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The η-Hermitian Solutions to Some Systems of Real Quaternion Matrix Equations

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Abstract. Let $\mathbb{H}^{m\times n}$ be the set of all $m \times n$ matrices over the real quaternion algebra. We call that $A \in \mathbb{H}^{n\times n}$ is η-Hermitian if $A = A^{\eta^*}$, where $A^{\eta^*} = -\eta A^*\eta$, $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternion units. In this paper, we derive some solvability conditions and the general solution to a system of real quaternion matrix equations. As an application, we present some necessary and sufficient conditions for the existence of an η-Hermitian solution to some systems of real quaternion matrix equations. We also give the expressions of the general η-Hermitian solutions to these systems when they are solvable. Some numerical examples are given to illustrate the results of this paper.

1. Introduction

Throughout, the set of all $m \times n$ matrices over the quaternion number field $\mathbb{H}$

$$\mathbb{H} = \left\{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \,\middle|\, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

by $\mathbb{H}^{m\times n}$. For a matrix A, A^* stands for the conjugate transpose of A. I denotes the identity matrix with appropriate sizes. The Moore-Penrose inverse $A^\dagger$ of A is defined to be the unique matrix $A^\dagger$, such that

$$\text{(i) } AA^\dagger A = A, \text{ (ii) } A^\dagger AA^\dagger = A^\dagger, \text{ (iii) } (AA^\dagger)^* = AA^\dagger, \text{ (iv) } (A^\dagger A)^* = A^\dagger A.$$

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by A, respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$. The symbol $r(A)$ stands for the rank of a given real quaternion matrix A. For a real quaternion matrix A, $r(A) = r(A^{\eta^*})$ ([4]). A quaternion matrix A is called an η-Hermitian matrix if $A = A^{\eta^*} = -\eta A^*\eta$, $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ ([22]).

Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton Nowadays quaternion matrices can be used in signal and color image processing, quantum physics, computer science, and so on (e.g. [1], [19]-[21], [27]). Many problems in systems and control theory can be reduced to solving systems of quaternion matrix equations (e.g. [6]-[16], [26]).

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The η-Hermitian matrices have some applications in widely linear modelling, convergence analysis in statistical signal processing ([21]). He and Wang ([4]) gave some solvability conditions and general solution to the real quaternion matrix equation involving η-Hermicity

$$A_1X + (A_1X)^{\eta*} + B_1YB_1^{\eta*} + C_1ZC_1^{\eta*} = D_1,$$

where Y and Z are η-Hermitian. Horn and Zhang ([17]) derived an analogous special singular value decomposition for η-Hermitian matrices. He and Wang ([2]) considered the η-Hermitian solution to a system of real quaternion matrix equations

$$\begin{cases} A_1X = C_1,\ XB_1 = D_1, \\ A_2Y = C_2,\ YB_2 = D_2, \\ C_3XC_3^{\eta*} + D_3YD_3^{\eta*} = A_3. \end{cases}$$

Very recently, He, Wang and Zhang ([5]) presented a simultaneous decomposition for a set of nine real quaternion matrices involving η-Hermicity: $A_i \in \mathbb{H}^{p_i\times t_i}, B_i \in \mathbb{H}^{p_i\times t_{i+1}}$, and $C_i \in \mathbb{H}^{p_i\times p_i}$, where C_i are η-Hermitian matrices, $(i = 1, 2, 3)$. The reference ([5]) gave some necessary and sufficient conditions for the existence of the general η-Hermitian solution to the system of coupled real quaternion matrix equations involving η-Hermicity

$$A_iX_iA_i^{\eta*} + B_iX_{i+1}B_i^{\eta*} = C_i,\ (i = 1, 2, 3),$$

where $A_i \in \mathbb{H}^{p_i\times t_i}, B_i \in \mathbb{H}^{p_i\times t_{i+1}}$, and $C_i \in \mathbb{H}^{p_i\times p_i}$, and C_i are η-Hermitian matrices.

Motivated by the work mentioned above and the recent increasing interests in η-Hermitian quaternion matrices and real quaternion matrix equations, we in this paper consider the η-Hermitian solution to the following system of real quaternion matrix equations

$$\begin{cases} A_1X = C_1, X = X^{\eta*}, \\ A_2XA_2^{\eta*} = C_2, \\ A_3XA_3^{\eta*} = C_3, \\ A_4XA_4^{\eta*} = C_4 \end{cases} \tag{1}$$

where $A_1, C_1, A_2, A_3, A_4, C_2 = C_2^{\eta*}, C_3 = C_3^{\eta*}, C_4 = C_4^{\eta*}$ be known over $\mathbb{H}$, and $X = X^{\eta*}$ be unknown. We aim to give some solvability conditions and general η-Hermitian solution to the system of real quaternion matrix equations (1). Observe that the following system of real quaternion matrix equations

$$\begin{cases} A_1X = C_1, \\ XB_1 = D_1, \\ A_2XB_2 = C_2, \\ A_3XB_3 = C_3, \\ A_4XB_4 = C_4 \end{cases} \tag{2}$$

plays an important role in investigating the η-Hermitian solution to (1). Another goal of this paper is to give some solvability conditions and the general solution to the system (2).

The remainder of the paper is organized as follows. In Section 2, we give some lemmas which are used in this paper. In Section 3, we present some necessary and sufficient conditions for the existence of a solution to the system of real quaternion matrix equations (2) and provide the general solution to system (2). In Section 4, we derive some solvability conditions and the general η-Hermitian solution to the system of real quaternion matrix equations (1).

2. Preliminaries

In this section, we review some lemmas which are used in this paper.

Lemma 2.1. *([23]) Let $A_1 \in \mathbb{H}^{m\times n}, B_1 \in \mathbb{H}^{r\times s}, C_1 \in \mathbb{H}^{m\times r}$, and $D_1 \in \mathbb{H}^{n\times s}$ be given and $X \in \mathbb{H}^{n\times r}$ be unknown. The the system of real quaternion matrix equations*

$$A_1X = C_1,\ XB_1 = D_1 \tag{3}$$

is consistent if and only if

$$R_{A_1}C_1 = 0,\ D_1L_{B_1} = 0,\ A_1D_1 = C_1B_1.$$

In this case, the general solution to (3) is

$$X = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}YR_{B_1},$$

where Y is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Lemma 2.2. *([3]) Let A_{ii}, B_{ii}, and C_{ii} $(i = 1, 2)$ be given with appropriate sizes. Set*

$$A = A_{22}L_{A_{11}},\ B = R_{B_{11}}B_{22},\ C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}B_{11}^{\dagger}B_{22},\ D = R_{A_{11}}A_{22}.$$

Then the system

$$A_{11}XB_{11} = C_{11},\ A_{22}XB_{22} = C_{22} \tag{4}$$

is consistent if and only if

$$R_ACL_B = 0, \quad R_{A_{ii}}C_{ii} = 0, \quad C_{ii}L_{B_{ii}} = 0,\ i = 1, 2.$$

In this case, the general solution of system (4) can be expressed as

$$\begin{aligned} X =& A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \end{aligned}$$

where U_1, U_2, U_3, and U_4 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 2.3. *([3], [25]) Let $A_1, B_1, C_3, D_3, C_4, D_4$, and E_1 be given. Set*

$$\begin{aligned} &A = R_{A_1}C_3, B = D_3L_{B_1}, C = R_{A_1}C_4, D = D_4L_{B_1}, \\ &E = R_{A_1}E_1L_{B_1}, M = R_AC, N = DL_B, S = CL_M. \end{aligned}$$

Then the real quaternion matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{5}$$

is consistent if and only if

$$R_MR_AE = 0, EL_BL_N = 0, R_AEL_D = 0, R_CEL_B = 0.$$

In this case, the general solution can be expressed as

$$\begin{aligned} X_1 =& A_1^{\dagger}(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^{\dagger}T_7B_1 + L_{A_1}T_6, \\ X_2 =& R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^{\dagger} + A_1A_1^{\dagger}T_7 + T_8R_{B_1}, \\ X_3 =& A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}ST_2R_NDB^{\dagger} + L_AT_4 + T_5R_B, \\ X_4 =& M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_D, \end{aligned}$$

where $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

The following lemma can be easily generalized to $\mathbb{H}$.

