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Inequalities of Singular Values and Unitarily Invariant Norms for Sums and Products of Matrices

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Abstract. In this work, we investigate inequalities of singular values and unitarily invariant norms for sums and products of matrices. First, we give an another more concise and clear proof to inequality obtained by Chen and Zhang [6, Theorem 5]. Then, we establish an inequality for singular values. In addition, we also give a singular values inequality for sums and products of matrices. As applications of this inequality, we present some unitarily invariant norms inequalities.

1. Introduction

Throughout, let $M_n(C)$ be the space of $n \times n$ complex matrices. $I_n \in M_n(C)$ is the identity matrix. For $A \in M_n(C)$, let $\lambda_j(A)$ be the eigenvalues of A. let $s_j(A)(j = 1, 2, \cdots, n)$ be the singular values of A (i.e., the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the transpose conjugate of A), arranged in decreasing order and repeated according to multiplicity. The notation $A \leq B$, as usual, means that both A and B are two Hermitian matrices in $M_n(C)$ and $B - A$ is a positive semidefinite matrix. The relation $\leq$ is a partial order on $M_n(C)$.

A norm $\|\cdot\|$, defined on $M_n(C)$, is called a unitarily invariant norm if $\|UAV\| = \|A\|$ for $A, U, V \in M_n(C)$ with U, V are unitary matrices. Examples in these classes are the operator norm defined by $\|A\|_\infty = s_1(A)$ and the Schatten $p-$norms $(p \geq 1)$ defined by $\|A\|_p = \left(\sum\limits_{i=1}^{n} s_i^p(A)\right)^{\frac{1}{p}}$ for all $A \in M_n(A)$.

The $m \times m$ block matrix $\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$ is a matrix in $M_m(M_n(C))$, where $A_{ij} \in M_n(C)(i, j = 1, 2, \cdots, m)$. When $A_{ij} = 0 (i \neq j)$, the $m \times m$ block matrix $\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$ is just the direct sum of

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matrices $A_i \in M_n(C)(i = 1, 2, \cdots, m)$, denoted by $\bigoplus_{i=1}^{m} A_i$. When $m = 2$, we write $A_1 \oplus A_2$ instead of $\bigoplus_{i=1}^{2} A_i$. The famous inequality between the real part and the singular values of $A \in M_n(C)$ is

$$\lambda_j(ReA) \leq s_j(A)$$

obtained by Fan and Hoffman [7], $j = 1, 2, \cdots, n$.

Recently, Chen and Zhang [6] studied some singular values inequalities among the real and imaginary parts of matrices and themselves. The authors [6, Theorem 5] obtained:

$$s_j(ReA) \leq \frac{1}{4} s_j\Big((|A| + |A^*| + A + A^*) \oplus (|A| + |A^*| - (A + A^*))\Big) \tag{1}$$

and

$$s_j(ImA) \leq \frac{1}{4} s_j\Big((|A| + |A^*| + i(A^* - A)) \oplus (|A| + |A^*| - i(A^* - A))\Big) \tag{2}$$

for $j = 1, 2, \cdots n$, where $A \in M_n(C)$ and $i^2 = -1$. As results of the above inequalities, they [6, Theorem 6] also got:

$$s_j(ReA) \leq \frac{1}{2} s_j\Big((|A| + |A^*|) \oplus (|A| + |A^*|)\Big)$$

and

$$s_j(ImA) \leq \frac{1}{2} s_j\Big((|A| + |A^*|) \oplus (|A| + |A^*|)\Big)$$

for $j = 1, 2, \cdots n$.

In this work, we investigate singular values and unitarily invariant norms inequalities for sums and products of matrices. First, we give an another more concise and clear proof to inequality (1). Then, we establish an inequality for singular values, which is a generalization of inequality (1). In addition, we also give a singular values inequality for sums and products of matrices. As applications of this inequality, we present some unitarily invariant norms inequalities.

2. Main results

In this section, we mainly study inequalities of singular values and unitarily invariant norms for sums and products of matrices. To achieve our goal, we need the following lemmas. The first lemma is the Corollary 1.3.7 and the second one is the Proposition 1.3.2 in [3].

Lemma 2.1. *If $A \in M_n(C)$, then the block matrix $\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}$ in $M_{2n}(C)$ is a positive semidefinite matrix.*

Lemma 2.2. *Let $A, B \in M_n(C)$ with $A \geq 0$ and $B \geq 0$. Then the block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$ if and only if $X = A^{\frac{1}{2}} W B^{\frac{1}{2}}$ for some contraction W.*

The next lemma was obtained by Tao [9].

