}(g;z)-g(z)\ |\\ &\leq 2\|q-g\| + 4\|\theta g'\|\sqrt{\frac{1+\varrho}{m\varrho}}. \end{aligned} \tag{18}$$

Taking infimum on the right hand side over all $g\in J_\theta$,

$$|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)| \leq 2\overline{K}_\theta\left(q;2\sqrt{\frac{1+\varrho}{m\varrho}}\right). \tag{19}$$

Using $\overline{K_\theta}(q,\lambda)\sim\varpi_\theta(q,\lambda)$, shows the relation. □

We compute a quantitative Voronovskaja type result for the operator $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}$ by using of Ditzian-Totik modulus of smoothness.

Theorem 2.8. *Let $q\in C^2(\hat{\mathcal{J}})$ and m sufficiently large the following inequality holds*

$$\left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)-\Lambda_{m,1}(z,\tau)q'(z)-\Lambda_{m,2}(z,\tau)q''(z)\right| \leq \frac{\mathscr{N}}{m}\varpi_\theta\left(q'',\sqrt{\frac{(1+\varrho)}{m\varrho}}\right),$$

where

$$\begin{aligned} \Lambda_{m,1}(z,\tau) &= \frac{\tau(z)(1-2z)}{(m\varrho+2\tau(z))},\\ \Lambda_{m,2}(z,\tau) &= \frac{z(1-z)\big(m\varrho(1+m+2m\varrho)-4(m+1)\tau^2(z)-2(m+1)\tau(z)\big)}{(m+1)(m\varrho+2\tau(z))(m\varrho+2\tau(z)+1)}\\ &\quad+\frac{\tau(z)(1+\tau(z))}{(m\varrho+2\tau(z))(m\varrho+2\tau(z)+1)}, \end{aligned}$$

and $\mathscr{N}>0$ depends on z and τ.

Proof. For $q\in C^2(\hat{\mathcal{J}})$, $t,z\in\hat{\mathcal{J}}$, by Taylor's expansion, we may write

$$q(t)-q(z)=(t-z)q'(z)+\int_z^t(t-y)q''(y)dy.$$

Hence,

$$q(t)-q(z)-(t-z)q'(z)-\frac{1}{2}(t-z)^2q''(z)=\int_z^t(t-y)\left(q''(y)-q''(z)\right)dy.$$

Using $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(\cdot;z)$ to both sides of the above relation, we obtain

$$\left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)-q(z)-\Lambda_{m,1}(z,\tau)q'(z)-\Lambda_{m,2}(z,\tau)q''(z)\right| \leq \mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(\left|\int_z^t|t-y|\ |q''(y)-q''(z)|dy\right|;z\right) \tag{20}$$

From [[19], p.337], we have

$$\left|\int_z^t |t-y|\,|q''(y)-q''(z)|dy\right| \leq 2\|q''-g\|(t-z)^2 + 2\|\theta g'\|\theta^{-1}(z)|t-z|^3, \tag{21}$$

where $g \in J_\theta$.
Applying Lemma 1.2 it follows that there exists

$$\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z) \leq \frac{\mathscr{N}(1+\varrho)}{m\varrho}\theta^2(z) \text{ and } \mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^4;z) \leq \frac{\mathscr{N}(1+\varrho)^2}{m^2\varrho^2}\theta^4(z). \tag{22}$$

Collecting (20-22) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z) - q(z) - \Lambda_{m,1}(z,\tau)q'(z) - \Lambda_{m,2}(z,\tau)q''(z)\right| \\
&\leq 2\|q''-g\|\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z) + 2\|\theta g'\|\theta^{-1}(z)\mathscr{G}^{(\mu)}_{m,\tau,\varrho}\left(|t-z|^3;z\right) \\
&\leq \frac{\mathscr{N}(1+\varrho)}{m\varrho}\theta^2(z)\|q''-g\| + 2\|\theta g'\|\theta^{-1}(z)\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^2;z)\right)^{1/2}\left(\mathscr{G}^{(\mu)}_{m,\tau,\varrho}((t-z)^4;z)\right)^{1/2} \\
&\leq \frac{\mathscr{N}(1+\varrho)}{m\varrho}\theta^2(z)\|q''-g\| + \theta^2(z)\frac{\mathscr{N}(1+\varrho)}{m\varrho}\sqrt{\frac{(1+\varrho)}{m\varrho}}\|\theta g'\| \\
&\leq \frac{\mathscr{N}(1+\varrho)}{m\varrho}\theta^2(z)\left(\|q''-g\| + \sqrt{\frac{(1+\varrho)}{m\varrho}}\|\theta g'\|\right).
\end{aligned}$$

Taking the infimum on the right hand side of the above relations over $g \in J_\theta$, the theorem is proved. □

3. Numerical Examples

Example 3.1. *Let $m = 20$, $\mu = 0.9$ and $\varrho = 15$. The convergence of the operator $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)$ (magenta) and Bernstein-Durrmeyer [17] (green) to the function is illustrated in Figure 1 for $\tau(z) = \dfrac{(z^3+\sin z+1)}{3}$ and $q(z) = z^3\cos\left(\dfrac{z^2\pi}{2}\right)$ (orange). We observe that the operator $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)$ gives a better approximation to $q(z)$ than Bernstein-Durrmeyer [17].*