Lemma 2.4. *([18]) Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times k}, C \in \mathbb{H}^{l\times n}, D \in \mathbb{H}^{m\times p}, E \in \mathbb{H}^{q\times n}, Q \in \mathbb{H}^{m_1\times k}$, and $P \in \mathbb{H}^{l\times n_1}$ be given. Then*

(1) $r(A) + r(R_AB) = r(B) + r(R_BA) = r(A,\ B)$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{pmatrix} A \\ C \end{pmatrix}$.

3. Solvability conditions and general solution to the system (2)

In this section, we consider the system of real quaternion matrix equations (2). We derive solvability conditions and general solution to the system (2). Now we give the fundamental theorem of this section.

Theorem 3.1. *Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, A_3, B_3, C_3, A_4, B_4, C_4$ be known over $\mathbb{H}$, and X be unknown. Set*

$$A_{ii} = A_{i+1}L_{A_1}, B_{ii} = R_{B_1}B_{i+1}, C_{ii} = C_{i+1} - A_{i+1}(A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger})B_{i+1},\ (i = 1, 2, 3), \tag{6}$$

$$A = A_{22}L_{A_{11}}, B = R_{B_{11}}B_{22}, C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}B_{11}^{\dagger}B_{22}, D = R_{A_{11}}A_{22}, \tag{7}$$

$$A_5 = (L_{A_{11}}L_A,\ L_{A_{33}}), B_5 = \begin{pmatrix} R_BR_{B_{11}} \\ R_{B_{33}} \end{pmatrix}, \tag{8}$$

$$C_5 = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - A_{11}^{\dagger}C_{11}B_{11}^{\dagger} - L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} + L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} - D^{\dagger}R_ACB^{\dagger}R_{B_{11}}, \tag{9}$$

$$A_6 = R_{A_5}L_{A_{11}}, B_6 = R_{B_{22}}L_{B_5}, C_6 = R_{A_5}L_{A_{22}}, D_6 = R_{B_{11}}L_{B_5}, \tag{10}$$

$$E = R_{A_5}C_5L_{B_5}, M = R_{A_6}C_6, N = D_6L_{B_6}, S = C_6L_M. \tag{11}$$

Then the following statements are equivalent:
(1) *The system of real quaternion matrix equations (2) is consistent.*
(2)

$$R_{A_1}C_1 = 0, D_1L_{B_1} = 0, A_1D_1 = C_1B_1, \tag{12}$$

$$R_{A_{ii}}C_{ii} = 0, C_{ii}L_{B_{ii}} = 0,\ (i = 1, 2, 3),\ R_ACL_B = 0, \tag{13}$$

$$R_MR_{A_6}E = 0, EL_{B_6}L_N = 0, R_{A_6}EL_{D_6} = 0, R_{C_6}EL_{B_6} = 0. \tag{14}$$

(3)

$$r(A_1,\ C_1) = r(A_1),\ r\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = r(B_1), A_1D_1 = C_1B_1, \tag{15}$$

$$r\begin{pmatrix} C_{i+1} & A_{i+1} \\ C_1B_{i+1} & A_1 \end{pmatrix} = r\begin{pmatrix} A_{i+1} \\ A_1 \end{pmatrix}, r\begin{pmatrix} A_{i+1}D_1 & C_{i+1} \\ B_1 & B_{i+1} \end{pmatrix} = r(B_1,\ B_{i+1}),\ (i = 1, 2, 3), \tag{16}$$

$$r\begin{pmatrix} -C_2 & A_2 & 0 & 0 \\ B_2 & 0 & B_3 & B_1 \\ 0 & A_3 & C_3 & A_3D_1 \\ 0 & A_1 & C_1B_3 & C_1B_1 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + r(B_1,\ B_2,\ B_3), \tag{17}$$

$$r\begin{pmatrix} 0 & 0 & B_2 & B_3 & B_4 & B_1 \\ A_2 & A_2 & -C_2 & 0 & 0 & 0 \\ A_3 & 0 & 0 & C_3 & 0 & A_3D_1 \\ 0 & A_4 & 0 & 0 & C_4 & 0 \\ A_1 & 0 & -C_1B_2 & 0 & -C_1B_4 & 0 \\ 0 & A_1 & 0 & 0 & C_1B_4 & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_2 \\ A_3 & 0 \\ 0 & A_4 \\ A_1 & 0 \\ 0 & A_1 \end{pmatrix} + r(B_1,\ B_2,\ B_3,\ B_4), \tag{18}$$

$$r\begin{pmatrix} 0 & B_2 & B_3 & 0 & B_1 & 0 \\ 0 & B_2 & 0 & B_4 & 0 & B_1 \\ A_2 & -C_2 & 0 & 0 & 0 & -A_2D_1 \\ A_3 & 0 & C_3 & 0 & A_3D_1 & -A_3D_1 \\ A_4 & 0 & 0 & C_4 & 0 & 0 \\ A_1 & 0 & 0 & C_1B_4 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} B_2 & B_3 & 0 & B_1 & 0 \\ B_2 & 0 & B_4 & 0 & B_1 \end{pmatrix} + r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}, \tag{19}$$

$$r\begin{pmatrix} 0 & B_2 & B_4 & B_1 \\ A_2 & -C_2 & 0 & 0 \\ A_4 & 0 & C_4 & A_4D_1 \\ A_1 & -C_1B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \\ A_4 \end{pmatrix} + r(B_1,\ B_2,\ B_4), \tag{20}$$

$$r\begin{pmatrix} 0 & B_3 & B_4 & B_1 \\ A_3 & -C_3 & 0 & 0 \\ A_4 & 0 & C_4 & A_4D_1 \\ A_1 & -C_1B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_3 \\ A_4 \end{pmatrix} + r(B_1,\ B_3,\ B_4). \tag{21}$$