Lemma 2.3. *If A, B and $X \in M_n(C)$ with $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$, then*

$$2 s_j(X) \leq s_j \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

for $j = 1, 2, \cdots, n$.

The fourth lemma was given by Kittaneh [8].

Lemma 2.4. *Let A, B and $X \in M_n(C)$ with $A \geq 0$, $B \geq 0$ and $BX = XA$ and f, g be two nonnegative continuous functions on $[0, +\infty)$ with $f(t)g(t) = t$ for $t \in [0, +\infty)$. If $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geq 0$, then so is $\begin{bmatrix} f^2(A) & X^* \\ X & g^2(B) \end{bmatrix} \geq 0$.*

The geometric mean of A and $B \in M_n(C)$, defined by $A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$, has the following extremal property[3, Theorem 4.1.3], where $A \geq 0$ and $B \geq 0$.

Lemma 2.5.

$$A \sharp B = \max\left\{X \middle| X^* = X, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\}.$$

The following two Lemmas were deduced by Audeh and Kittaneh [2, Theorems 2.1, 2.4].

Lemma 2.6. *Let A, B and $X \in M_n(C)$ with $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$, then*

$$s_j(X) \leq s_j(A \oplus B)$$

for $j = 1, 2, \cdots, n$.

Lemma 2.7. *Let $A, B \in M_n(C)$, where A is a Hermitian, $B \geq 0$ and $\pm A \leq B$, Then*

$$s_j(A) \leq s_j((A + B) \oplus (B - A)),$$

for $j = 1, 2, \cdots, n$.

The last lemma was due to Bourin and Uchiyama [5, Theorem 1.1].

Lemma 2.8. *Let $A_i \in M_n$ $(i = 1, 2)$ with $A_i \geq 0$ and $f : [0, +\infty) \to [0, +\infty)$ be concave function. Then, for all unitarily invariant norms $\|\cdot\|$,*

$$\|f(A_1 + A_2)\| \leq \|f(A_1) + f(A_2)\|.$$

First, we present a more concise and clear proof to inequality (1). In fact, since

$$\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix} \geq 0 \quad and \begin{bmatrix} |A^*| & A \\ A^* & |A| \end{bmatrix} \geq 0,$$

then

$$\begin{bmatrix} |A| + |A^*| & A + A^* \\ A + A^* & |A| + |A^*| \end{bmatrix} \geq 0. \tag{3}$$

Putting $J = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}$, then J is a unitary matrix. Using the unitarily invariance property for singular values of matrices and Lemma 2.3 for inequality (3), we have

$$\begin{aligned} 2s_j(A + A^*) &\leq s_j\left(\begin{bmatrix} |A| + |A^*| & A + A^* \\ A + A^* & |A| + |A^*| \end{bmatrix}\right) \\ &= s_j\left(J^*\begin{bmatrix} |A| + |A^*| & A + A^* \\ A + A^* & |A| + |A^*| \end{bmatrix}J\right) \\ &= s_j\left(\begin{pmatrix} |A| + |A^*| + A + A^* & 0 \\ 0 & |A| + |A^*| - (A + A^*) \end{pmatrix}\right), \end{aligned}$$

or equivalently,

$$s_j\Big(\frac{A+A^*}{2}\Big) \leq \frac{1}{4}s_j\Big((|A|+|A^*|+A+A^*)\oplus(|A|+|A^*|-(A+A^*))\Big).$$

Next, we give a generalization of inequality (1).

Theorem 2.9. *Let $A, B, X \in M_n(\mathbb{C})$. Then*

$$2s_j(V) \leq s_j((U\sharp W+V)\oplus(U\sharp W-V)), \tag{4}$$

for $j = 1, 2, \cdots, n$, where $U = Af^2(|X|)A^* + Bf^2(|X^*|)B^*$, $V = AX^*B^* + BXA^*$ and $W = Bg^2(|X^*|)B^* + Ag^2(|X|)A^*$.