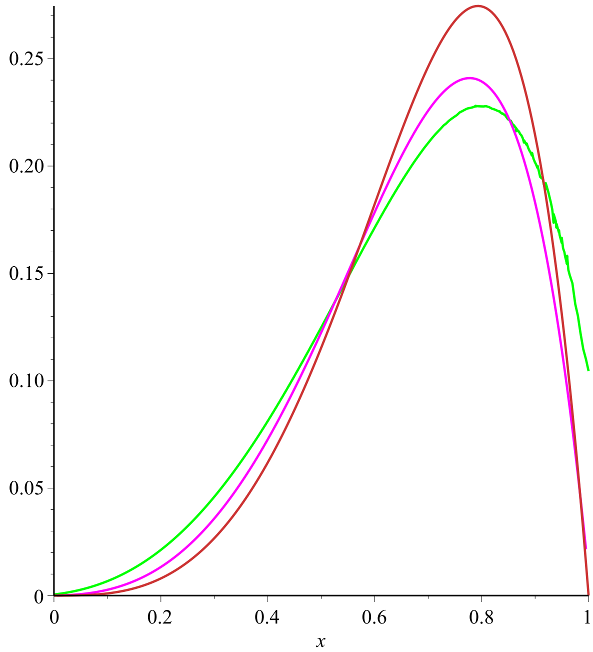

Figure 1: Approximation Process

Example 3.2. *Let $m = 20$, $\mu = 0.9$ and $\varrho = 10$. The convergence of the operator $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)$ (magenta) and Bernstein-Durrmeyer [17] (green) to the function is illustrated in Figure 2 for $\tau(z) = \dfrac{(z^3+z^2+1)}{3}$ and $q(z) = z^2 \sin\left(\dfrac{z\pi}{2}\right)$ (orange). We notice that the operator $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)$ gives a better approximation to $q(z)$ than Bernstein-Durrmeyer [17].*

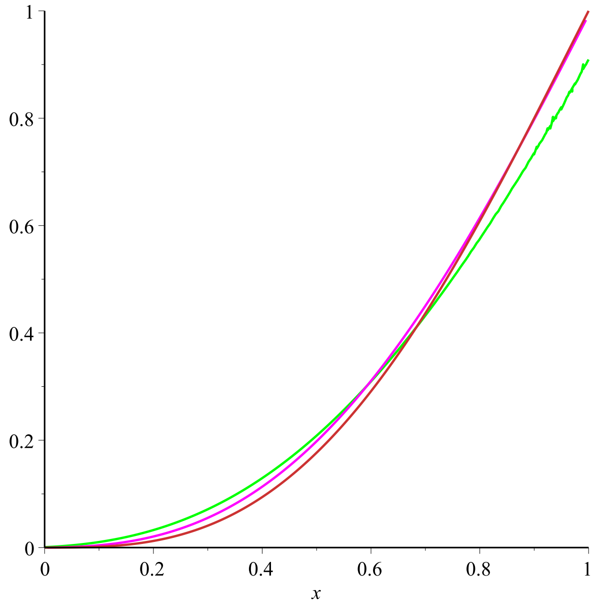

Figure 2: Approximation Process

Example 3.3. *Let $m = 10, 20, 30, 40$, $\mu = 0.5$ and $\varrho = 5$. The convergence of the operator $\mathscr{G}^{(0.5)}_{10,\tau,5}(q;z)$(blue), $\mathscr{G}^{(0.5)}_{20,\tau,5}(q;z)$(green), $\mathscr{G}^{(0.5)}_{30,\tau,5}(q;z)$(yellow) and $\mathscr{G}^{(0.5)}_{40,\tau,5}(q;z)$ (magenta) to the function is illustrated in Figure 3 for $\tau(z) = \dfrac{(z^3 + \cos z + 1)}{5}$ and $q(z) = z^2 \cos\left(\dfrac{z\pi}{2}\right)$ (red). This example explains the convergence of the operators $\mathscr{G}^{(\mu)}_{m,\tau,\varrho}(q;z)$ that are going to the function $q(z)$ if the values of m are increasing.*

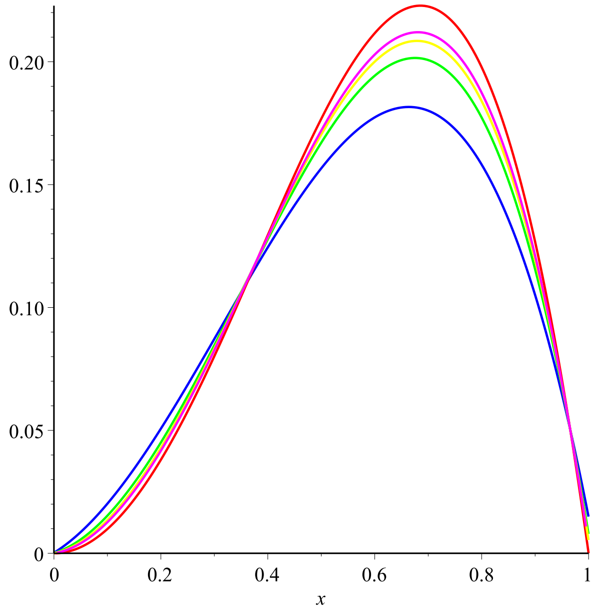

Figure 3: Approximation Process

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