In this case, the general solution to system (2) can be expressed as

$$X = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}YR_{B_1}, \tag{22}$$

where

$$\begin{aligned} Y =& A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \end{aligned} \tag{23}$$

or

$$Y = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - L_{A_{33}}U_5 - U_6R_{B_{33}}, \tag{24}$$

$$\begin{pmatrix} U_1 \\ U_5 \end{pmatrix} = A_5^{\dagger}(C_5 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - A_5^{\dagger}T_7B_5 + L_{A_5}T_6, \tag{25}$$

$$(U_2,\ U_6) = R_{A_5}(C_5 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})B_5^{\dagger} + A_5A_5^{\dagger}T_7 + T_8R_{B_5}, \tag{26}$$

$$U_3 = A_6^{\dagger}EB_6^{\dagger} - A_6^{\dagger}C_6M^{\dagger}EB_6^{\dagger} - A_6^{\dagger}SC_6^{\dagger}EL_BN^{\dagger}D_6B_6^{\dagger} - A_6^{\dagger}ST_2R_ND_6B_6^{\dagger} + L_{A_6}T_4 + T_5R_{B_6}, \tag{27}$$

$$U_4 = M^{\dagger}ED_6^{\dagger} + S^{\dagger}SC_6^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_{D_6}, \tag{28}$$

and $T_1, \ldots, T_8$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Proof. (1) $\Longleftrightarrow$ (2) : We separate the real quaternion matrix equations in system (2) into three groups

$$A_1X = C_1,\ XB_1 = D_1, \tag{29}$$

$$A_2XB_2 = C_2,\ A_3XB_3 = C_3, \tag{30}$$

and

$$A_4XB_4 = C_4. \tag{31}$$

It follows from Lemma 2.1 that the system of real quaternion matrix equations (29) is consistent if and only if

$$R_{A_1}C_1 = 0, D_1L_{B_1} = 0, A_1D_1 = C_1B_1. \tag{32}$$

In this case, the general solution to the system (29) can be expressed as

$$X = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}YR_{B_1}, \tag{33}$$

where Y is an arbitrary matrix over $\mathbb{H}$ with appropriate size. Substituting (33) into (30) and (31) gives

$$\begin{aligned} &A_2(A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger})B_2 + A_2L_{A_1}YR_{B_1}B_2 = C_2,\\ &A_3(A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger})B_3 + A_3L_{A_1}YR_{B_1}B_3 = C_3 \end{aligned} \tag{34}$$

and

$$A_4(A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger})B_4 + A_4L_{A_1}YR_{B_1}B_4 = C_4, \tag{35}$$

i.e.,

$$\begin{aligned} &A_{11}YB_{11} = C_{11},\\ &A_{22}YB_{22} = C_{22}, \end{aligned} \tag{36}$$

and

$$A_{33}YB_{33} = C_{33}, \tag{37}$$

where A_{ii}, B_{ii}, C_{ii} are defined in (6). Hence, the system (2) is consistent if and only if the matrix equations (36) and (37) are consistent, respectively. By Lemma 2.2, we know that the system of real quaternion matrix equations (36) is consistent if and only if

$$R_ACL_B = 0,\ R_{A_{11}}C_{11} = 0,\ C_{11}L_{B_{11}} = 0,\ R_{A_{22}}C_{22} = 0,\ C_{22}L_{B_{22}} = 0. \tag{38}$$

In this case, the general solution to the system of real quaternion matrix equations (36) can be expressed as

$$\begin{aligned} Y =& A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}}\\ &+ L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \end{aligned} \tag{39}$$

where A, B, C, D are defined in (7), U_1, U_2, U_3, and U_4 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes. It follows from Lemma 2.2 that the real quaternion matrix equation (37) is consistent if and only if

$$R_{A_{33}}C_{33} = 0,\ C_{33}L_{B_{33}} = 0. \tag{40}$$

In this case, the general solution to the real quaternion matrix equation (37) can be expressed as

$$Y = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - L_{A_{33}}U_5 - U_6R_{B_{33}}, \tag{41}$$

where U_5 and U_6 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes. Equating Y in (39) and Y in (41) gives

$$\begin{aligned} &A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}} + L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}}\\ &+ L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}} = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - L_{A_{33}}U_5 - U_6R_{B_{33}}, \end{aligned}$$

i.e.,

$$A_5\begin{pmatrix} U_1\\ U_5 \end{pmatrix} + (U_2,\ U_6)B_5 + L_{A_{11}}U_3R_{B_{22}} + L_{A_{11}}U_4R_{B_{11}} = C_5, \tag{42}$$

where A_5, B_5, C_5 are defined in (8) and (9). Now we want to solve the real quaternion matrix equation (42). It follows from Lemma 2.3 that the real quaternion matrix equation (42) is consistent if and only if

$$R_M R_{A_6} E = 0, E L_{B_6} L_N = 0, R_{A_6} E L_{D_6} = 0, R_{C_6} E L_{B_6} = 0, \tag{43}$$

where $A_6, B_6, C_6, D_6, E, M, N, S$ are defined in (10) and (11). In this case, the general solution to the real quaternion matrix equation (42) can be expressed as

$$\begin{pmatrix} U_1 \\ U_5 \end{pmatrix} = A_5^{\dagger}(C_5 - L_{A_{11}} U_3 R_{B_{22}} - L_{A_{22}} U_4 R_{B_{11}}) - A_5^{\dagger} T_7 B_5 + L_{A_5} T_6, \tag{44}$$

$$(U_2,\ U_6) = R_{A_5}(C_5 - L_{A_{11}} U_3 R_{B_{22}} - L_{A_{22}} U_4 R_{B_{11}}) B_5^{\dagger} + A_5 A_5^{\dagger} T_7 + T_8 R_{B_5}, \tag{45}$$

$$U_3 = A_6^{\dagger} E B_6^{\dagger} - A_6^{\dagger} C_6 M^{\dagger} E B_6^{\dagger} - A_6^{\dagger} S C_6^{\dagger} E L_B N^{\dagger} D_6 B_6^{\dagger} - A_6^{\dagger} S T_2 R_N D_6 B_6^{\dagger} + L_{A_6} T_4 + T_5 R_{B_6}, \tag{46}$$

$$U_4 = M^{\dagger} E D_6^{\dagger} + S^{\dagger} S C_6^{\dagger} E N^{\dagger} + L_M L_S T_1 + L_M T_2 R_N + T_3 R_{D_6}, \tag{47}$$

and $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

(2) $\Longleftrightarrow$ (3) : It follows from Lemma 2.4 that

$$R_{A_1} C_1 = 0 \Longleftrightarrow r(C_1,\ A_1) = r(A_1),\ D_1 L_{B_1} = 0 \Longleftrightarrow r\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = r(B_1). \tag{48}$$

Hence, (12) $\Longleftrightarrow$ (15). Then, the real quaternion matrix equations (29) has a solution, say X_0. So we have

$$A_1 X_0 = C_1,\ X_0 B_1 = D_1. \tag{49}$$

Now we want to prove (13) $\Longleftrightarrow$ (16) and (17). Note that

$$R_{A_{11}} C_{11} = 0 \Longleftrightarrow r(A_{11},\ C_{11}) = r(A_{11}) \Longleftrightarrow r(A_2 L_{A_1},\ C_{11}) = r(A_{11})$$

$$\Longleftrightarrow r\begin{pmatrix} C_{11} & A_2 \\ 0 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} \Longleftrightarrow r\begin{pmatrix} C_2 - A_2 X_0 B_2 & A_2 \\ 0 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}$$

$$\Longleftrightarrow r\begin{pmatrix} C_2 & A_2 \\ A_1 X_0 B_2 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} \Longleftrightarrow r\begin{pmatrix} C_2 & A_2 \\ C_1 B_2 & A_1 \end{pmatrix} = r\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}.$$

