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The λ-Aluthge Transform and its Applications to Some Classes of Operators

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Abstract. Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$, and let $T = U|T|$ be its polar decomposition. Then, for every $\lambda \in [0,1]$ the λ-Aluthge transform of T is defined by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$. In this paper, we characterize the invertible, binormal, and EP operators and its intersection with a special class of introduced operators via the λ-Aluthge transform.

1. Introduction and preliminaries

let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* for the range, the null subspace and the adjoint operator of T, respectively. For any closed subspace M of $\mathcal{H}$, let P_M denote the orthogonal projection onto M.
Recall that for $T \in \mathcal{B}(\mathcal{H})$, there is a unique factorization $T = U|T|$, where $\mathcal{N}(U) = \mathcal{N}(T) = \mathcal{N}(|T|)$, U is a partial isometry, i.e. $UU^*U = U$ and $|T| = (T^*T)^{\frac{1}{2}}$ is the modulus of T. This factorization is called the polar decomposition of T. It is known that if T is invertible then U is unitary and $|T|$ is also invertible. From the polar decomposition, the Aluthge transform of T is defined by

$$\Delta(T) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(\mathcal{H}).$$

This transform was introduced in [1] by Aluthge, in order to study p-hyponormal and log-hyponormal operators. In [16], Okubo introduced a more general notion called λ-Aluthge transform which has later been studied also in detail. This is defined for any $\lambda \in [0,1]$ by

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}, \quad T \in \mathcal{B}(\mathcal{H}).$$

Clearly, for $\lambda = \frac{1}{2}$ we obtain the usual Aluthge transform. Also, $\Delta_1(T) = |T|U$ is known as Duggal's transform.
These transforms have been studied in many different contexts and considered by a number of authors (see for instance, [1, 9, 11, 13, 16, 17]). One of the interests of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example T has a nontrivial invariant subspace if an only if

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$\Delta(T)$ does (see [11]). Another important property is that T and $\Delta_\lambda(T)$ have the same spectrum (see [11]). So T is invertible if and only if $\Delta_\lambda(T)$ is invertible, and in this case they are similar. It would be certainly interesting to know which invertible operators in $\mathcal{B}(\mathcal{H})$ satisfy $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$. Recently, the answer to this problem in the case of matrices was given in [17]. In this paper we obtain new results related to this problem for bounded linear operators.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $TT^* = T^*T$, quasinormal if T commutes with T^*T or equivalently $U|T| = |T|U$, where $T = U|T|$ is the polar decomposition of T. The operator T is called binormal if TT^* and T^*T commute. Binormality of operators was defined by Campbell in [2], It is easy to see that normal $\Longrightarrow$ quasinormal $\Longrightarrow$ binormal and the inverse implications do not hold. However, every invertible quasinormal operator is normal. Also, in finite dimensional spaces every quasinormal operator is normal. The Aluthge transform is designed as a measure for the normality of an operator, this is justified by the fact that $\Delta_\lambda(T) = T$ if and only if T is quasi-normal [11].
Let $T = U|T|$, be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. Throughout the remainder of this paper, we denote by $\delta(\mathcal{H})$ the class of operator $T \in \mathcal{B}(\mathcal{H})$ which satisfies $U^2|T| = |T|U^2$. This class was introduced in [12], in order to study the relationship between a hyponormal operator and its mean transform. Clearly, quasinormal operators belong to $\delta(\mathcal{H})$ but the converse is not true in genaral. In section 2 of this paper, firstly, we provide a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal. Secondly, we show that an invertible operator T belongs to the class $\delta(\mathcal{H})$ if and only if $\Delta_1(T^{-1}) = (\Delta_1(T))^{-1}$. Afterwards, we give examples and discuss how this class of operators is distinct from the class of binormal operators. We prove that, if T is invertible in $\delta(\mathcal{H})$, then T is binormal if and only if $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, for $\lambda \in]0, 1[$. In [9] , Ito, Yamazaki, Yanagida prove that The binormality of an operator in $\mathcal{B}(\mathcal{H})$ does not imply the binormality of its Aluthge transform. However, the binormality of an invertible operator implies the binormality of its Duggal transform [13] . In the last part of this section, we show that if T is binormal in $\delta(\mathcal{H})$ such that the partial isometry factor U of its polar decomposition is unitary, then $\Delta_\lambda(T)$ is binormal, for any $\lambda \in]0, 1[$.