Proof. Let $X = U|X|$ be the polar decomposition of $X \in M_n(\mathbb{C})$, where U is a unitary matrix. Then $|X| = U^*X$, $|X^*| = U|X|U^*$ and $\pm|X^*|X = \pm U|X|^2 = \pm X|X|$. By Lemma 2.1, we know $\begin{bmatrix} |X| & \pm X^* \\ \pm X & |X^*| \end{bmatrix} \geq 0$. Thus, Lemma 2.4 gives $\begin{bmatrix} f^2(|X|) & \pm X^* \\ \pm X & g^2(|X^*|) \end{bmatrix} \geq 0$, then so is

$$\begin{bmatrix} Af^2(|X|)A^* & \pm AX^*B^* \\ \pm BXA^* & Bg^2(|X^*|)B^* \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} f^2(|X|) & \pm X^* \\ \pm X & g^2(|X^*|) \end{bmatrix}\begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \geq 0. \tag{5}$$

Similarly, since

$$\begin{bmatrix} |X^*| & \pm X \\ \pm X^* & |X| \end{bmatrix} \geq 0,$$

then

$$\begin{aligned} 0 \leq \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} f^2(|X^*|) & \pm X \\ \pm X^* & g^2(|X|) \end{bmatrix}\begin{bmatrix} B^* & 0 \\ 0 & A^* \end{bmatrix} &= \begin{bmatrix} Bf^2(|X^*|) & \pm BX \\ \pm AX^* & Ag^2(|X|) \end{bmatrix}\begin{bmatrix} B^* & 0 \\ 0 & A^* \end{bmatrix} \\ &= \begin{bmatrix} Bf^2(|X^*|)B^* & \pm BXA^* \\ \pm AX^*B^* & Ag^2(|X|)A^* \end{bmatrix}. \end{aligned} \tag{6}$$

Inequalities (5)and (6) give

$$0 \leq \begin{bmatrix} Af^2(|X|)A^* + Bf^2(|X^*|)B^* & \pm(AX^*B^* + BXA^*) \\ \pm(BXA^* + AX^*B^*) & Bg^2(|X^*|)B^* + Ag^2(|X|)A^* \end{bmatrix} = \begin{bmatrix} U & \pm V \\ \pm V & W \end{bmatrix}. \tag{7}$$

Applying Lemma 2.5 to inequality (7), we have

$$\pm V \leq U\sharp W.$$

Thus, the desired inequality (4) follows from Lemma 2.7. This completes the proof. □

Remark 2.10. *Putting $f(t) = g(t) = t^{\frac{1}{2}}$ in inequality (4), we get*

$$2s_j(V) \leq s_j((U_1+V)\oplus(U_1-V)), \tag{8}$$

for $j = 1, 2, \cdots, n$, where $U_1 = A|X|A^* + B|X^*|B^*$ and $V = AX^*B^* + BXA^*$. It should be mentioned that inequality (8) was obtained by Audeh and Kittaneh [2, Theorem 2.7]. Thus, inequality (4) is a generalization of inequality (8).

Remark 2.11. *Let $A \in M_n(\mathbb{C})$ and $A = U\Sigma V^*$ be the singular values decomposition of A, where $\Sigma = diag(s_1(A), s_2(A), \cdots, s_n(A))$, U and V are unitary matrices. Then by inequality (8), we get*

$$\begin{aligned} 4s_j\Big(\frac{A+A^*}{2}\Big) &= 2s_j(U\Sigma V^* + V\Sigma U^*) \\ &\leq s_j\Big(\big(U\Sigma U^* + V\Sigma V^* + U\Sigma V^* + V\Sigma U^*\big)\oplus\big(U\Sigma U^* + V\Sigma V^* - (U\Sigma V^* + V\Sigma U^*)\big)\Big) \\ &= s_j\Big(\big(|A|+|A^*|+A+A^*\big)\oplus\big(|A|+|A^*|-(A+A^*)\big)\Big). \end{aligned}$$

This is just inequality (1). In this sense, inequality (4) is also a generalization of inequality (1).

Remark 2.12. *Putting* $f(t) = g(t) = t^{\frac{1}{2}}$ *in inequality (7), we can easily get*

$$\pm(AX^*B^* + BXA^*) \leq A|X|A^* + B|X^*|B^*,$$

then by (4), we obtain

$$s_j(AX^*B^* + BXA^*) \leq s_j((A|X|A^* + B|X^*|B^*) \oplus (A|X|A^* + B|X^*|B^*)) \tag{9}$$

for $j = 1, 2, \cdots, n$*. Thus, inequality (4) can be considered as a refinement of* $s_j(AB^* + B^*A) \leq s_j\Big((AA^* + BB^*) \oplus (AA^* + BB^*)\Big)$ *obtained by Bhatia and Kittaneh [4].*

Replacing X by $-iX$ in inequality (4), we obtain the following corollary.

