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Lupaş Post Quantum Blending Functions and Bézier Curves Over Arbitrary Intervals

Asif Khan[a], M.S. Mansoori[a], Mohammad Iliyas[a], M. Mursaleen[a,b]

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[b]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

Abstract. In this paper, we extend the properties of rational Lupaş-Bernstein blending functions, Lupaş-Bézier curves and surfaces over arbitrary compact intervals $[\alpha, \beta]$ in the frame of post quantum-calculus and derive the de-Casteljau's algorithm based on post quantum-integers. We construct a two parameter family as Lupaş post quantum Bernstein functions over arbitrary compact intervals and establish their degree elevation and reduction properties. We also discuss some fundamental properties over arbitrary intervals for these curves such as de Casteljau algorithm and degree evaluation properties. Further we construct post quantum Lupaş Bernstein operators over arbitrary compact intervals with the help of rational Lupaş-Bernstein functions. At the end some graphical representations are added to demonstrate consistency of theoretical findings.

1. Essential preliminaries and review of previous results

Computer aided geometric design (CAGD) is a discipline which deals with computational aspects of geometric objects. It emphasizes on the mathematical development of curves and surfaces such that it becomes compatible with computers. In [2], Bernstein constructed polynomials called as Bernstein polynomials. The Bernstein bases play a significant role in preserving the shape of the curves or surfaces. Many popular programs utilize Bernstein polynomials to form what are known as Bézier curves [6, 8, 9, 15, 18, 30, 35, 36].

Quantum calculus [37] has led to a new generalizations of Bernstein polynomials which was first initiated by Lupaş [20] and later on by Phillips [34].

Recently, extension of quantum calculus to post quantum calculus in Approximation Theory has been initiated by Mursaleen et al [24]. They constructed and studied post quantum analogue of Phillips Bernstein operators (polynomials) [34]. These generalizations of Phillips operators (polynomials) reduces to classical Bernstein operators [2], for parameters $\mathfrak{p} = \mathfrak{q} = 1$. For more relevant works, we refer the reader to

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Corresponding author: M. Mursaleen
Email addresses: asifjnu07@gmail.com (Asif Khan), shanawaz110592@gmail.com (M.S. Mansoori), iliyas2695@gmail.com (Mohammad Iliyas), mursaleenm@gmail.com (M. Mursaleen)

[1, 7, 13, 14, 21–23, 25–27, 29, 31, 38–42, 44].

The post quantum integers $[h]_{\mathfrak{p},\mathfrak{q}}$ for any $\mathfrak{p} > 0$ and $\mathfrak{q} > 0$ are defined by

$$[h]_{\mathfrak{p},\mathfrak{q}} = \mathfrak{p}^{h-1} + \mathfrak{p}^{h-2}\mathfrak{q} + \mathfrak{p}^{h-3}\mathfrak{q}^2 + ... + \mathfrak{p}\mathfrak{q}^{h-2} + \mathfrak{q}^{h-1} = \begin{cases} \frac{\mathfrak{p}^h-\mathfrak{q}^h}{\mathfrak{p}-\mathfrak{q}}, & \text{when } \mathfrak{p} \neq \mathfrak{q} \neq 1 \\ h\ \mathfrak{p}^{h-1}, & \text{when } \mathfrak{p} = \mathfrak{q} \neq 1 \\ [h]_{\mathfrak{q}}, & \text{when } \mathfrak{p} = 1 \\ h, & \text{when } \mathfrak{p} = \mathfrak{q} = 1, \end{cases}$$

where $[h]_{\mathfrak{q}}$ denotes the $\mathfrak{q}$-integer of non-negative integer h.

The post quantum-binomial expansion is given by

$$(au + bv)^h_{\mathfrak{p},\mathfrak{q}} := \sum_{k=0}^{h} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} \left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} a^{h-k} b^k u^{h-k} v^k,$$

$$(u + v)^h_{\mathfrak{p},\mathfrak{q}} = (u + v)(\mathfrak{p}u + \mathfrak{q}v)(\mathfrak{p}^2u + \mathfrak{q}^2v) \cdots (\mathfrak{p}^{h-1}u + \mathfrak{q}^{h-1}v),$$

$$(1 - u)^h_{\mathfrak{p},\mathfrak{q}} = (1 - u)(\mathfrak{p} - \mathfrak{q}u)(\mathfrak{p}^2 - \mathfrak{q}^2u) \cdots (\mathfrak{p}^{h-1} - \mathfrak{q}^{h-1}u),$$

where the post quantum binomial coefficients are defined by

$$\left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} = \frac{[h]_{\mathfrak{p},\mathfrak{q}}!}{[k]_{\mathfrak{p},\mathfrak{q}}![h-k]_{\mathfrak{p},\mathfrak{q}}!}.$$

Details on post quantum-calculus can be found in [10, 11, 24].
The post quantum Bernstein operators [24] are defined as follows. For $0 < \mathfrak{q} < \mathfrak{p} \leq 1$,

$$B_{h,\mathfrak{p},\mathfrak{q}}(f; u) = \frac{1}{\mathfrak{p}^{\frac{h(h-1)}{2}}} \sum_{k=0}^{h} \left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{k(k-1)}{2}} u^k \prod_{s=0}^{h-k-1} (\mathfrak{p}^s - \mathfrak{q}^s u)\ f\left(\frac{[k]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{k-h}[h]_{\mathfrak{p},\mathfrak{q}}}\right), \quad u \in [0, 1]. \tag{1}$$

Post quantum Bernstein operators (1) reduce to Phillips $\mathfrak{q}$-Bernstein operators [34] for $\mathfrak{p} = 1$.

Herein, we recall and review some preliminary results of [16] for the sake of completeness.

With the development of Post quantum calculus, Khalid and Lobiyal [16] constructed and studied $(\mathfrak{p}, \mathfrak{q})$-analogue of Lupaş Bernstein operators which is an extension to the work of Lupaş [20]. They studied and derived various results for rational Lupaş Bernstein blending functions, Lupaş Bèzier curves and surfaces. The operators $L^h_{\mathfrak{p},\mathfrak{q}} : C[0, 1] \to C[0, 1]$ defined by

$$L^h_{\mathfrak{p},\mathfrak{q}}(f; u) = \sum_{k=0}^{h} \frac{f\left(\frac{\mathfrak{p}^{h-k}\ [k]_{\mathfrak{p},\mathfrak{q}}}{[h]_{\mathfrak{p},\mathfrak{q}}}\right) \left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}}\ u^k\ (1-u)^{h-k}}{\prod_{j=1}^{h} \{\mathfrak{p}^{j-1}(1-u) + \mathfrak{q}^{j-1}u\}}, \tag{2}$$

are post quantum analogue of Lupaş Bernstein operators.

For $\mathfrak{p} = 1$, these turn out to be $\mathfrak{q}$-analogue of Lupaş operators [31].

These operators (rational) reduces to classical Bernstein operators [2], if one chooses parameters $\mathfrak{p} = \mathfrak{q} = 1$.

All these operators can be used to approximate any continuous function via Korovkin type approximation.

For other application of Bernstein polynomials, one can refer to [19] and [30].

The post quantum generalization by Khalid and Lobiyal [16] of Lupaş operators $L^h_{\mathfrak{p},\mathfrak{q}}(f;u)$ have an advantage of generating positive linear operators for all $\mathfrak{p} > 0$, $\mathfrak{q} > 0$, whereas post quantum generalization of Phillips polynomials [24] generate positive linear operators only if $\mathfrak{p}, \mathfrak{q} \in (0, 1)$.

Motivated by work of Khalid and Lobiyal [16], we study these on arbitrary compact intervals.

For other works related to Bézier curves and Approximation theory, one can refer [9, 15, 17, 30, 31, 33]. Mainly de-Casteljau's algorithm has been used in this paper. The derived results and constructions are important from computational point.

We have formulated this paper as follows: In Section 2, Lupaş post quantum analogue of Bernstein functions over $[\alpha, \beta]$ is defined and its various properties has been established. In Section 3, the Lupaş post quantum Bézier curves are studied. In Section 4, Bézier surfaces over the generalized tensor product on the rectangular domain from the Lupaş post quantum analogue of the Bernstein functions are discussed. In section 5, post quantum-analogue of Lupaş operators over arbitrary intervals are constructed and its endpoints interpolation properties are presented. The effects of the shape parameters on the shape of the curves and surfaces are shown in Section 6.