Similarly, we can prove

$$R_{A_{22}} C_{22} = 0 \Longleftrightarrow r\begin{pmatrix} C_3 & A_3 \\ C_1 B_3 & A_1 \end{pmatrix} = r\begin{pmatrix} A_3 \\ A_1 \end{pmatrix},$$

$$R_{A_{33}} C_{33} = 0 \Longleftrightarrow r\begin{pmatrix} C_4 & A_4 \\ C_1 B_4 & A_1 \end{pmatrix} = r\begin{pmatrix} A_4 \\ A_1 \end{pmatrix},$$

$$C_{ii} L_{B_{ii}} = 0 \Longleftrightarrow r\begin{pmatrix} A_{i+1} D_1 & C_{i+1} \\ B_1 & B_{i+1} \end{pmatrix} = r(B_1,\ B_{i+1}),\ (i = 1, 2, 3).$$

We now pay attention to $R_ACL_B = 0$. Note that

$$A_{11}YB_{11} = C_{11}$$

has a specila solution Y_0

$$Y_0 = A_{11}^{\dagger}C_{11}B_{11}^{\dagger}.$$

Then we have

$$A_{11}Y_0B_{11} = C_{11}. \tag{50}$$

It follows from Lemma 2.4 and (50) that

$$R_ACL_B = 0 \Longleftrightarrow r\begin{pmatrix} C & A \\ B & 0 \end{pmatrix} = r(A) + r(B)$$

$$\Longleftrightarrow r\begin{pmatrix} C & A_{22}L_{A_{11}} \\ R_{B_{11}}B_{22} & 0 \end{pmatrix} = r(A_{22}L_{A_{11}}) + r(R_{B_{11}}B_{22})$$

$$\Longleftrightarrow r\begin{pmatrix} C & A_{22} & 0 \\ B_{22} & 0 & B_{11} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} A_{22} \\ A_{11} \end{pmatrix} + r(B_{11},\ B_{22})$$

$$\Longleftrightarrow r\begin{pmatrix} C_{22} - A_{22}Y_0B_{22} & A_{22} & 0 \\ B_{22} & 0 & B_{11} \\ 0 & A_{11} & 0 \end{pmatrix} = r\begin{pmatrix} A_{22} \\ A_{11} \end{pmatrix} + r(B_{11},\ B_{22})$$

$$\Longleftrightarrow r\begin{pmatrix} C_{22} & A_{22} & 0 \\ B_{22} & 0 & B_{11} \\ 0 & A_{11} & -C_{11} \end{pmatrix} = r\begin{pmatrix} A_{22} \\ A_{11} \end{pmatrix} + r(B_{11},\ B_{22})$$

$$\Longleftrightarrow r\begin{pmatrix} -C_2 + A_2X_0B_2 & A_2L_{A_1} & 0 \\ R_{B_1}B_2 & 0 & R_{B_1}B_3 \\ 0 & A_3L_{A_1} & C_3 - A_3X_0B_3 \end{pmatrix} = r\begin{pmatrix} A_2L_{A_1} \\ A_3L_{A_1} \end{pmatrix} + r(R_{B_1}B_2,\ R_{B_1}B_3)$$

$$\Longleftrightarrow r\begin{pmatrix} -C_2 & A_2 & 0 & 0 \\ B_2 & 0 & B_3 & B_1 \\ 0 & A_3 & C_3 & A_3D_1 \\ 0 & A_1 & C_1B_3 & C_1B_1 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + r(B_1,\ B_2,\ B_3) \Longleftrightarrow (17).$$

Similarly, we can prove

$$R_MR_{A_6}E = 0 \Longleftrightarrow (18),\ EL_{B_6}L_N = 0 \Longleftrightarrow (19),$$

$$R_{A_6}EL_{D_6} = 0 \Longleftrightarrow (20),\ R_{C_6}EL_{B_6} = 0 \Longleftrightarrow (21).$$

□

Now we give an example to illustrate Theorem 3.1.

Example 3.2. *Let*

$$A_1=\begin{pmatrix}1+\mathbf{j} & \mathbf{i}-\mathbf{j} & 1+\mathbf{i}+\mathbf{k}\\ -1-\mathbf{j} & -\mathbf{i}+\mathbf{j} & -1-\mathbf{i}-\mathbf{k}\end{pmatrix}, B_1=\begin{pmatrix}\mathbf{j}-\mathbf{k} & 1\\ 1+2\mathbf{k} & \mathbf{j}\\ 1+\mathbf{i} & 1+\mathbf{k}\end{pmatrix},$$

$$C_1=\begin{pmatrix}1+3\mathbf{i}+3\mathbf{j}-\mathbf{k} & \mathbf{i}+\mathbf{j}-\mathbf{k} & -1+\mathbf{i}-3\mathbf{j}+\mathbf{k}\\ -2-4\mathbf{i}-\mathbf{j}+\mathbf{k} & 1-2\mathbf{i}-\mathbf{j}+\mathbf{k} & 2\mathbf{j}-2\mathbf{k}\end{pmatrix}, D_1=\begin{pmatrix}\mathbf{i}+\mathbf{k} & -1-\mathbf{j}+\mathbf{k}\\ 2-3\mathbf{k} & 2+\mathbf{i}-\mathbf{k}\\ 1+2\mathbf{k} & \mathbf{j}\end{pmatrix},$$

$$A_2=\begin{pmatrix}\mathbf{j}+\mathbf{k} & 1+2\mathbf{i}+\mathbf{j} & 1-\mathbf{i}\\ \mathbf{i} & \mathbf{k} & 1+\mathbf{j}\end{pmatrix}, B_2=\begin{pmatrix}\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{j}\\ 1+\mathbf{k} & \mathbf{i}+\mathbf{k}\\ 2\mathbf{i} & \mathbf{k}\end{pmatrix},$$

$$A_3=\begin{pmatrix}1+\mathbf{j}+\mathbf{k} & 2+\mathbf{j}-\mathbf{k} & \mathbf{i}+\mathbf{k}\\ -1-\mathbf{j}-\mathbf{k} & -2-\mathbf{j}+\mathbf{k} & -\mathbf{i}-\mathbf{k}\end{pmatrix}, B_3=\begin{pmatrix}2-3\mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{k}\\ \mathbf{i}-\mathbf{k} & -\mathbf{k}\\ 1+\mathbf{j} & \mathbf{j}\end{pmatrix},$$

$$A_4=\begin{pmatrix}\mathbf{j}-2\mathbf{k} & \mathbf{i}+\mathbf{k} & 1\\ \mathbf{i} & \mathbf{j} & \mathbf{i}\end{pmatrix}, B_4=\begin{pmatrix}\mathbf{i}+\mathbf{j} & \mathbf{k}\\ 1+2\mathbf{i}+\mathbf{k} & 1-\mathbf{j}\\ 1-\mathbf{i}+\mathbf{k} & -\mathbf{j}\end{pmatrix}.$$