Now, we recall the notion of the Moore-Penrose inverse that will be used in section 3 of this paper. For $T \in \mathcal{B}(\mathcal{H})$, the Moore-Penrose inverse of T is the unique operator $T^+ \in \mathcal{B}(\mathcal{H})$ which satisfies:

$$TT^+T = T, \ \ T^+TT^+ = T^+, \ \ (TT^+)^* = TT^+, \ \ (T^+T)^* = T^+T.$$

It is well known that the Moore-Penrose inverse of T exists if and only if $\mathcal{R}(T)$ is closed. It is easy to see that $\mathcal{R}(T^+) = \mathcal{R}(T^*)$, TT^+ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$ and that T^+T is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T^*)$. The operator T is said to be EP operator, if $\mathcal{R}(T)$ is closed and $TT^+ = T^+T$. Clearly

$$T \text{ EP} \Longleftrightarrow \mathcal{R}(T) = \mathcal{R}(T^*) \Longleftrightarrow \mathcal{N}(T) = \mathcal{N}(T^*).$$

Obviously, every normal operator with closed range is EP but the converse is not true even in a finite dimensional space. For more details about on EP operators see [3, 5].
Most results on the λ-Aluthge transform show that it generally has better properties than its original operator. However, an operator $T \in \mathcal{B}(\mathcal{H})$ may have a closed range without $\Delta_\lambda(T)$ having a closed range as shown in example 3.2. In section 3, firstly, we shall show a necessary and sufficient condition for the range of $\Delta_\lambda(T)$ to be closed. Secondly, we investigate when an operator and its λ-Aluthge transform both are EP. Finally, we give a formula for the Moore-Penrose inverse of $\Delta_\lambda(T)$ when T is a binormal operator with closed range and then show under some conditions that T^+ is nilpotent of order $d + 1$ if and only if $(\Delta_\lambda(T)^+)^d = 0$.

Now we state some known properties of the polar decomposition, needed in the sequel. If $T = U|T|$ is the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$, then

$$UU^* = P_{\overline{\mathcal{R}(T)}} = P_{\overline{\mathcal{R}(|T^*|)}} \quad \text{and} \quad U^*U = P_{\overline{\mathcal{R}(T^*)}} = P_{\overline{\mathcal{R}(|T|)}}.$$

Moreover, we have

$P(1)$ $T^* = U^*|T^*|$ is the polar decomposition of T^*;

$P(2)$ $U|T|^{\alpha} = |T^*|^{\alpha}U$, for any $\alpha \geq 0$. Indeed let (q_n) be a sequence of polynomials such that $q_n(t) \to t^{\frac{1}{\alpha}}$ uniformly on $\sigma(|T|) \cup \sigma(|T^*|)$ as $n \to 0$. From $P(1)$, we have $U|T| = |T^*|U$ and so $Uq_n(|T|) = q_n(|T^*|)U$. Hence $U|T|^{\alpha} = |T^*|^{\alpha}U$. This property is trivial in case $\alpha = 0$.

$P(3)$ If T is invertible,
- (i) $T^{-1} = U^*|T^{-1}|$ is the polar decomposition of T^{-1};
- (ii) $|T^{-1}| = |T^*|^{-1}$;
- (iii) $|T|^{-\alpha} = U^*|T^{-1}|^{\alpha}U$, for $\alpha > 0$, (it follows from $P(2)$ and $P(3)$ (ii)).

2. On the class $\delta(\mathcal{H})$, binormal operators and λ-Aluthge transform

In this section, first we give a condition under which an operator in $\delta(\mathcal{H})$ becomes quasinormal.