2. Construction of post quantum Lupaş basis functions on $[\alpha, \beta]$

We present here an extension of Lupaş type post quantum analogue (rational) of the Bernstein functions over arbitrary compact intervals $[\alpha, \beta]$:

for any $\mathfrak{p} > 0$ and $\mathfrak{q} > 0$, we set

$$b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \frac{\left[\begin{array}{c} h \\ k \end{array}\right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} \,(u-\alpha)^k\,(\beta-u)^{h-k}}{\prod\limits_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}, \tag{3}$$

where $b^{0,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta), b^{1,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta), \cdots, b^{h,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)$ are the post-quantum analogue of the Lupaş $\mathfrak{q}$-Bernstein functions [9] of degree h on the interval $[\alpha, \beta]$.

When $\mathfrak{p} = 1$ and $\alpha = 0$, $\beta = 1$, Lupaş post quantum Bernstein functions over $[\alpha, \beta]$ turns out to be Lupaş $\mathfrak{q}$-Bernstein functions as given in [9], whereas when $\mathfrak{p} = \mathfrak{q} = 1$, and $\alpha = 0$, $\beta = 1$ Lupaş post quantum Bernstein functions turns out to be classical Bernstein functions.

The Lupaş post quantum Bernstein blending functions over the interval $[\alpha, \beta]$ for $h = 3$ are as follows:

$$b^{0,3}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \frac{\mathfrak{p}^3(\beta-u)^3}{(\beta-\alpha)(\mathfrak{p}(\beta-u)+\mathfrak{q}(u-\alpha))\ (\mathfrak{p}^2(\beta-u)+\mathfrak{q}^2(u-\alpha))}$$

$$b^{1,3}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \frac{(\mathfrak{p}^2+\mathfrak{p}\mathfrak{q}+\mathfrak{q}^2)\ \mathfrak{p}(u-\alpha)(\beta-u)^2}{(\beta-\alpha)(\mathfrak{p}(\beta-u)+\mathfrak{q}(u-\alpha))\ (\mathfrak{p}^2(\beta-u)+\mathfrak{q}^2(u-\alpha))}$$

$$b^{2,3}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \frac{(\mathfrak{p}^2+\mathfrak{p}\mathfrak{q}+\mathfrak{q}^2)\ \mathfrak{q}(u-\alpha)^2(\beta-u)}{(\beta-\alpha)(\mathfrak{p}(\beta-u)+\mathfrak{q}(u-\alpha)\ (\mathfrak{p}^2(\beta-u)+\mathfrak{q}^2(u-\alpha))}$$

$$b^{3,3}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \frac{\mathfrak{q}^3(u-\alpha)^3}{(\beta-\alpha)(\mathfrak{p}(\beta-u)+\mathfrak{q}(u-\alpha))\ (\mathfrak{p}^2(\beta-u)+\mathfrak{q}^2(u-\alpha))}$$

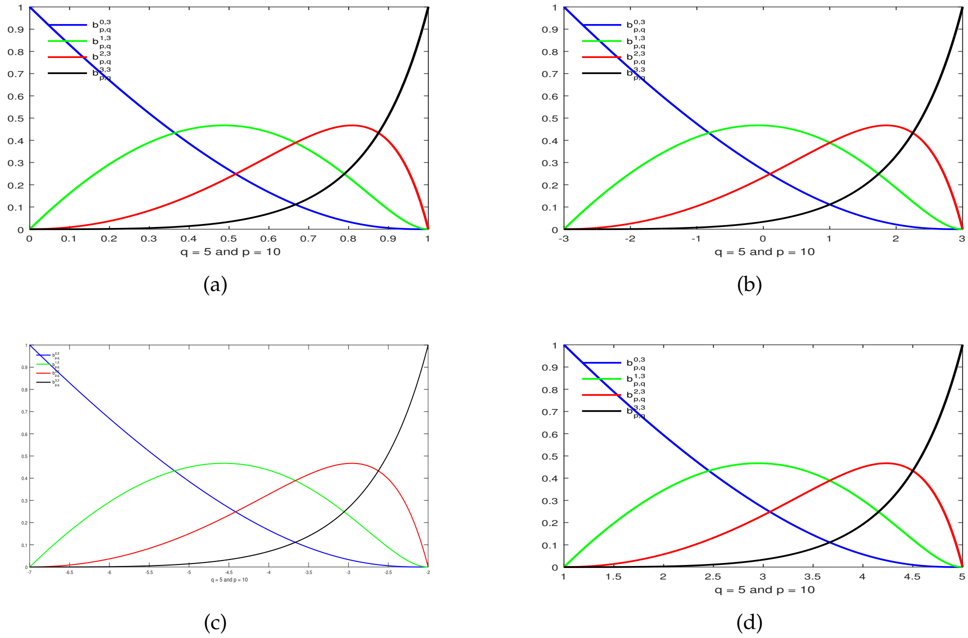


Figure 1: Lupaş cubic Bèzier blending functions on arbitrary intervals

Figure 1 and 2 show the Lupaş post quantum-Bernstein blending functions of degree 3 for different values of $\mathfrak{p}$ and $\mathfrak{q}$. Here one can observe that at each point of the interval, sum of blending functions is unity.

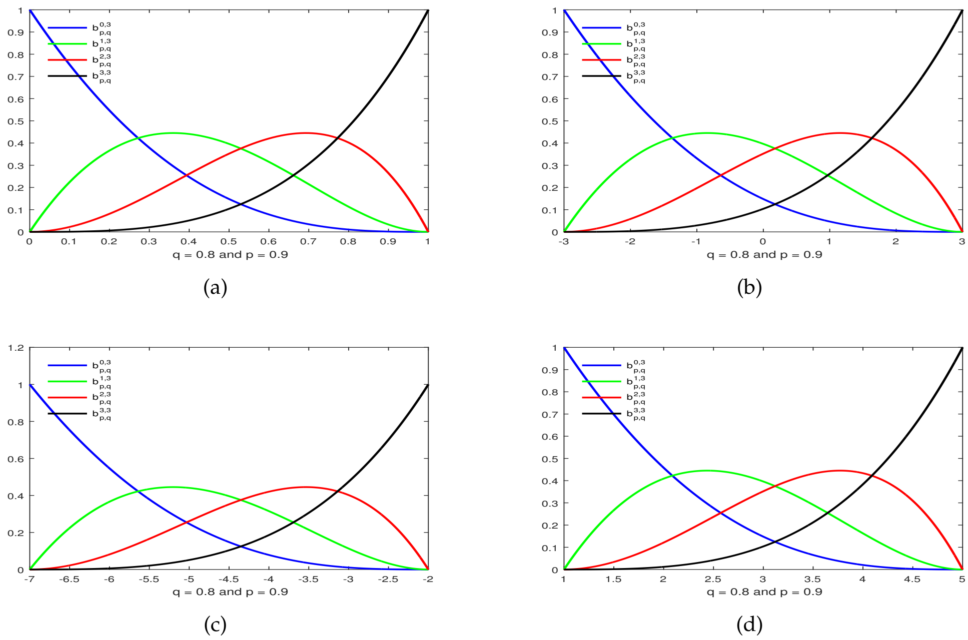


Figure 2: Lupaş cubic Bèzier blending functions on arbitrary intervals

Theorem 2.1. *The Lupaş post quantum-analogue of the Bernstein functions over the interval* $[\alpha, \beta]$ *possess the following properties:*

(1.) Non-negativity: $b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) \geq 0, \quad k = 0, 1, \cdots, h, \quad u \in [\alpha, \beta].$

(2.) Partition of unity:

$$\sum_{k=0}^{h} b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) = 1, \quad u \in [\alpha, \beta].$$

(3.) End-point property:

$$b^{k,h}_{\mathfrak{p},\mathfrak{q}}(\alpha; \alpha, \beta) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & k \neq 0 \end{cases}$$

$$b^{k,h}_{\mathfrak{p},\mathfrak{q}}(\beta; \alpha, \beta) = \begin{cases} 1, & \text{if } k = h \\ 0, & k \neq h \end{cases}$$

(4.) Post quantum-inverse symmetry:

$$b^{h-k,h}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) = b^{h-k,h}_{\frac{1}{\mathfrak{q}},\frac{1}{\mathfrak{p}}}(u; \alpha, \beta) = b^{k,h}_{\frac{1}{\mathfrak{p}},\frac{1}{\mathfrak{q}}}(\alpha + \beta - u; \alpha, \beta)$$

for $k = 0, 1, \cdots, h$.
(5.) Reducibility: by choosing, $\mathfrak{p} = 1$ *and* $\alpha = 0,\ \beta = 1$, *formula 3 will turnout to be the Lupaş* $\mathfrak{q}$*-Bernstein bases.*

Proof:
Here we only present the proofs of properties 2 and 4 as the properties 1, 3 and 5 are obvious.