Now we consider the system of real quaternion matrix equations (2). Check that

$$r(A_1,\ C_1)=r(A_1)=2,\ r\begin{pmatrix}B_1\\ D_1\end{pmatrix}=r(B_1)=2, A_1D_1=C_1B_1,$$

$$r\begin{pmatrix}C_{i+1} & A_{i+1}\\ C_1B_{i+1} & A_1\end{pmatrix}=r\begin{pmatrix}A_{i+1}\\ A_1\end{pmatrix}=3, r\begin{pmatrix}A_{i+1}D_1 & C_{i+1}\\ B_1 & B_{i+1}\end{pmatrix}=r(B_1,\ B_{i+1})=3,\ (i=1,2,3),$$

$$r\begin{pmatrix}-C_2 & A_2 & 0 & 0\\ B_2 & 0 & B_3 & B_1\\ 0 & A_3 & C_3 & A_3D_1\\ 0 & A_1 & C_1B_3 & C_1B_1\end{pmatrix}=r\begin{pmatrix}A_1\\ A_2\\ A_3\end{pmatrix}+r(B_1,\ B_2,\ B_3)=6,$$

$$r\begin{pmatrix}0 & 0 & B_2 & B_3 & B_4 & B_1\\ A_2 & A_2 & -C_2 & 0 & 0 & 0\\ A_3 & 0 & 0 & C_3 & 0 & A_3D_1\\ 0 & A_4 & 0 & 0 & C_4 & 0\\ A_1 & 0 & -C_1B_2 & 0 & -C_1B_4 & 0\\ 0 & A_1 & 0 & 0 & C_1B_4 & 0\end{pmatrix}=r\begin{pmatrix}A_2 & A_2\\ A_3 & 0\\ 0 & A_4\\ A_1 & 0\\ 0 & A_1\end{pmatrix}+r(B_1,\ B_2,\ B_3,\ B_4)=9,$$

$$r\begin{pmatrix}0 & B_2 & B_3 & 0 & B_1 & 0\\ 0 & B_2 & 0 & B_4 & 0 & B_1\\ A_2 & -C_2 & 0 & 0 & 0 & -A_2D_1\\ A_3 & 0 & C_3 & 0 & A_3D_1 & -A_3D_1\\ A_4 & 0 & 0 & C_4 & 0 & 0\\ A_1 & 0 & 0 & C_1B_4 & 0 & 0\end{pmatrix}=r\begin{pmatrix}B_2 & B_3 & 0 & B_1 & 0\\ B_2 & 0 & B_4 & 0 & B_1\end{pmatrix}+r\begin{pmatrix}A_1\\ A_2\\ A_3\\ A_4\end{pmatrix}=9,$$

$$r\begin{pmatrix} 0 & B_2 & B_4 & B_1 \\ A_2 & -C_2 & 0 & 0 \\ A_4 & 0 & C_4 & A_4D_1 \\ A_1 & -C_1B_2 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \\ A_4 \end{pmatrix} + r(B_1,\ B_2,\ B_4) = 6,$$

$$r\begin{pmatrix} 0 & B_3 & B_4 & B_1 \\ A_3 & -C_3 & 0 & 0 \\ A_4 & 0 & C_4 & A_4D_1 \\ A_1 & -C_1B_3 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_3 \\ A_4 \end{pmatrix} + r(B_1,\ B_3,\ B_4) = 6.$$

Hence, the system of real quaternion matrix equations (2) is consistent.

Now we consider some special cases of the system (2). Let A_1, B_1, C_1, D_1 vanish in Theorem 3.1. Then we can give solvability conditions and general solution to the system

$$\begin{cases} A_{11}XB_{11} = C_{11}, \\ A_{22}XB_{22} = C_{22}, \\ A_{33}XB_{33} = C_{33}. \end{cases} \tag{51}$$

He and Wang considered the system (51) over complex field ([3]).

Corollary 3.3. *Let A_{ii}, B_{ii}, C_{ii} be known over $\mathbb{H}$, and X be unknown, $(i = 1, 2, 3)$. Set*

$$A = A_{22}L_{A_{11}}, B = R_{B_{11}}B_{22}, C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}B_{11}^{\dagger}B_{22}, D = R_{A_{11}}A_{22},$$

$$A_5 = (L_{A_{11}}L_A,\ L_{A_{33}}), B_5 = \begin{pmatrix} R_BR_{B_{11}} \\ R_{B_{33}} \end{pmatrix},$$

$$C_5 = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - A_{11}^{\dagger}C_{11}B_{11}^{\dagger} - L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} + L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} - D^{\dagger}R_ACB^{\dagger}R_{B_{11}},$$

$$A_6 = R_{A_5}L_{A_{11}}, B_6 = R_{B_{22}}L_{B_5}, C_6 = R_{A_5}L_{A_{22}}, D_6 = R_{B_{11}}L_{B_5},$$

$$E = R_{A_5}C_5L_{B_5}, M = R_{A_6}C_6, N = D_6L_{B_6}, S = C_6L_M.$$

Then the system of real quaternion matrix equations (51) is consistent if and only if

$$R_{A_{ii}}C_{ii} = 0, C_{ii}L_{B_{ii}} = 0,\ (i = 1, 2, 3),\ R_ACL_B = 0,$$

$$R_MR_{A_6}E = 0, EL_{B_6}L_N = 0, R_{A_6}EL_{D_6} = 0, R_{C_6}EL_{B_6} = 0.$$

In this case, the general solution to system (51) can be expressed as

$$\begin{aligned} X =& A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \end{aligned}$$

or

$$X = A_{33}^{\dagger}C_{33}B_{33}^{\dagger} - L_{A_{33}}U_5 - U_6R_{B_{33}},$$

where

$$\begin{pmatrix} U_1 \\ U_5 \end{pmatrix} = A_5^{\dagger}(C_5 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}}) - A_5^{\dagger}T_7B_5 + L_{A_5}T_6,$$

$$(U_2,\ U_6) = R_{A_5}(C_5 - L_{A_{11}}U_3R_{B_{22}} - L_{A_{22}}U_4R_{B_{11}})B_5^{\dagger} + A_5A_5^{\dagger}T_7 + T_8R_{B_5},$$

$$U_3 = A_6^{\dagger}EB_6^{\dagger} - A_6^{\dagger}C_6M^{\dagger}EB_6^{\dagger} - A_6^{\dagger}SC_6^{\dagger}EL_BN^{\dagger}D_6B_6^{\dagger} - A_6^{\dagger}ST_2R_ND_6B_6^{\dagger} + L_{A_6}T_4 + T_5R_{B_6},$$

$$U_4 = M^{\dagger}ED_6^{\dagger} + S^{\dagger}SC_6^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_{D_6},$$

and $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Let A_4, B_4, C_4 vanish in Theorem 3.1. Then we can give solvability conditions and general solution to the system

$$\begin{cases} A_1X = C_1, \\ XB_1 = D_1, \\ A_2XB_2 = C_2, \\ A_3XB_3 = C_3. \end{cases} \tag{52}$$

Wang, Chang and Ning considered the system of real quaternion matrix equations (52) ([24]).