Corollary 2.13. *Let* $A, B, X \in M_n(C)$*. Then*

$$2s_j(V_1) \leq s_j((U\sharp W + iV_1) \oplus (U\sharp W - iV_1)), \tag{10}$$

for $j = 1, 2, \cdots, n$*, where* $U = Af^2(|X|)A^* + Bf^2(|X^*|)B^*$*,* $V_1 = AX^*B^* - BXA^*$*,* $W = Bg^2(|X^*|)B^* + Ag^2(|X|)A^*$ *and* $i^2 = -1$*.*

Remark 2.14. *Let* $A \in M_n(C)$ *and* $A = U\Sigma V^*$ *be the singular values decomposition of A, where* $\Sigma = diag(s_1(A), s_2(A), \cdots, s_n(A))$*, U and V are unitary matrices. Putting* $f(t) = g(t) = t^{\frac{1}{2}}$ *in inequality (10), then we get*

$$\begin{aligned} 4s_j\Big(\frac{A - A^*}{2i}\Big) &= 4s_j\Big(\frac{(-iA) + (-iA)^*}{2}\Big) \\ &= 2s_j\Big(i(A^* - A)\Big) \\ &= 2s_j\Big(i(V\Sigma U^* - U\Sigma V^*)\Big) \\ &= 2s_j\Big(V\Sigma U^* - U\Sigma V^*\Big) \\ &\leq s_j\Big(\Big(U\Sigma U^* + V\Sigma V^* + i(V\Sigma U^* - U\Sigma V^*)\Big) \oplus \Big(U\Sigma U^* + V\Sigma V^* - i(V\Sigma U^* - U\Sigma V^*)\Big)\Big) \\ &= s_j\Big(\Big(|A| + |A^*| + i(A^* - A)\Big) \oplus \Big(|A| + |A^*| - i(A^* - A)\Big)\Big). \end{aligned}$$

This is just inequality (2). Therefore, inequality (10) is a generalization of inequality (2).

Remark 2.15. *Chen and Zhang [6] pointed out that the inequality*

$$s_j\Big(\frac{A + A^*}{2}\Big) \leq \frac{1}{2} s_j\Big(|A| + |A^*|\Big)$$

does not hold for $j = 1, 2, \cdots, n$*, where* $A \in M_n(C)$*. However, we have the following weak form:*

$$\prod_{j=1}^{k} s_j(A + A^*) \leq \prod_{j=1}^{k} s_j(|A| + |A^*|), \tag{11}$$

for $k = 1, 2, \cdots, n$*. Actually, by Lemma 2.2 and inequality (3), there exists a contraction W such that* $A + A^* = (|A| + |A^*|)^{\frac{1}{2}} W (|A| + |A^*|)^{\frac{1}{2}}$*, then by Horn's result [10, Theorem 4.6], we get the above inequality (11).*

In the sequel, we present a singular values inequality of $m-$tuples for sums and products of matrices.

Theorem 2.16. *Let* A_j*,* B_j *and* $X_j \in M_n(C)$ *(*$j = 1, 2, \cdots, m$*) and* f*,* g *be two nonnegative continuous functions on* $[0, +\infty)$ *with* $f(t)g(t) = t$ *for* $t \in [0, +\infty)$*. Then*

$$s_j\Big(\sum_{j=1}^{m} A_j^* X_j^* B_j\Big) \leq s_j\Big(\sum_{j=1}^{m} \Big(A_j^* f^2(|X_j|) A_j \oplus B_j^* g^2(|X_j^*|) B_j\Big)\Big),$$

for $j = 1, 2, \cdots, n$*.*

Proof. Let $X = U|X|$ be the polar decomposition of $X \in M_n(C)$, where U is a unitary matrix. Then $|X| = U^*X$, $|X^*| = U|X|U^*$ and $|X^*|X = U|X|^2 = X|X|$. Since $\begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \geq 0$, by Lemma 2.4, we get $\begin{bmatrix} f^2(|X|) & X^* \\ X & g^2(|X^*|) \end{bmatrix} \geq 0$, then so is

$$\begin{bmatrix} A^*f^2(|X|)A & A^*X^*B \\ B^*XA & B^*g^2(|X^*|)B \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} f^2(|X|) & X^* \\ X & g^2(|X^*|) \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \geq 0.$$

Hence, Lemma 2.6 gives

$$s_j(A^*X^*B) \leq s_j(A^*f^2(|X|)A \oplus B^*g^2(|X^*|)B), \tag{12}$$

for $j = 1, 2, \cdots, n$, where $A, B \in M_n(C)$.