Property 2:
When $u = \alpha$ or β, the result is clear. In case $u \neq \alpha, \beta$, the post quantum analogue of Newton's Binomial formula will be applied:
Consider

$$\sum_{k=0}^{h} \left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} (u-\alpha)^k (\beta-u)^{h-k}$$

$$= \sum_{k=0}^{h} \left[\begin{array}{c} h \\ k \end{array} \right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} (\beta-u)^h \left(\frac{u-a}{\beta-u}\right)^k$$

$$= \Big((\beta-u) + (u-\alpha)\Big)\Big(\mathfrak{p}(\beta-u) + \mathfrak{q}(u-\alpha)\Big) \cdots \Big(\mathfrak{p}^{h-1}(\beta-u) + \mathfrak{q}^{h-1}(u-\alpha)\Big)$$

$$= \prod_{s=1}^{h} \Big(\mathfrak{p}^{s-1}(\beta-u) + \mathfrak{q}^{s-1}(u-\alpha)\Big).$$

Hence

$$\sum_{k=0}^{h} b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) = 1.$$

Property (4) We need following relations to prove this result, :

$$[h]_{\mathfrak{p},\mathfrak{q}} = [h]_{\mathfrak{q},\mathfrak{p}} \text{ and } \begin{bmatrix} h \\ k \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} = \begin{bmatrix} h \\ k \end{bmatrix}_{\frac{1}{\mathfrak{q}},\frac{1}{\mathfrak{p}}} \frac{(\mathfrak{p}\mathfrak{q})^{\frac{k(2h-k+1)}{2}}}{(\mathfrak{p}\mathfrak{q})^{\frac{k(k+1)}{2}}}.$$

Consider

$$\begin{aligned}
b^{h-k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) &= \frac{\begin{bmatrix} h \\ h-k \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(k)(k-1)}{2}} \mathfrak{q}^{\frac{(h-k)(h-k-1)}{2}} (u-\alpha)^{h-k} (\beta-u)^{k}}{\prod_{j=1}^{n}\{\mathfrak{p}^{j-1}(\beta-u) + \mathfrak{q}^{j-1}(u-\alpha)\}} \\
&= \frac{\begin{bmatrix} h \\ h-k \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(k)(k-1)}{2}} \mathfrak{q}^{\frac{(h-k)(h-k-1)}{2}} (u-\alpha)^{h-k} (\beta-u)^{k}}{\mathfrak{p}^{\frac{(h)(h-1)}{2}} \mathfrak{q}^{\frac{(h)(h-1)}{2}} \prod_{j=1}^{h}\{\frac{1}{\mathfrak{p}^{j-1}}(u-\alpha) + \frac{1}{\mathfrak{q}^{j-1}}(\beta-u)\}} \\
&= \frac{\begin{bmatrix} h \\ k \end{bmatrix}_{\frac{1}{\mathfrak{q}},\frac{1}{\mathfrak{p}}} \frac{1}{\mathfrak{p}}^{\frac{(h-k)(h-k-1)}{2}} \frac{1}{\mathfrak{q}}^{\frac{(k)(k-1)}{2}} (u-\alpha)^{h-k} (\beta-u)^{k}}{\prod_{j=1}^{h}\{\frac{1}{\mathfrak{p}^{j-1}}(u-\alpha) + \frac{1}{\mathfrak{q}^{j-1}}(\beta-u)\}} \\
&= b^{k,h}_{\frac{1}{\mathfrak{p}},\frac{1}{\mathfrak{q}}}(\alpha+\beta-u) \\
&= b^{h-k,h}_{\frac{1}{\mathfrak{q}},\frac{1}{\mathfrak{p}}}(u;\alpha,\beta).
\end{aligned}$$

3. Degree evaluation for Lupaş post quantum Bernstein functions over $[\alpha, \beta]$

With the help of this algorithm one can construct a new control polygon by taking a convex combination of the old control points which retains the previous points. For this, the identities (4),(5) and Theorem (3.1) will be useful.

$$\frac{\mathfrak{q}^{h}\,(u-\alpha)}{\mathfrak{p}^{h}(\beta-u) + \mathfrak{q}^{h}\,(u-\alpha)} b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \Big(1 - \frac{\mathfrak{p}^{k+1}[h-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\Big) b^{k+1,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta), \tag{4}$$

$$\frac{\mathfrak{p}^{h}(\beta-u)}{\mathfrak{p}^{h}(\beta-u) + \mathfrak{q}^{h}\,(u-\alpha)} b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \Big(\frac{\mathfrak{p}^{k}[h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\Big) b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta). \tag{5}$$

Theorem 3.1. *Each Lupaş post quantum analogue of the corresponding Bernstein function of degree h over the interval $[\alpha,\beta]$ is a linear combination of two Lupaş post quantum analogues of the Bernstein functions of degree $h+1$ over the interval $[\alpha,\beta]$.*

$$b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) = \Big(\frac{\mathfrak{p}^{k}\,[h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\Big) b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) + \Big(1 - \frac{\mathfrak{p}^{k+1}\,[h-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\Big) b^{k+1,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta). \tag{6}$$

Proof:

$$
\begin{aligned}
b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta) &= b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)\Big(1-\frac{\mathfrak{q}^h\,(u-\alpha)}{\mathfrak{p}^h(\beta-u)+\mathfrak{q}^h\,(u-\alpha)}+\frac{\mathfrak{q}^h\,(u-\alpha)}{\mathfrak{p}^h(\beta-u)+\mathfrak{q}^h\,(u-\alpha)}\Big)\\
&=\frac{\mathfrak{p}^h(\beta-u)}{\mathfrak{p}^h(\beta-u)+\mathfrak{q}^h\,(u-\alpha)}\left(\frac{\begin{bmatrix} h\\ k\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}}\mathfrak{q}^{\frac{k(k-1)}{2}}\,(u-\alpha)^k\,(\beta-u)^{h-k}}{\prod_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}\right)\\
&+\frac{\mathfrak{q}^h\,(u-\alpha)}{\mathfrak{p}^h(\beta-u)+\mathfrak{q}^h\,(u-\alpha)}\left(\frac{\begin{bmatrix} h\\ k\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}}\mathfrak{q}^{\frac{k(k-1)}{2}}\,(u-\alpha)^k\,(\beta-u)^{h-k}}{\prod_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}\right).
\end{aligned}
$$

Using 4 and 5, we have

$$
b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)=\left(\frac{\mathfrak{p}^k\,[h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right)b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)+\left(1-\frac{\mathfrak{p}^{k+1}\,[h-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right)b^{k+1,h+1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta).
$$

Theorem 3.2. *Each Lupaş post quantum analogue of the Bernstein function of degree h over the interval $[\alpha,\beta]$ can be expressed as a linear combination of two Lupaş post quantum analogues of the Bernstein functions of degree $h-1$ over the interval $[\alpha,\beta]$ as follows:*

$$
b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)=\frac{\mathfrak{q}^{h-1}\,(u-\alpha)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\,b^{k-1,h-1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)+\frac{\mathfrak{p}^{h-1}(\beta-u)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\,b^{k,h-1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta), \tag{7}
$$

$$
b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)=\frac{\mathfrak{p}^{h-k}\mathfrak{q}^{k-1}(u-\alpha)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\,b^{k-1,h-1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)+\frac{\mathfrak{p}^{h-k-1}\mathfrak{q}^{k}(\beta-u)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\,b^{k,h-1}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta). \tag{8}
$$