Corollary 3.4. *Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, A_3, B_3, C_3$ be known over $\mathbb{H}$, and X be unknown. Set*

$$A_{ii} = A_{i+1}L_{A_1}, B_{ii} = R_{B_1}B_{i+1}, C_{ii} = C_{i+1} - A_{i+1}(A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger})B_{i+1},\ (i = 1, 2),$$

$$A = A_{22}L_{A_{11}}, B = R_{B_{11}}B_{22}, C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}B_{11}^{\dagger}B_{22}, D = R_{A_{11}}A_{22}.$$

Then the following statements are equivalent:
(1) *The system of real quaternion matrix equations (52) is consistent.*
(2)

$$R_{A_1}C_1 = 0, D_1L_{B_1} = 0, A_1D_1 = C_1B_1,$$

$$R_{A_{ii}}C_{ii} = 0, C_{ii}L_{B_{ii}} = 0,\ (i = 1, 2),\ R_ACL_B = 0.$$

(3)

$$r(A_1,\ C_1) = r(A_1),\ r\begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = r(B_1), A_1D_1 = C_1B_1,$$

$$r\begin{pmatrix} C_{i+1} & A_{i+1} \\ C_1B_{i+1} & A_1 \end{pmatrix} = r\begin{pmatrix} A_{i+1} \\ A_1 \end{pmatrix}, r\begin{pmatrix} A_{i+1}D_1 & C_{i+1} \\ B_1 & B_{i+1} \end{pmatrix} = r(B_1,\ B_{i+1}),\ (i = 1, 2),$$

$$r\begin{pmatrix} -C_2 & A_2 & 0 & 0 \\ B_2 & 0 & B_3 & B_1 \\ 0 & A_3 & C_3 & A_3D_1 \\ 0 & A_1 & C_1B_3 & C_1B_1 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} + r(B_1,\ B_2,\ B_3).$$

In this case, the general solution to system (52) can be expressed as

$$X = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}YR_{B_1},$$

where

$$\begin{aligned} Y =& A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}A^{\dagger}CB_{22}^{\dagger} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_ACB_{22}^{\dagger} + D^{\dagger}R_ACB^{\dagger}R_{B_{11}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_BR_{B_{11}} + L_{A_{11}}U_3R_{B_{22}} + L_{A_{22}}U_4R_{B_{11}}, \end{aligned}$$

where U_1, U_2, U_3, and U_4 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

4. The η-Hermitian solution to system of real quaternion matrix equations (1)

In this section, we consider the general η-Hermitian solution to system of real quaternion matrix equations (1).

Theorem 4.1. *Let $A_1, C_1, A_2, A_3, A_4, C_2 = C_2^{\eta*}, C_3 = C_3^{\eta*}, C_4 = C_4^{\eta*}$ be known over $\mathbb{H}$, and $X = X^{\eta*}$ be unknown. Set*

$$A_{ii} = A_{i+1}L_{A_1}, C_{ii} = C_{i+1} - A_{i+1}(A_1^{\dagger}C_1 + L_{A_1}C_1^{\eta*}(A_1^{\dagger})^{\eta*})A_{i+1}^{\eta*},\ (i = 1, 2, 3),$$

$$A = A_{22}L_{A_{11}}, C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*}A_{22}^{\eta*}, D = R_{A_{11}}A_{22},$$

$$A_5 = (L_{A_{11}}L_A,\ L_{A_{33}}), A_6 = R_{A_5}L_{A_{11}}, B_6 = R_{A_{22}^{\eta*}}L_{A_5^{\eta*}},$$

$$\begin{aligned}C_5 =& A_{33}^{\dagger}C_{33}(A_{33}^{\dagger})^{\eta*} - A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*} - L_{A_{11}}A^{\dagger}C(A_{22}^{\dagger})^{\eta*}\\ &+ L_{A_{11}^{\eta*}}A^{\dagger}A_{22}D^{\dagger}R_AC(A_{22}^{\dagger})^{\eta*} - D^{\dagger}R_AC(A^{\dagger})^{\eta*}R_{A_{11}^{\eta*}},\end{aligned}$$

$$E = R_{A_5}C_5L_{A_5^{\eta*}}, M = R_{A_6}B_6^{\eta*}, N = A_6^{\eta*}L_{B_6}, S = B_6^{\eta*}L_M.$$

Then the following statements are equivalent:
(1) *The system of real quaternion matrix equations (1) has an η-Hermitian solution.*
(2)

$$R_{A_1}C_1 = 0,\ A_1C_1^{\eta*} = C_1A_1^{\eta*},\ R_{A_{ii}}C_{ii} = 0,\ (i = 1, 2, 3),$$

$$R_ACL_{A^{\eta*}} = 0,\ R_MR_{A_6}E = 0,\ R_{A_6}EL_{A_6^{\eta*}} = 0.$$

(3)

$$r(A_1,\ C_1) = r(A_1),\ A_1C_1^{\eta*} = C_1A_1^{\eta*},$$

$$r\begin{pmatrix} C_{i+1} & A_{i+1} \\ C_1A_{i+1}^{\eta*} & A_1 \end{pmatrix} = r\begin{pmatrix} A_{i+1} \\ A_1 \end{pmatrix},\ (i = 1, 2, 3),$$

$$r\begin{pmatrix} -C_2 & A_2 & 0 & 0 \\ A_2^{\eta*} & 0 & A_3^{\eta*} & A_1^{\eta*} \\ 0 & A_3 & C_3 & A_3C_1^{\eta*} \\ 0 & A_1 & C_1A_3^{\eta*} & C_1A_1^{\eta*} \end{pmatrix} = 2r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & 0 & A_2^{\eta*} & A_3^{\eta*} & A_4^{\eta*} & A_1^{\eta*} \\ A_2 & A_2 & -C_2 & 0 & 0 & 0 \\ A_3 & 0 & 0 & C_3 & 0 & A_3C_1^{\eta*} \\ 0 & A_4 & 0 & 0 & C_4 & 0 \\ A_1 & 0 & -C_1A_2^{\eta*} & 0 & -C_1A_4^{\eta*} & 0 \\ 0 & A_1 & 0 & 0 & C_1A_4^{\eta*} & 0 \end{pmatrix} = r\begin{pmatrix} A_2 & A_2 \\ A_3 & 0 \\ 0 & A_4 \\ A_1 & 0 \\ 0 & A_1 \end{pmatrix} + r\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & A_2^{\eta*} & A_4^{\eta*} & A_1^{\eta*} \\ A_2 & -C_2 & 0 & 0 \\ A_4 & 0 & C_4 & A_4C_1^{\eta*} \\ A_1 & -C_1A_2^{\eta*} & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} A_1 \\ A_2 \\ A_4 \end{pmatrix}.$$