Putting $A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_m & 0 & \cdots & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_m & 0 & \cdots & 0 \end{bmatrix}$ and $X = \bigoplus_{j=1}^{m} X_j$,

then $A^*X^*B = \begin{bmatrix} \sum_{j=1}^{m} A_j^*X_j^*B_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $A^*f^2(|X|)A = \begin{bmatrix} \sum_{j=1}^{m} A_j^*f^2(|X_j|)A_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

and $B^*g^2(|X^*|)B = \begin{bmatrix} \sum_{j=1}^{m} B_j^*g^2(|X_j^*|)B_j & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$,

by inequality (12), we have

$$s_j\Big(\sum_{j=1}^{m} A_j^*X_j^*B_j\Big) \leq s_j\Big(\sum_{j=1}^{m} \Big(A_j^*f^2(|X_j|)A_j \oplus B_j^*g^2(|X_j^*|)B_j\Big)\Big).$$

This completes the proof. □

As an application of Theorem 2.16 and Lemma 2.8, we have the following inequality for unitarily invariant norms.

Corollary 2.17. *Let A_j, B_j and $X_j \in M_n(C)$ $(j = 1, 2, \cdots, m)$ and f, g be two nonnegative continuous functions on $[0, +\infty)$ with $f(t)g(t) = t$ for $t \in [0, +\infty)$. If $h : [0, +\infty) \to [0, +\infty)$ be an increasing concave function. Then, for all unitarily invariant norms $\|\cdot\|$,*

$$\Big\|h\Big(\Big|\sum_{j=1}^{m} A_j^*X_j^*B_j\Big|\Big)\Big\| \leq \Big\|\sum_{j=1}^{m} \Big(h\Big(A_j^*f^2(|X_j|)A_j\Big) \oplus h\Big(B_j^*g^2(|X_j^*|)B_j\Big)\Big)\Big\|. \tag{13}$$

The following results are the special case of inequality (13).

Corollary 2.18. *Let A_j, B_j and $X_j \in M_n(C)(j = 1, 2, \cdots, m)$ and f, g be two nonnegative continuous functions on $[0, +\infty)$ with $f(t)g(t) = t$ for $t \in [0, +\infty)$. Then, for all unitarily invariant norms $\|\cdot\|$,*

$$\Big\|\Big|\sum_{j=1}^{m} A_j^*X_j^*B_j\Big|^r\Big\| \leq \Big\|\sum_{j=1}^{m} \Big(\Big(A_j^*f^2(|X_j|)A_j\Big)^r \oplus \Big(B_j^*g^2(|X_j^*|)B_j\Big)^r\Big)\Big\|,$$

in particular,

$$\Big\|\Big|\sum_{j=1}^{m} A_j^*X_j^*B_j\Big|^r\Big\|_\infty \le \max\Big\{\Big\|\sum_{j=1}^{m}\Big(A_j^*f^2(|X_j|)A_j\Big)^r\Big\|_\infty, \Big\|\sum_{j=1}^{m}\Big(B_j^*g^2(|X_j^*|)B_j\Big)^r\Big\|_\infty\Big\},$$

and

$$\Big\|\Big|\sum_{j=1}^{m} A_j^*X_j^*B_j\Big|^r\Big\|_p \le \max\Big(\Big\|\sum_{j=1}^{m}\Big(A_j^*f^2(|X_j|)A_j\Big)^r\Big\|_p^p + \Big\|\sum_{j=1}^{m}\Big(B_j^*g^2(|X_j^*|)B_j\Big)^r\Big\|_p^p\Big)^{\frac{1}{p}},$$

where $0 < r \le 1$ *and* $p \ge 1$.

Remark 2.19. *Taking* $r = 1$ *in Corollary 2.18, we get Corollary 2.7 obtained by Alfakhr and Omidvar [1, Corollary 2.7].*

Corollary 2.20. *Let* $A_j, B_j \in M_n(C)(j = 1, 2, \cdots, m)$ *and* $h : [0, +\infty) \to [0, +\infty)$ *be an increasing concave function. Then, for all unitarily invariant norms* $\|\cdot\|$,

$$\Big\|h\Big(\Big|\sum_{j=1}^{m} A_j^*B_j\Big|\Big)\Big\| \le \Big\|\sum_{j=1}^{m}\Big(h\Big(A_j^*A_j\Big) \oplus h\Big(B_j^*B_j\Big)\Big)\Big\|.$$