Proof: From the Pascal's type relations of the post quantum Binomial coefficient, we have

$$
b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)=\frac{\left(\mathfrak{p}^{h-k}\begin{bmatrix} h-1\\ k-1\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}+\mathfrak{q}^k\begin{bmatrix} h-1\\ k\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\right)\mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}}\mathfrak{q}^{\frac{k(k-1)}{2}}\,(u-\alpha)^k\,(\beta-u)^{h-k}}{\prod_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}
$$

or

$$b_{\mathfrak{p},\mathfrak{q}}^{k,h}(u;\alpha,\beta)=\frac{\mathfrak{p}^{h-k}\mathfrak{q}^{k-1}(u-\alpha)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\frac{\begin{bmatrix}h-1\\k-1\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\mathfrak{p}^{\frac{(n-k)(h-k-1)}{2}}\mathfrak{q}^{\frac{(k-1)(k-2)}{2}}(u-\alpha)^{k-1}(\beta-u)^{h-k}}{\prod_{j=1}^{h-1}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}$$
$$+\frac{\mathfrak{p}^{h-k-1}\mathfrak{q}^{k}(\beta-u)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}\frac{\begin{bmatrix}h-1\\k\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\mathfrak{p}^{\frac{(h-1-k)(h-k-2)}{2}}\mathfrak{q}^{\frac{k(k-1)}{2}}(u-\alpha)^{k}(\beta-u)^{h-k-1}}{\prod_{j=1}^{h-1}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}$$
$$=\frac{\mathfrak{p}^{h-k}\mathfrak{q}^{k-1}(u-\alpha)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}b_{\mathfrak{p},\mathfrak{q}}^{k-1,h-1}(u;\alpha,\beta)+\frac{\mathfrak{p}^{h-k-1}\mathfrak{q}^{k}(\beta-u)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}b_{\mathfrak{p},\mathfrak{q}}^{k,h-1}(u;\alpha,\beta)$$

or

$$b_{\mathfrak{p},\mathfrak{q}}^{k,h}(u;\alpha,\beta)=\frac{\left(\mathfrak{q}^{h-k}\begin{bmatrix}h-1\\k-1\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}+\mathfrak{p}^{k}\begin{bmatrix}h-1\\k\end{bmatrix}_{\mathfrak{p},\mathfrak{q}}\right)\mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}}\mathfrak{q}^{\frac{k(k-1)}{2}}(u-\alpha)^{k}(\beta-u)^{h-k}}{\prod_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u)+\mathfrak{q}^{j-1}(u-\alpha)\}}$$
$$=\frac{\mathfrak{q}^{h-1}(u-\alpha)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}b_{\mathfrak{p},\mathfrak{q}}^{k-1,h-1}(u;\alpha,\beta)+\frac{\mathfrak{p}^{h-1}(\beta-u)}{\mathfrak{p}^{h-1}(\beta-u)+\mathfrak{q}^{h-1}(u-\alpha)}b_{\mathfrak{p},\mathfrak{q}}^{k,h-1}(u;\alpha,\beta).$$

4. Lupaş post quantum Bézier curves over arbitrary compact intervals $[\alpha,\beta]$

We define the Lupaş post quantum Bézier curves of degree h over the interval $[\alpha,\beta]$ using the Lupaş post quantum analogues of the Bernstein functions over $[\alpha,\beta]$, as follows:

$$\mathcal{P}(u;\mathfrak{p},\mathfrak{q})=\sum_{i=0}^{h}\mathcal{P}_{\mathbf{i}}\,b_{\mathfrak{p},\mathfrak{q}}^{k,h}(u;\alpha,\beta) \tag{9}$$

where $\mathcal{P}_i\in R^3$ $(i=0,1,\cdots,h)$, $\mathfrak{p}>0$ and $\mathfrak{q}>0$. $\mathcal{P}_i$ are control points. Joining up adjacent points $\mathcal{P}_i$, $i=0,1,2,\cdots,h$ to obtain a polygon which is called the control polygon of Lupaş post quantum Bézier curves over $[\alpha,\beta]$.

4.1. Properties

Theorem 4.1. *Some basic properties from the definition of Lupaş post quantum Bézier curves over the interval* $[\alpha,\beta]$ *are as follows:*
1. Lupaş post quantum Bézier curves on the interval $[\alpha,\beta]$ *have geometric and affine invariance.*
2. Lupaş post quantum Bèzier on the interval $[\alpha,\beta]$ *lie inside the convex hull of its control polygon.*
3. $\mathcal{P}(\alpha;\mathfrak{p},\mathfrak{q})=\mathcal{P}_{\mathbf{0}}$, $\mathcal{P}(\beta;\mathfrak{p},\mathfrak{q})=\mathcal{P}_{\mathbf{h}}$ *(End-point interpolation property).*
4. The Lupaş post quantum Bèzier on the interval $[\alpha,\beta]$ *obtained by reversing the order of the control points is the same as the Lupaş post quantum Bézier curves with* $\mathfrak{q}$ *replaced by* $\frac{1}{\mathfrak{q}}$ *and* $\mathfrak{p}$ *replaced by* $\frac{1}{\mathfrak{p}}$ *(Post quantum inverse symmetry).*
5. If $\mathfrak{p}=1$, $\alpha=0$, $\beta=1$, *then (9) reduces to the Lupaş* $\mathfrak{q}$*-Bèzier curves.*

Proof: Above properties of Lupaş post quantum Bézier curves over the interval $[\alpha,\beta]$ can be obtained from corresponding properties of the Lupaş post quantum analogue of the Bernstein basis functions over

the interval $[\alpha,\beta]$. Here we only discuss the proof of property 4.
Let $\mathcal{P}^*_{\mathbf{i}} = \mathcal{P}_{\mathbf{h-i}}, \quad i = 0, 1, \cdots, h$, then we have

$$\begin{aligned}\mathcal{P}^*(u;\mathfrak{p},\mathfrak{q}) &= \sum_{k=0}^{h} \mathcal{P}^*_{\mathbf{i}}\, b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u;\alpha,\beta)\\ &= \sum_{k=0}^{h} \mathcal{P}^*_{\mathbf{i}}\, b^{k,h}_{\frac{1}{\mathfrak{p}},\frac{1}{\mathfrak{q}}}(\alpha+\beta-u)\\ &= \mathcal{P}(\alpha+\beta-u;\frac{1}{\mathfrak{p}},\frac{1}{\mathfrak{q}}).\end{aligned}$$

Theorem 4.2. *The end-point property of derivative:*

$$\mathcal{P}'(\alpha;\mathfrak{p},\mathfrak{q}) = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{h-1}}(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_{\mathbf{0}})$$

$$\mathcal{P}'(\beta;\mathfrak{p},\mathfrak{q}) = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{q}^{h-1}}(\mathcal{P}_{\mathbf{h}} - \mathcal{P}_{\mathbf{h-1}})$$

i.e., Lupaş post quantum-Bézier over the interval $[\alpha,\beta]$ *are tangent to fore-and-aft edges of its control polygon at end points.*

Proof: Let

$$\mathcal{P}(u;\mathfrak{p},\mathfrak{q}) = \sum_{k=0}^{h} \mathcal{P}_{\mathbf{k}}\, b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u) = \frac{\sum\limits_{k=0}^{h} \mathcal{P}_{\mathbf{k}} \left[\begin{array}{c} h \\ k \end{array}\right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} (u-\alpha)^k (\beta-u)^{h-k}}{\prod\limits_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u) + \mathfrak{q}^{j-1}(u-\alpha)\}} \tag{10}$$

$$= \frac{\mathbf{V}(u;\mathfrak{p},\mathfrak{q})}{\mathbf{W}(u;\mathfrak{p},\mathfrak{q})} \tag{11}$$

or

$$\mathcal{P}(u;\mathfrak{p},\mathfrak{q})\ \mathbf{W}(u;\mathfrak{p},\mathfrak{q}) = \mathbf{V}(u;\mathfrak{p},\mathfrak{q}).$$

On differentiating both hand side with respect to 'u', we have

$$\mathcal{P}'(u;\mathfrak{p},\mathfrak{q})\ \mathbf{W}(u;\mathfrak{p},\mathfrak{q}) + \mathcal{P}(u;\mathfrak{p},\mathfrak{q})\ \mathbf{W}'(u;\mathfrak{p},\mathfrak{q}) = \mathbf{V}'(u;\mathfrak{p},\mathfrak{q}).$$

Let

$$A^{h,\alpha,\beta}_k(u;\mathfrak{p},\mathfrak{q}) = \left[\begin{array}{c} h \\ k \end{array}\right]_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} (u-\alpha)^k (b-u)^{h-k},$$

then

$$\mathbf{V}(u;\mathfrak{p},\mathfrak{q}) = \sum_{k=0}^{h} \mathcal{P}_{\mathbf{k}} A^{h,\alpha,\beta}_k(u;\mathfrak{p},\mathfrak{q})$$