In this case, the general η-Hermitian solution to system (1) can be expressed as

$$X = \frac{\widehat{X} + \widehat{X}^{\eta*}}{2},$$

where

$$\widehat{X} = A_1^{\dagger}C_1 + L_{A_1}C_1^{\eta*}(A_1^{\dagger})^{\eta*} + L_{A_1}YR_{A_1^{\eta*}},$$

$$\begin{aligned} Y =& A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*} + L_{A_{11}}A^{\dagger}C(A_{22}^{\dagger})^{\eta*} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_AC(A_{22}^{\dagger})^{\eta*} + D^{\dagger}R_AC(A^{\dagger})^{\eta*}R_{A_{11}^{\eta*}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_{A^{\eta*}}R_{A_{11}^{\eta*}} + L_{A_{11}}U_3R_{A_{22}^{\eta*}} + L_{A_{22}}U_4R_{A_{11}^{\eta*}}, \end{aligned}$$

or

$$Y = A_{33}^{\dagger}C_{33}(A_{33}^{\dagger})^{\eta*} - L_{A_{33}}U_5 - U_6R_{A_{33}^{\eta*}},$$

$$\begin{pmatrix} U_1 \\ U_5 \end{pmatrix} = A_5^{\dagger}(C_5 - L_{A_{11}}U_3R_{A_{22}^{\eta*}} - L_{A_{22}}U_4R_{A_{11}^{\eta*}}) - A_5^{\dagger}T_7A_5^{\eta*} + L_{A_5}T_6,$$

$$(U_2,\ U_6) = R_{A_5}(C_5 - L_{A_{11}}U_3R_{A_{22}^{\eta*}} - L_{A_{22}}U_4R_{A_{11}^{\eta*}})(A_5^{\dagger})^{\eta*} + A_5A_5^{\dagger}T_7 + T_8R_{A_5^{\eta*}},$$

$$\begin{aligned} U_3 =& A_6^{\dagger}EB_6^{\dagger} - A_6^{\dagger}B_6^{\eta*}M^{\dagger}EB_6^{\dagger} - A_6^{\dagger}S(B_6^{\dagger})^{\eta*}EL_{A^{\eta*}}N^{\dagger}A_6^{\eta*}B_6^{\dagger} \\ &- A_6^{\dagger}ST_2R_NA_6^{\eta*}B_6^{\dagger} + L_{A_6}T_4 + T_5R_{B_6}, \end{aligned}$$

$$U_4 = M^{\dagger}E(A_6^{\dagger})^{\eta*} + S^{\dagger}S(B_6^{\dagger})^{\eta*}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_{A_6^{\eta*}},$$

and $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Proof. We first prove that the system of real quaternion matrix equations (1) has an η-Hermitian solution if and only if the system of real quaternion matrix equations

$$\begin{cases} A_1\widehat{X} = C_1, \\ \widehat{X}A_1^{\eta*} = C_1^{\eta*}, \\ A_2\widehat{X}A_2^{\eta*} = C_2, \\ A_3\widehat{X}A_3^{\eta*} = C_3, \\ A_4\widehat{X}A_4^{\eta*} = C_4 \end{cases} \tag{53}$$

has a solution $\widehat{X}$. If the system of real quaternion matrix equations (1) has an η-Hermitian solution, say, X_0, then the system (53) clearly has a solution $\widehat{X} = X_0$. Conversely, if the system (53) has a solution $\widehat{X}$, then

$$X = \frac{\widehat{X} + \widehat{X}^{\eta*}}{2}$$

is an η-Hermitian solution to (1). We can derive the solvability conditions to the system of real quaternion matrix equations (1) by Theorem 3.1.

□

Now we give an example to illustrate Theorem 4.1.

Example 4.2. *Let*

$$A_1=\begin{pmatrix}-\mathbf{i}-\mathbf{j}-\mathbf{k} & 1+\mathbf{i}+\mathbf{j}+2\mathbf{k} & \mathbf{j}\\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 1-\mathbf{j}-2\mathbf{k} & 1+\mathbf{k}\\ \mathbf{k} & 2+\mathbf{i} & 1+\mathbf{j}+\mathbf{k}\end{pmatrix}, A_2=\begin{pmatrix}1 & \mathbf{i}+\mathbf{j} & \mathbf{i}+\mathbf{k}\\ 1+\mathbf{i} & -1+\mathbf{i}+\mathbf{j}+\mathbf{k} & -1+\mathbf{i}-\mathbf{j}+\mathbf{k}\\ \mathbf{i} & -1+\mathbf{k} & -1-\mathbf{j}\end{pmatrix},$$

$$A_3=\begin{pmatrix}\mathbf{j}+\mathbf{k} & 2\mathbf{i}+2\mathbf{j} & 1\\ \mathbf{i}+\mathbf{j} & -\mathbf{i}+\mathbf{k} & -1\\ \mathbf{i}+2\mathbf{j}+\mathbf{k} & \mathbf{i}+2\mathbf{j}+\mathbf{k} & 0\end{pmatrix}, A_4=\begin{pmatrix}-1 & 2\mathbf{i}+\mathbf{j} & -\mathbf{i}+\mathbf{k}\\ -\mathbf{i}+\mathbf{j} & -2+\mathbf{k} & 1\\ 0 & 1+\mathbf{j} & \mathbf{j}+\mathbf{k}\end{pmatrix},$$

$$C_1=\begin{pmatrix}2+2\mathbf{i}-\mathbf{j}-\mathbf{k} & 2\mathbf{i}+\mathbf{j}-3\mathbf{k} & -1+2\mathbf{i}-2\mathbf{j}+2\mathbf{k}\\ 0 & -2-\mathbf{j}+5\mathbf{k} & -\mathbf{i}+2\mathbf{j}+\mathbf{k}\\ 2+2\mathbf{i}-\mathbf{j}-\mathbf{k} & -2+2\mathbf{i}+2\mathbf{k} & -1+\mathbf{i}+3\mathbf{k}\end{pmatrix},$$

$$C_2=C_2^{\mathbf{j}*}=\begin{pmatrix}-1-4\mathbf{i} & 3-5\mathbf{i} & 4-\mathbf{i}\\ 3-5\mathbf{i} & 8-2\mathbf{i} & 5+3\mathbf{i}\\ 4-\mathbf{i} & 5+3\mathbf{i} & 1+4\mathbf{i}\end{pmatrix},$$

$$C_3=C_3^{\mathbf{j}*}=\begin{pmatrix}8-2\mathbf{i}-\mathbf{k} & -1+3\mathbf{i}-\mathbf{j}-8\mathbf{k} & 7+\mathbf{i}-\mathbf{j}-9\mathbf{k}\\ -1+3\mathbf{i}+\mathbf{j}-8\mathbf{k} & 4+4\mathbf{i}+3\mathbf{k} & 3+7\mathbf{i}+\mathbf{j}-5\mathbf{k}\\ 7+\mathbf{i}+\mathbf{j}-9\mathbf{k} & 3+7\mathbf{i}-\mathbf{j}-5\mathbf{k} & 10+8\mathbf{i}-14\mathbf{k}\end{pmatrix},$$

$$C_4=C_4^{\mathbf{j}*}=\begin{pmatrix}-1-11\mathbf{i} & 12-3\mathbf{i}-2\mathbf{k} & -5-2\mathbf{i}+5\mathbf{j}-2\mathbf{k}\\ 12-3\mathbf{i}-2\mathbf{k} & 6+11\mathbf{i}-\mathbf{k} & 3-5\mathbf{i}+3\mathbf{j}+3\mathbf{k}\\ -5-2\mathbf{i}-5\mathbf{j}-2\mathbf{k} & 3-5\mathbf{i}-3\mathbf{j}+3\mathbf{k} & -2+2\mathbf{i}-6\mathbf{k}\end{pmatrix}.$$