Corollary 2.21. *Let* $X_j \in M_n(C)(j = 1, 2, \cdots, m)$ *and* f, g *be two nonnegative continuous functions on* $[0, +\infty)$ *with* $f(t)g(t) = t$ *for* $t \in [0, +\infty)$. *If* $h : [0, +\infty) \to [0, +\infty)$ *be an increasing concave function. Then, for all unitarily invariant norms* $\|\cdot\|$,

$$\Big\|h\Big(\Big|\sum_{j=1}^{m} X_j^*\Big|\Big)\Big\| \le \Big\|\sum_{j=1}^{m}\Big(h\Big(|X_j|\Big) \oplus h\Big(|X_j^*|\Big)\Big)\Big\|,$$

especially,

$$\Big\|\sum_{j=1}^{m} X_j^*\Big\| \le \Big\|\sum_{j=1}^{m}\Big(|X_j| \oplus |X_j^*|\Big)\Big\|.$$

Proof. This result will follows by putting $f(t) = g(t) = t^{\frac{1}{2}}$ and $A_j = B_j = I_n(j = 1, 2, \cdots, m)$ in Corollary 2.17. This completes the proof. □

Corollary 2.22. *Let* A_j *and* $X_j \in M_n(C)$ $(j = 1, 2, \cdots, m)$ *and* $h : [0, +\infty) \to [0, +\infty)$ *be an increasing concave function. Then, for all unitarily invariant norms* $\|\cdot\|$,

$$\Big\|h\Big(\Big|\sum_{j=1}^{m} A_j^*X^*A_{j+1}\Big|\Big)\Big\| \le \sum_{j=1}^{m}\Big\|h\Big(|X_j|^{\frac{1}{2}}|A_j^*|^2|X_j|^{\frac{1}{2}}\Big) \oplus h\Big(|X_j^*|^{\frac{1}{2}}|A_j^*|^2|X_j^*|^{\frac{1}{2}}\Big)\Big\|,$$

where $A_{m+1} = A_1$. *In particular, If* $X_j = X \in M_n(C)(j = 1, 2, \cdots, m)$ *are positive semidefinite matrices, then*

$$\Big\|h\Big(\Big|\sum_{j=1}^{m} A_j^*XA_{j+1}\Big|\Big)\Big\| \le \sum_{j=1}^{m}\Big\|h\Big(X^{\frac{1}{2}}|A_j^*|^2X^{\frac{1}{2}}\Big) \oplus h\Big(X^{\frac{1}{2}}|A_j^*|^2X^{\frac{1}{2}}\Big)\Big\|.$$

Proof. Let $T = U|T|$ be the polar decomposition of $T \in M_n(C)$, where U is a unitary matrix. Then $TT^* = U|T||T|U^* = UT^*TU^*$, and hence $h(TT^*) = Uh(T^*T)U^*$. So inequality (13) entails

$$\begin{aligned}
\left\|h\Big(\Big|\sum_{j=1}^{m} A_j^* X^* A_{j+1}\Big|\Big)\right\| &\leq \left\|\sum_{j=1}^{m}\Big(h\big(A_j^*|X_j|A_j\big) \oplus h\big(A_j^*|X_j^*|A_j\big)\Big)\right\| \\
&\leq \sum_{j=1}^{m}\left\|h\big(A_j^*|X_j|A_j\big) \oplus h\big(A_j^*|X_j^*|A_j\big)\right\| \\
&= \sum_{j=1}^{m}\left\|h\big(A_j^*|X_j|A_j \oplus A_j^*|X_j^*|A_j\big)\right\| \\
&= \sum_{j=1}^{m}\left\|h\Big(|X_j|^{\frac{1}{2}}|A_j^*|^2|X_j|^{\frac{1}{2}} \oplus |X_j^*|^{\frac{1}{2}}|A_j^*|^2|X_j^*|^{\frac{1}{2}}\Big)\right\| \\
&= \sum_{j=1}^{m}\left\|h\Big(|X_j|^{\frac{1}{2}}|A_j^*|^2|X_j|^{\frac{1}{2}}\Big) \oplus h\Big(|X_j^*|^{\frac{1}{2}}|A_j^*|^2|X_j^*|^{\frac{1}{2}}\Big)\right\|.
\end{aligned}$$

This completes the proof. □

Remark 2.23. *Putting $h(x) = x$ in Corollary 2.22, we get Corollary 2.10 in [1].*

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