From property 2 of the Lupaş post quantum Bernstein functions, we have

$$\mathbf{W}(u;\mathfrak{p},\mathfrak{q}) = \sum_{k=0}^{h} A_k^{h,\alpha,\beta}(u;\mathfrak{p},\mathfrak{q})$$

as

$$\begin{aligned}
(A_k^{h,\alpha,\beta}(u;\mathfrak{p},\mathfrak{q}))' &= \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[k]_{\mathfrak{p},\mathfrak{q}}} \begin{bmatrix} h-1 \\ k-1 \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}}\, k\,(u-\alpha)^{k-1}\,(\beta-u)^{h-k} \\
&- \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[h-k]_{\mathfrak{p},\mathfrak{q}}} \begin{bmatrix} h-1 \\ k \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}}\,(h-k)\,(u-\alpha)^{k}\,(\beta-u)^{h-k-1} \\
&= \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[k]_{\mathfrak{p},\mathfrak{q}}}\, \mathfrak{q}^{k-1} k\, A_{k-1}^{h-1}(u;\mathfrak{p},\mathfrak{q}) - \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[h-k]_{\mathfrak{p},\mathfrak{q}}}\, \mathfrak{p}^{h-k-1}(h-k)\, A_k^{h-1}(u;\mathfrak{p},\mathfrak{q}) \\
&= C_k^h\, A_{k-1}^{h-1,\alpha,\beta}(u;\mathfrak{p},\mathfrak{q}) - D_{h-k}^h\, A_k^{h-1,\alpha,\beta}(u;\mathfrak{p},\mathfrak{q})
\end{aligned}$$

where

$$C_k^h = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[k]_{\mathfrak{p},\mathfrak{q}}}\, \mathfrak{q}^{k-1} k, \quad D_{h-k}^h = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{[h-k]_{\mathfrak{p},\mathfrak{q}}}\, \mathfrak{p}^{h-k-1}(h-k).$$

Then

$$\mathbf{V}(\alpha;\mathfrak{p},\mathfrak{q}) = \mathcal{P}_{\mathbf{0}}\, \mathfrak{p}^{\frac{\mathbf{h(h-1)}}{2}}, \quad \mathbf{W}(\alpha;\mathfrak{p},\mathfrak{q}) = \mathfrak{p}^{\frac{h(h-1)}{2}}$$

$$\mathbf{V}'(\alpha;\mathfrak{p},\mathfrak{q}) = (C_1^h\, \mathcal{P}_{\mathbf{1}} - D_h^h\, \mathcal{P}_{\mathbf{0}})\, \mathfrak{p}^{\frac{(h-1)(h-2)}{2}}$$

$$\mathbf{W}'(\alpha;\mathfrak{p},\mathfrak{q}) = (C_1^h - D_h^h)\, \mathfrak{p}^{\frac{(h-1)(h-2)}{2}}$$

hence

$$\mathcal{P}'(\alpha;\mathfrak{p},\mathfrak{q}) = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{p}^{h-1}}(\mathcal{P}_{\mathbf{1}} - \mathcal{P}_{\mathbf{0}}).$$

By similar computation, we have

$$\mathbf{V}(\beta;\mathfrak{p},\mathfrak{q}) = \mathcal{P}_{\mathbf{h}}\, \mathfrak{q}^{\frac{\mathbf{h(h-1)}}{2}}, \quad \mathbf{W}(\beta;\mathfrak{p},\mathfrak{q}) = \mathfrak{q}^{\frac{h(h-1)}{2}}$$

$$\mathbf{V}'(\beta;\mathfrak{p},\mathfrak{q}) = (C_h^h\, \mathcal{P}_{\mathbf{h}} - D_1^h\, \mathcal{P}_{\mathbf{h-1}})\, \mathfrak{q}^{\frac{(h-1)(h-2)}{2}}$$

$$\mathbf{W}'(\beta;\mathfrak{p},\mathfrak{q}) = (C_h^h - D_1^h)\, \mathfrak{q}^{\frac{(h-1)(h-2)}{2}}$$

hence

$$\mathcal{P}'(\beta;\mathfrak{p},\mathfrak{q}) = \frac{[h]_{\mathfrak{p},\mathfrak{q}}}{\mathfrak{q}^{h-1}}(\mathcal{P}_{\mathbf{h}} - \mathcal{P}_{\mathbf{h-1}}).$$

4.2. Degree elevation for post quantum Lupaş Bézier curves over the interval $[\alpha, \beta]$

Using the technique of degree elevation for Post quantum Lupaş Bézier curves over the interval $[\alpha, \beta]$, one can attain more control over the shape of a given curve and the parameters will provide the flexibility.

$$\mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) = \sum_{k=0}^{h} \mathcal{P}_{\mathbf{k}}\, b^{k,h}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta)$$

$$\mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) = \sum_{k=0}^{h+1} \mathcal{P}^{*}_{\mathbf{k}}\, b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta),$$

where

$$\mathcal{P}^{*}_{\mathbf{k}} = \left(1 - \frac{\mathfrak{p}^{k}\, [h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}_{\mathbf{k-1}} + \left(\frac{\mathfrak{p}^{k}\, [h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}_{\mathbf{k}} \;\; for\ \mathfrak{q} \leq \mathfrak{p}. \tag{12}$$

Using the identities (4) and (5), the above statements can easily derived. Consider

$$\mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) = \frac{\mathfrak{p}^{h}(\beta - u)}{\mathfrak{p}^{h}(\beta - u) + \mathfrak{q}^{h}\,(u - \alpha)}\, \mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) + \frac{\mathfrak{q}^{h}\,(u - \alpha)}{\mathfrak{p}^{h}(\beta - u) + \mathfrak{q}^{h}\,(u - \alpha)}\, \mathcal{P}(u; \mathfrak{p}, \mathfrak{q}),$$

we obtain

$$\mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) = \sum_{k=0}^{h} \left(\mathfrak{p}^{k}\;\; \frac{[h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}^{\mathbf{0}}_{\mathbf{k}} b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) + \sum_{k=0}^{h} \left(1 - \frac{\mathfrak{p}^{k+1}\, [h-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}^{\mathbf{0}}_{\mathbf{k}} b^{k+1,h+1}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta).$$

Now by shifting the limits, we have

$$\mathcal{P}(u; \mathfrak{p}, \mathfrak{q}) = \sum_{k=0}^{h+1} \left(\mathfrak{p}^{k}\;\; \frac{[h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}^{\mathbf{0}}_{\mathbf{k}} b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta) + \sum_{k=0}^{h+1} \left(1 - \frac{\mathfrak{p}^{k}\, [h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}^{\mathbf{0}}_{\mathbf{k-1}} b^{k,h+1}_{\mathfrak{p},\mathfrak{q}}(u; \alpha, \beta),$$

where the zero vector is denoted by $\mathcal{P}^{\mathbf{0}}_{\mathbf{-1}}$. Comparing coefficients on both side, we have

$$\mathcal{P}^{*}_{\mathbf{k}} = \left(1 - \frac{\mathfrak{p}^{k}\, [h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}_{\mathbf{k-1}} + \left(\frac{\mathfrak{p}^{k}\, [h+1-k]_{\mathfrak{p},\mathfrak{q}}}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\right) \mathcal{P}_{\mathbf{k}},$$

where $k = 0, 1, 2, \cdots, h+1$ and $\mathcal{P}_{\mathbf{-1}} = \mathcal{P}_{\mathbf{h+1}} = 0$.
When $\mathfrak{p} = 1$, $\alpha = 0$, *and* $\beta = 1$ Formula (12) reduces to the degree evaluation formula of the Lupaş $\mathfrak{q}$-Bèzier curves. If $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_h)^T$ denotes the vector of control points of the initial post quantum Lupaş Bèzier curve of degree h over the interval $[\alpha, \beta]$; and $\mathcal{P}^{(\mathbf{1})} = (\mathcal{P}^{*}_{0}, \mathcal{P}^{*}_{1}, \cdots, \mathcal{P}^{*}_{h+1})$ be the vector of control points of the degree elevated post quantum Lupaş Bèzier curve of degree $h+1$ over the interval $[\alpha, \beta]$, then we can represent the degree elevation procedure as follows:

$$\mathcal{P}^{(\mathbf{1})} = T_{h+1}\mathcal{P},$$

where

$$T_{h+1} = \frac{1}{[h+1]_{\mathfrak{p},\mathfrak{q}}}\begin{bmatrix} [h+1]_{\mathfrak{p},\mathfrak{q}} & 0 & \dots & 0 & 0 \\ [h+1]_{\mathfrak{p},\mathfrak{q}}-\mathfrak{p}[h]_{\mathfrak{p},\mathfrak{q}} & \mathfrak{p}[h]_{\mathfrak{p},\mathfrak{q}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & [h+1]_{\mathfrak{p},\mathfrak{q}}-\mathfrak{p}^{h-1}[2]_{\mathfrak{p},\mathfrak{q}} & \mathfrak{p}^{h-1}[2]_{\mathfrak{p},\mathfrak{q}} & 0 \\ 0 & 0 & \dots & [h+1]_{\mathfrak{p},\mathfrak{q}}-\mathfrak{p}^{h}[1]_{\mathfrak{p},\mathfrak{q}} & \mathfrak{p}^{h}[1]_{\mathfrak{p},\mathfrak{q}} \\ 0 & 0 & \dots & 0 & [h+1]_{\mathfrak{p},\mathfrak{q}} \end{bmatrix}_{(h+2)\times(h+1)}$$

For any $l \in \mathbb{N}$, the vector of control points of the degree elevated post quantum Lupaş Bézier curves of degree $h+l$ over the interval $[\alpha,\beta]$ is: $\mathcal{P}^{(l)} = T_{h+l}\ \cdots\ T_{h+2}\ T_{h+1}\mathcal{P}$. As $l \longrightarrow \infty$, the control polygon $\mathcal{P}^{(\mathbf{l})}$ converges to a post quantum Lupaş Bézier curve over the interval $[\alpha,\beta]$.