Now we consider the system (1) where X is $\mathbf{j}$-Hermitian. Check that

$$r(A_1,\ C_1)=r(A_1)=2,\ A_1C_1^{\eta*}=C_1A_1^{\eta*},$$

$$r\begin{pmatrix}C_{i+1} & A_{i+1}\\ C_1A_{i+1}^{\eta*} & A_1\end{pmatrix}=r\begin{pmatrix}A_{i+1}\\ A_1\end{pmatrix}=3,\ (i=1,2,3),$$

$$r\begin{pmatrix}-C_2 & A_2 & 0 & 0\\ A_2^{\eta*} & 0 & A_3^{\eta*} & A_1^{\eta*}\\ 0 & A_3 & C_3 & A_3C_1^{\eta*}\\ 0 & A_1 & C_1A_3^{\eta*} & C_1A_1^{\eta*}\end{pmatrix}=2r\begin{pmatrix}A_1\\ A_2\\ A_3\end{pmatrix}=6,$$

$$r\begin{pmatrix}0 & 0 & A_2^{\eta*} & A_3^{\eta*} & A_4^{\eta*} & A_1^{\eta*}\\ A_2 & A_2 & -C_2 & 0 & 0 & 0\\ A_3 & 0 & 0 & C_3 & 0 & A_3C_1^{\eta*}\\ 0 & A_4 & 0 & 0 & C_4 & 0\\ A_1 & 0 & -C_1A_2^{\eta*} & 0 & -C_1A_4^{\eta*} & 0\\ 0 & A_1 & 0 & 0 & C_1A_4^{\eta*} & 0\end{pmatrix}=r\begin{pmatrix}A_2 & A_2\\ A_3 & 0\\ 0 & A_4\\ A_1 & 0\\ 0 & A_1\end{pmatrix}+r\begin{pmatrix}A_1\\ A_2\\ A_3\\ A_4\end{pmatrix}=9,$$

$$r\begin{pmatrix}0 & A_2^{\eta*} & A_4^{\eta*} & A_1^{\eta*}\\ A_2 & -C_2 & 0 & 0\\ A_4 & 0 & C_4 & A_4C_1^{\eta*}\\ A_1 & -C_1A_2^{\eta*} & 0 & 0\end{pmatrix}=2r\begin{pmatrix}A_1\\ A_2\\ A_4\end{pmatrix}=6.$$

Hence, the system (1) has a $\mathbf{j}$-Hermitian solution.

Let A_1 and C_1 vanish in Theorem 4.1. Then we obtain some necessary and sufficient conditions for the existence of an η-Hermitian solution to the following system of real quaternion matrix equations

$$\begin{cases} A_{11}XA_{11}^{\eta*} = C_{11}, \\ A_{22}XA_{22}^{\eta*} = C_{22}, \\ A_{33}XA_{33}^{\eta*} = C_{33}. \end{cases} \tag{54}$$

We can also give the general η-Hermitian solution to the system (54).

Corollary 4.3. *Let $A_{ii}, C_{ii} = C_{ii}^{\eta*}$ be known over $\mathbb{H}$, and $X = X^{\eta*}$ be unknown, $(i = 1, 2, 3)$. Set*

$$A = A_{22}L_{A_{11}}, C = C_{22} - A_{22}A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*}A_{22}^{\eta*}, D = R_{A_{11}}A_{22},$$

$$A_5 = (L_{A_{11}}L_A,\ L_{A_{33}}), A_6 = R_{A_5}L_{A_{11}}, B_6 = R_{A_{22}^{\eta*}}L_{A_5^{\eta*}},$$

$$\begin{aligned} C_5 =& A_{33}^{\dagger}C_{33}(A_{33}^{\dagger})^{\eta*} - A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*} - L_{A_{11}}A^{\dagger}C(A_{22}^{\dagger})^{\eta*} \\ &+ L_{A_{11}^{\eta*}}A^{\dagger}A_{22}D^{\dagger}R_AC(A_{22}^{\dagger})^{\eta*} - D^{\dagger}R_AC(A^{\dagger})^{\eta*}R_{A_{11}^{\eta*}}, \end{aligned}$$

$$E = R_{A_5}C_5L_{A_5^{\eta*}}, M = R_{A_6}B_6^{\eta*}, N = A_6^{\eta*}L_{B_6}, S = B_6^{\eta*}L_M.$$

Then the system of real quaternion matrix equations (54) has an η-Hermitian solution if and only if

$$R_{A_{ii}}C_{ii} = 0,\ (i = 1, 2, 3), R_ACL_{A^{\eta*}} = 0,\ R_MR_{A_6}E = 0,\ R_{A_6}EL_{A_6^{\eta*}} = 0.$$

In this case, the general η-Hermitian solution to system (54) can be expressed as

$$X = \frac{\widehat{X} + \widehat{X}^{\eta*}}{2},$$

where

$$\begin{aligned} \widehat{X} =& A_{11}^{\dagger}C_{11}(A_{11}^{\dagger})^{\eta*} + L_{A_{11}}A^{\dagger}C(A_{22}^{\dagger})^{\eta*} - L_{A_{11}}A^{\dagger}A_{22}D^{\dagger}R_AC(A_{22}^{\dagger})^{\eta*} + D^{\dagger}R_AC(A^{\dagger})^{\eta*}R_{A_{11}^{\eta*}} \\ &+ L_{A_{11}}L_AU_1 + U_2R_{A^{\eta*}}R_{A_{11}^{\eta*}} + L_{A_{11}}U_3R_{A_{22}^{\eta*}} + L_{A_{22}}U_4R_{A_{11}^{\eta*}}, \end{aligned}$$

or

$$\widehat{X} = A_{33}^{\dagger}C_{33}(A_{33}^{\dagger})^{\eta*} - L_{A_{33}}U_5 - U_6R_{A_{33}^{\eta*}},$$

$$\begin{pmatrix} U_1 \\ U_5 \end{pmatrix} = A_5^{\dagger}(C_5 - L_{A_{11}}U_3R_{A_{22}^{\eta*}} - L_{A_{22}}U_4R_{A_{11}^{\eta*}}) - A_5^{\dagger}T_7A_5^{\eta*} + L_{A_5}T_6,$$

$$(U_2,\ U_6) = R_{A_5}(C_5 - L_{A_{11}}U_3R_{A_{22}^{\eta*}} - L_{A_{22}}U_4R_{A_{11}^{\eta*}})(A_5^{\dagger})^{\eta*} + A_5A_5^{\dagger}T_7 + T_8R_{A_5^{\eta*}},$$

$$\begin{aligned} U_3 =& A_6^{\dagger}EB_6^{\dagger} - A_6^{\dagger}B_6^{\eta*}M^{\dagger}EB_6^{\dagger} - A_6^{\dagger}S(B_6^{\dagger})^{\eta*}EL_{A^{\eta*}}N^{\dagger}A_6^{\eta*}B_6^{\dagger} \\ &- A_6^{\dagger}ST_2R_NA_6^{\eta*}B_6^{\dagger} + L_{A_6}T_4 + T_5R_{B_6}, \end{aligned}$$

$$U_4 = M^{\dagger}E(A_6^{\dagger})^{\eta*} + S^{\dagger}S(B_6^{\dagger})^{\eta*}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_{A_6^{\eta*}},$$

and $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

5. Conclusions

We have presented necessary and sufficient conditions for the existence and the general solution to the system of real quaternion matrix equations (2). As an application of the system (51), we have also given necessary and sufficient conditions for the existence and the general η-Hermitian solution to the system of real quaternion matrix equations (1). Some numerical examples are presented to illustrate the results.

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