4.3. Post quantum de Casteljau algorithm for Lupaş Bézier curves over $[\alpha,\beta]$

We can get the two selectable algorithms to evaluate post quantum Lupaş Bézier curves over the interval $[\alpha,\beta]$. The algorithms are as follows:

Algorithm 1.

$$\begin{cases} \mathcal{P}^{\mathbf{0}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) \equiv \mathcal{P}^{\mathbf{0}}_{\mathbf{i}} \equiv \mathcal{P}_{\mathbf{i}} \quad i=0,1,2\cdots,h \\ \mathcal{P}^{\mathbf{r}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) = \frac{q^{h-r}\ (u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)}\ \mathbf{P}^{\mathbf{r-1}}_{\mathbf{i+1}}(u;\mathfrak{p},\mathfrak{q}) + \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)}\ \mathcal{P}^{\mathbf{r-1}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) \\ r=1,\cdots,h, \quad i=0,1,2\cdots,h-r., \end{cases} \tag{13}$$

or

$$\begin{cases} \mathcal{P}^{\mathbf{0}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) \equiv \mathcal{P}^{\mathbf{0}}_{\mathbf{i}} \equiv \mathcal{P}_{\mathbf{i}} \quad i=0,1,2\cdots,h \\ \mathcal{P}^{\mathbf{r}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) = \frac{\mathfrak{p}^{h-i-r}\mathfrak{q}^{i}\ (u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)}\ \mathcal{P}^{\mathbf{r-1}}_{\mathbf{i+1}}(u;\mathfrak{p},\mathfrak{q}) + \frac{\mathfrak{p}^{h-i-r}\ \mathfrak{q}^{i}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)}\ \mathcal{P}^{\mathbf{r-1}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) \\ r=1,\cdots,h, \quad i=0,1,2\cdots,h-r., \end{cases} \tag{14}$$

Then

$$\mathcal{P}(u;\mathfrak{p},\mathfrak{q}) = \sum_{i=0}^{h-1}\mathcal{P}^{\mathbf{1}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q}) = \ldots = \sum \mathcal{P}^{\mathbf{r}}_{\mathbf{i}}(u;\mathfrak{p},\mathfrak{q})\ b^{i,h-r}_{\mathfrak{p},\mathfrak{q}}(u) = \ldots = \mathcal{P}^{\mathbf{h}}_{\mathbf{0}}\ (u;\mathfrak{p},\mathfrak{q}) \tag{15}$$

Obvious results can be derived from Theorem 3.2. When $\mathfrak{p}=1$, formula (13) and (14) recover the de Casteljau algorithms of Lupaş $\mathfrak{q}$-Bézier curves. Let $\mathcal{P}^0 = (\mathcal{P}_0,\mathcal{P}_1,\cdots,\mathcal{P}_h)^T$ and $\mathcal{P}^r = (\mathcal{P}^r_0,\mathcal{P}^r_1,\cdots,\mathcal{P}^r_{h-r})^T$ then **Algorithm 2.**

$$\mathcal{P}^{\mathbf{r}}(u;\mathfrak{p},\mathfrak{q}) = M^{\alpha,\beta}_r(u;\mathfrak{p},\mathfrak{q})\cdots M^{\alpha,\beta}_2(u;\mathfrak{p},\mathfrak{q})M^{\alpha,\beta}_1(u;\mathfrak{p},\mathfrak{q})\ \mathcal{P}^{\mathbf{0}} \tag{16}$$

where $M^{\alpha,\beta}_r(u;\mathfrak{p},\mathfrak{q})$ is a $(h-r+1)\times(h-r+2)$ matrix and

$$M_r^{\alpha,\beta}(u;\mathfrak{p},\mathfrak{q}) =$$

$$\begin{bmatrix} \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{q}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \cdots & 0 & 0 \\ 0 & \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{q}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{q}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & 0 \\ 0 & 0 & \cdots & \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{q}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} \end{bmatrix}$$

or

$$M_r^{\alpha,\beta}(u;\mathfrak{p},\mathfrak{q}) =$$

$$\begin{bmatrix} \frac{\mathfrak{p}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{p}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \cdots & 0 & 0 \\ 0 & \frac{\mathfrak{p}^{h-r-1}\ \mathfrak{q}(\beta-u)}{(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{p}^{h-r-1}\ \mathfrak{q}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\mathfrak{p}\ \mathfrak{q}^{h-r-1}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{p}\mathfrak{q}^{h-r-1}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & 0 \\ 0 & 0 & \cdots & \frac{\mathfrak{q}^{h-r}(\beta-u)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} & \frac{\mathfrak{q}^{h-r}(u-\alpha)}{\mathfrak{p}^{h-r}(\beta-u)+\mathfrak{q}^{h-r}(u-\alpha)} \end{bmatrix}$$

5. Tensor product post quantum Lupaş Bézier surfaces on $[\alpha,\beta]\times[\alpha,\beta]$

We define

$$\mathcal{P}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{h} \mathcal{P}_{i,j}\ b^{i,m}_{\mathfrak{p}_1,\mathfrak{q}_1}(u;\alpha,\beta)\ \ b^{j,h}_{\mathfrak{p}_2,\mathfrak{q}_2}(v;\alpha,\beta), \quad (u,v)\in[\alpha,\beta]\times[\alpha,\beta], \tag{17}$$

a two-parameter family $\mathcal{P}(u,v)$ of tensor product surfaces of degree $m\times h$, where $\mathcal{P}_{i,j}\in\mathbb{R}^3$ $(i=0,1,\cdots,m,\ j=0,1,\cdots,h)$, where $b^{i,m}_{\mathfrak{p}_1,\mathfrak{q}_1}(u)$, $b^{j,h}_{\mathfrak{p}_2,\mathfrak{q}_2}(v)$ are Lupaş post quantum-analogue of Bernstein functions with the parameters $\mathfrak{p}_1,\mathfrak{q}_1$ and $\mathfrak{p}_2,\mathfrak{q}_2$, respectively. We call the parameter surface tensor product as the Lupaş post quantum-Bèzier surface with degree $m\times h$. Here $\mathcal{P}_{i,j}$ denotes the control points.

5.1. Properties

1. **Affine invariance and geometric property:** Since

$$\sum_{i=0}^{m}\sum_{j=0}^{h} b^{i,m}_{\mathfrak{p}_1,\mathfrak{q}_1}(u;\alpha,\beta)\ \ b^{j,h}_{\mathfrak{p}_2,\mathfrak{q}_2}(v;\alpha,\beta) = 1, \tag{18}$$

$\mathcal{P}(u,v)$ is an affine combination of its control points.
2. **Convex hull property:** $\mathcal{P}(u,v)$ represents convex combination of $\mathcal{P}_{i,j}$ which lies in the convex hull of its control net.
3. **Isoparametric curves property:** The iso-parametric curves $v=v^*$ and $u=u^*$ of a tensor product post quantum Lupaş Bézier surface on $[\alpha,\beta]\times[\alpha,\beta]$ are the post quantum Lupaş Bèzier curves of degree m and degree h over the interval $[\alpha,\beta]$, respectively. Namely

$$\mathcal{P}(u,v^*) = \sum_{i=0}^{m}\Big(\sum_{j=0}^{h}\mathcal{P}_{i,j}\ b^{j,h}_{\mathfrak{p}_2,\mathfrak{q}_2}(v^*;\alpha,\beta)\Big)\, b^{i,m}_{\mathfrak{p}_1,\mathfrak{q}_1}(u;\alpha,\beta), \quad u\in[\alpha,\beta];$$

$$\mathcal{P}(u^*, v) = \sum_{j=0}^{h} \left(\sum_{i=0}^{m} \mathcal{P}_{i,j}\, b^{j,h}_{\mathfrak{p}_1,\mathfrak{q}_1}(u^*;\alpha,\beta) \right) b^{i,m}_{\mathfrak{p}_2,\mathfrak{q}_2}(v;\alpha,\beta), \quad v \in [\alpha,\beta],$$

$\mathcal{P}(u,\alpha), \mathcal{P}(u,\beta), \mathcal{P}(\alpha,v)$ and $\mathcal{P}(\beta,v)$ denote the boundary curves of $\mathcal{P}(u,v)$.
4. **Interpolation property at corner point:** The corner control net coincides with the four corners of the surface. Namely, $\mathcal{P}(\alpha,\alpha) = \mathcal{P}_{0,0}, \mathcal{P}(\alpha,\beta) = \mathcal{P}_{0,h}, \mathcal{P}(\beta,\alpha) = \mathcal{P}_{m,0}, \mathcal{P}(\beta,\beta) = \mathcal{P}_{m,h}$.
5. **Reducibility:** For $\mathfrak{p}_1 = \mathfrak{p}_2 = 1$, $\alpha = 0$, and $\beta = 1$, the formula (17) reduces to a tensor product Lupaş $\mathfrak{q}$-Bézier patch.

5.2. Degree elevation and post quantum de Casteljau algorithm for Lupaş Bézier surface on $[\alpha,\beta] \times [\alpha,\beta]$

Let $\mathcal{P}(u,v)$ be a tensor product post quantum Lupaş Bézier surface of degree $m \times h$ on $[\alpha,\beta] \times [\alpha,\beta]$.

$$\mathcal{P}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{h} \mathcal{P}_{i,j}\, b^{i,m}_{\mathfrak{p}_1,\mathfrak{q}_1}(u;\alpha,\beta)\ \ b^{j,h}_{\mathfrak{p}_2,\mathfrak{q}_2}(v;\alpha,\beta) = \sum_{i=0}^{m+1}\sum_{j=0}^{h+1} \mathcal{P}^*{}_{\mathbf{i},\mathbf{j}}\, b^{i,m+1}_{\mathfrak{p}_1,\mathfrak{q}_1}(u;\alpha,\beta)\ \ b^{j,h+1}_{\mathfrak{p}_2,\mathfrak{q}_2}(v;\alpha,\beta). \tag{19}$$

Let $\alpha_i = 1 - \frac{\mathfrak{p}_1^i\ [m+1-i]_{\mathfrak{p}_1,\mathfrak{q}_1}}{[m+1]_{\mathfrak{p}_1,\mathfrak{q}_1}}, \quad \beta_j = 1 - \frac{\mathfrak{p}_2^j\ [h+1-j]_{\mathfrak{p}_2,\mathfrak{q}_2}}{[h+1]_{\mathfrak{p}_2,\mathfrak{q}_2}}$. Then

$$\mathcal{P}^*_{i,j} = \alpha_i\, \beta_j\, \mathcal{P}_{i-1,j-1} + \alpha_i\, (1-\beta_j)\, \mathcal{P}_{i-1,j} + (1-\alpha_i)\, \beta_j\, \mathcal{P}_{i,j-1} + (1-\alpha_i)\, (1-\beta_j)\, \mathcal{P}_{i,j} \tag{20}$$

and its matrix form is

$$\begin{bmatrix} 1 - \frac{\mathfrak{p}_1^i\ [m+1-i]_{\mathfrak{p}_1,\mathfrak{q}_1}}{[m+1]_{\mathfrak{p}_1,\mathfrak{q}_1}} & \frac{\mathfrak{p}_1^i\ [m+1-i]_{\mathfrak{p}_1,\mathfrak{q}_1}}{[m+1]_{\mathfrak{p}_1,\mathfrak{q}_1}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_{i-1,j-1} & \mathcal{P}_{i-1,j} \\ \mathcal{P}_{i,j-1} & \mathcal{P}_{i,j} \end{bmatrix} \begin{bmatrix} 1 - \frac{\mathfrak{p}_2^j\ [h+1-j]_{\mathfrak{p}_2,\mathfrak{q}_2}}{[h+1]_{\mathfrak{p}_2,\mathfrak{q}_2}} \\ \frac{\mathfrak{p}_2^j\ [h+1-j]_{\mathfrak{p}_2,\mathfrak{q}_2}}{[h+1]_{\mathfrak{p}_2,\mathfrak{q}_2}} \end{bmatrix}.$$

The de Casteljau algorithms can be extended to evaluate points on a post quantum Lupaş Bézier surface over $[\alpha,\ \beta]$. Given the control net $\mathcal{P}_{i,j} \in \mathbb{R}^3, i = 0,1,\cdots,m,\ \ j = 0,1,\cdots,h$.

$$\begin{cases} \mathcal{P}^{\mathbf{0,0}}_{\mathbf{i,j}}(u,v;\alpha,\beta) \equiv \mathcal{P}^{\mathbf{0,0}}_{\mathbf{i,j}} \equiv \mathcal{P}_{\mathbf{i,j}} \quad i = 0,1,2\cdots,m;\ \ j = 0,1,2\cdots h. \\ \mathcal{P}^{\mathbf{r,r}}_{\mathbf{i,j}}(u,v;\alpha,\beta) = \begin{bmatrix} \frac{\mathfrak{p}_1^{m-r}(1-u)}{\mathfrak{p}_1^{m-r}(\beta-u)+\mathfrak{q}_1^{m-r}(u-\alpha)} & \frac{\mathfrak{q}_1^{m-r}\ (u-\alpha)}{\mathfrak{p}_1^{m-r}(\beta-u)+\mathfrak{q}_1^{m-r}(u-\alpha)} \end{bmatrix} \begin{bmatrix} \mathcal{P}^{r-1,r-1}_{i,j} & \mathcal{P}^{r-1,r-1}_{i,j+1} \\ \mathcal{P}^{r-1,r-1}_{i+1,j} & \mathcal{P}^{r-1,r-1}_{i+1,j+1} \end{bmatrix} \begin{bmatrix} \frac{\mathfrak{p}_2^{h-r}(\beta-v)}{\mathfrak{p}_2^{m-r}(\beta-v)+\mathfrak{q}_2^{h-r}(v-\alpha)} \\ \frac{\mathfrak{q}_2^{h-r}\ (v-\alpha)}{\mathfrak{p}_2^{h-r}(b-v)+\mathfrak{q}_2^{h-r}(v-\alpha)} \end{bmatrix} \\ r = 1,\cdots,k,\ k = \min(m,h) \quad i = 0,1,2\cdots,m-r;\ \ j = 0,1,\cdots h-r \end{cases} \tag{21}$$

or

$$\begin{cases} \mathcal{P}^{\mathbf{0,0}}_{\mathbf{i,j}}(u,v;\alpha,\beta) \equiv \mathcal{P}^{\mathbf{0,0}}_{\mathbf{i,j}} \equiv \mathcal{P}_{\mathbf{i,j}} \quad i = 0,1,2\cdots,m;\ \ j = 0,1,2\cdots h. \\ \mathcal{P}^{\mathbf{r,r}}_{\mathbf{i,j}}(u,v;\alpha,\beta) = \begin{bmatrix} \frac{\mathfrak{p}_1^{m-i-r}\mathfrak{q}_1^i(\beta-u)}{\mathfrak{p}_1^{m-r}(\beta-u)+\mathfrak{q}_1^{m-r}(u-\alpha)} & \frac{\mathfrak{p}_1^{m-i-r}\mathfrak{q}_1^i\ (u-\alpha)}{\mathfrak{p}_1^{m-r}(\beta-u)+\mathfrak{q}_1^{m-r}(u-\alpha)} \end{bmatrix} \begin{bmatrix} \mathcal{P}^{r-1,r-1}_{i,j} & \mathcal{P}^{r-1,r-1}_{i,j+1} \\ \mathcal{P}^{r-1,r-1}_{i+1,j} & \mathcal{P}^{r-1,r-1}_{i+1,j+1} \end{bmatrix} \begin{bmatrix} \frac{\mathfrak{p}_2^{h-j-r}\mathfrak{q}_2^j(\beta-v)}{\mathfrak{p}_2^{h-r}(\beta-v)+\mathfrak{q}_2^{h-r}(v-\alpha)} \\ \frac{\mathfrak{p}_2^{h-j-r}\mathfrak{q}_2^j\ (v-\alpha)}{\mathfrak{p}_2^{h-r}(\beta-v)+\mathfrak{q}_2^{h-r}(v-\alpha)} \end{bmatrix} \\ r = 1,\cdots,k,\ k = \min(m,h) \quad i = 0,1,2\cdots,m-r;\ \ j = 0,1,\cdots h-r \end{cases} \tag{22}$$

For $m = h$, one can use the algorithms above directly to evaluate a point on the surface. When $m \neq h$, to evaluate a point on the surface after k applications of formula (21) or (22), we perform formula (16) for the intermediate point $\mathcal{P}^{k,k}_{i,j}$.
Note: We get Lupaş $\mathfrak{q}$-Bézier curves and surfaces for $(u,v) \in [\alpha,\beta] \times [\alpha,\beta]$ when we set the parameters $\mathfrak{p}_1 = \mathfrak{p}_2 = 1$, $\alpha = 0$, and $\beta = 1$ as proved in [9].

6. Some observations and concluding remarks

6.1. Post quantum analogue of Lupaş operators over $[\alpha, \beta]$

In this section, we present post quantum analogue of Lupaş Bernstein operators over $[\alpha, \beta]$ as follows:

For any $\mathfrak{p} > 0$ and $\mathfrak{q} > 0$, the linear operators $L^{h,\alpha,\beta}_{\mathfrak{p},\mathfrak{q}} : C[\alpha, \beta] \to C[\alpha, \beta]$

$$L^{h,\alpha,\beta}_{\mathfrak{p},\mathfrak{q}}(f; u) = \sum_{k=0}^{h} \frac{f\left(\alpha + \frac{(\beta-\alpha)\mathfrak{p}^{h-k}\,[k]_{\mathfrak{p},\mathfrak{q}}}{[h]_{\mathfrak{p},\mathfrak{q}}}\right) \begin{bmatrix} h \\ k \end{bmatrix}_{\mathfrak{p},\mathfrak{q}} \mathfrak{p}^{\frac{(h-k)(h-k-1)}{2}} \mathfrak{q}^{\frac{k(k-1)}{2}} (u-\alpha)^k (\beta-u)^{h-k}}{\prod_{j=1}^{h}\{\mathfrak{p}^{j-1}(\beta-u) + \mathfrak{q}^{j-1}(u-\alpha)\}}, \tag{23}$$

is post quantum analogue of Lupaş Bernstein operators on $[\alpha, \beta]$.

Again when $\mathfrak{p} = 1$, $\alpha = 0$ *and* $\beta = 1$, the post quantum Lupaş Bernstein operators turn out to be Lupaş $\mathfrak{q}$-Bernstein operators as given in [31].

When $\mathfrak{p} = \mathfrak{q} = 1$, $\alpha = 0$ *and* $\beta = 1$, the post quantum Lupaş Bernstein operators on $[\alpha, \beta]$ turn out to be classical Bernstein operators.

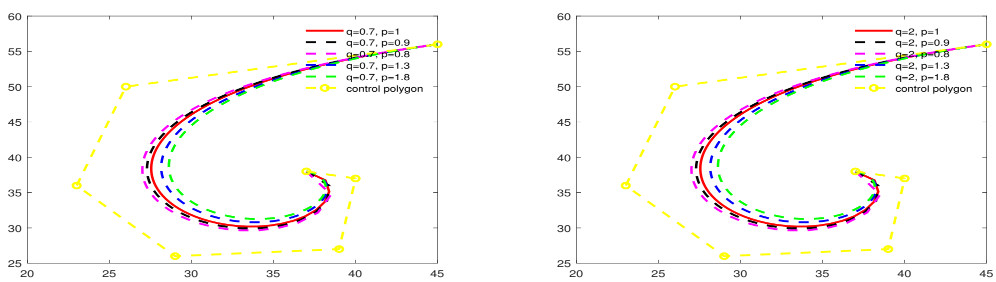
From the definition of the operators $L^{h,\alpha,\beta}_{\mathfrak{p},\mathfrak{q}}(f, u)$, it is clear that they posses the end point interpolation property, that is

$$L^{h,\alpha,\beta}_{\mathfrak{p},\mathfrak{q}}(f, \alpha) = f(\alpha), \quad L^{h,\alpha,\beta}_{\mathfrak{p},\mathfrak{q}}(f, \beta) = f(\beta)$$

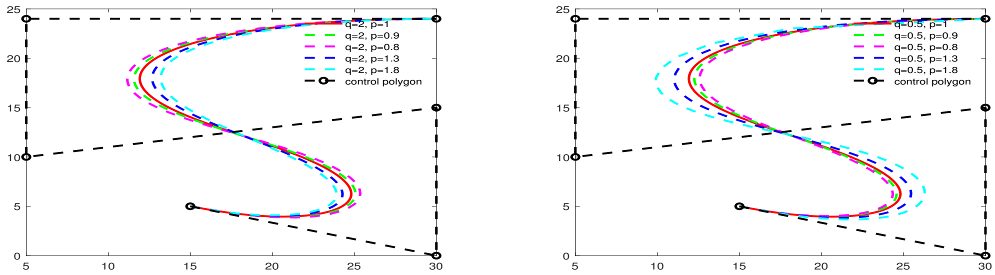
for all $\mathfrak{p} > 0$ and $\mathfrak{q} > 0$, and all $h = 1, 2, \cdots$.

6.2. Shape control of post quantum Lupaş Bézier curves on $[\alpha, \beta]$

We have constructed post quantum Lupaş type Bernstein functions and curves over $[\alpha, \beta]$ which holds the end point interpolation property. It is clear from the figures that generated curve will be within the convex hull of the control net (control polygon) for different values of $\mathfrak{p}$ and $\mathfrak{q}$. For a given $\mathfrak{q}$ such that $0 < \mathfrak{q} < 1$, if one chooses $\mathfrak{p} > 1$ then the curve will move towards the control net (control polygon) with further increase in the value of $\mathfrak{p}$. Similarly for same $\mathfrak{q}$, if one chooses $\mathfrak{p} < 1$ then the curve moves away from control net (control polygon) as the value of $\mathfrak{p}$ decreases. On the other hand, for $\mathfrak{q} > 1$ the effect of $\mathfrak{p}$ and $\mathfrak{q}$ will be opposite.



(a) (b)



(c) (d)

Figure 3: The effect of the shape of Lupaş post quantum-Bèzier curves on arbitrary intervals

6.3. *Importance of* $(\mathfrak{p},\mathfrak{q})$*-analogues*

A relationship between the post quantum integers $[h]_{\mathfrak{p},\mathfrak{q}}$ and quantum integers $[h]_{\mathfrak{q}}$ is $[h]_{\mathfrak{p},\mathfrak{q}} = \mathfrak{p}^{h-1}[h]_{\frac{\mathfrak{q}}{\mathfrak{p}}}$. But it is not true in general that the results for the $\mathfrak{q}$-analogues can trivially be translated into the corresponding results for the $(\mathfrak{p},\mathfrak{q})$-analogues $(0 < \mathfrak{p} < \mathfrak{q} \leq 1)$. Most of the $(\mathfrak{p},\mathfrak{q})$-analogues operators and their properties can not be obtained directly from $\mathfrak{q}$-analogues by simply substituting $[h]_{\mathfrak{q}} = \mathfrak{p}^{1-h}[h]_{\mathfrak{p},\mathfrak{p}\mathfrak{q}}$. On the other hand, $(\mathfrak{p},\mathfrak{q})$-analogues have some advantages over $\mathfrak{q}$-analogues, e.g. (i) for $\mathfrak{p} = 1$, the $(\mathfrak{p},\mathfrak{q})$-analogues are reduced directly to the respective $\mathfrak{q}$-analogues, (ii) the choice of $\mathfrak{p} > 1$ gives that the upper estimates of geometric order in Theorem 2.1 [26] hold in larger disks than those in the case when $\mathfrak{p} = 1$, (iii) for simulation purposes through computers and CAGD, this extra parameter $\mathfrak{p}$ has some advantages in modeling flexibility etc. [16, 17].

Conclusion: Post quantum-Lupaş-Bernstein operator as well as Bèzier curves and surfaces over arbitrary compact intervals constructed with the help of rational Lupaş-Bernstein basis functions are important from computational point of view. The extra parameters $\mathfrak{p}$ and $\mathfrak{q}$ provide more flexibility in approximation for simulation purposes. In this paper, we have extended the properties of rational Lupaş-Bernstein blending functions, Lupaş-Bézier curves and surfaces over arbitrary compact intervals $[\alpha,\beta]$ in the frame of post quantum-calculus. The de-Casteljau's algorithm based on post quantum-integers is derived. A two parameter family as Lupaş post quantum Bernstein functions over arbitrary compact intervals are constructed to establish their degree elevation and reduction properties. Some of their basic properties for Lupaş post quantum Bézier curves are studied. Some fundamental properties over arbitrary intervals for these curves as de Casteljau algorithm and degree evaluation properties are discussed. Post quantum-Lupaş-Bernstein operators over arbitrary compact intervals are constructed with the help of rational Lupaş-Bernstein functions.

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