Proposition 2.1. *Let n be a positive integer and $T \in \delta(\mathcal{H})$, with polar decomposition $T = U|T|$. If $U^{2n+1} = I$, then T is quasinormal.*

Proof. From $U^2|T| = |T|U^2$, we get $U^{2n}|T| = |T|U^{2n}$. This implies $U^{2n+1}|T|U = U|T|U^{2n+1}$. If $U^{2n+1} = I$, then $U|T| = |T|U$. Hence, T is quasinormal.
□

The following is a characterization of invertible operators in $\delta(\mathcal{H})$ via Duggal transform.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$T \in \delta(\mathcal{H}) \iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}.$$

Proof. Suppose that $T = U|T|$ is the polar decomposition of T. Since T is invertible, it follows that

$$T \in \delta(\mathcal{H}) \iff U^2|T| = |T|U^2 \iff U^2|T|^{-1} = |T|^{-1}U^2.$$

By $P(3)$ (iii), $U^2|T|^{-1} = U^2U^*|T^{-1}|U$. Since U is unitary, then

$$\begin{aligned} T \in \delta(\mathcal{H}) &\iff U^2U^*|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}|U = |T|^{-1}U^2 \\ &\iff U|T^{-1}| = |T|^{-1}U \\ &\iff |T^{-1}|U^* = U^*|T|^{-1} \\ &\iff \Delta_1(T^{-1}) = (\Delta_1(T))^{-1}. \end{aligned}$$

□

Example 2.3. *Proposition 2.2 is not valid when the Duggal transform is replaced by the Aluthge transform. To see this let $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where A and B are invertible positive operators such that $AB \neq BA$. Then T is invertible and*

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = U|T|$$

is the polar decomposition of T. Since $U^2 = I$, it follows that $U^2|T| = |T|U^2$ and so $T \in \delta(\mathcal{H} \oplus \mathcal{H})$. On the other hand, since

$$\Delta(T) = \begin{pmatrix} 0 & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & 0 \end{pmatrix}, \text{ we obtain } (\Delta(T))^{-1} = \begin{pmatrix} 0 & B^{-\frac{1}{2}}A^{-\frac{1}{2}} \\ A^{-\frac{1}{2}}B^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Using $P(3)$ (i) and (ii), we have

$$\Delta(T^{-1}) = |T^{-1}|^{\frac{1}{2}}U^*|T^{-1}|^{\frac{1}{2}} = |T^*|^{-\frac{1}{2}}U^*|T^*|^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}}A^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Hence $\Delta(T^{-1}) \neq (\Delta(T))^{-1}$.

It is well known that every quasi-normal operator is binormal. Hence one might expect that there is a relationship between $\delta(\mathcal{H})$ and binormal operators. But in the example 2.3, $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \delta(\mathcal{H} \oplus \mathcal{H})$ and T is not binormal because $AB \neq BA$.
Next, we shall show the following result on the binormality of an invertible operator T in $\delta(\mathcal{H})$.

Theorem 2.4. *Let $T \in \delta(\mathcal{H})$ be an invertible operator. Then the following statements are equivalent.*

(1) *T is binormal.*
(2) *$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for all $\lambda \in]0,1[$.*
(3) *$\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for some $\lambda \in]0,1[$.*

Proof. First, if $T \in \delta(\mathcal{H})$, then by the functional calculus, we obtain $U^2|T|^\lambda = |T|^\lambda U^2$ for all $\lambda > 0$. This implies $U|T^*|^\lambda U = |T|^\lambda U^2$, by $P(2)$. Multiplying this equality by U^* on the right side and since U is unitary, we get

$$U|T^*|^\lambda = |T|^\lambda U \quad for\ all\ \lambda \in]0,1[. \tag{1}$$

(1) $\Longrightarrow$ (2). Suppose that T is binormal and invertible. From $P(2)$, we get

$$\begin{aligned}\left(\Delta_\lambda(T^{-1})\right)^{-1} &= \left(|T^*|^{-\lambda}U^*|T^*|^{-(1-\lambda)}\right)^{-1} \\ &= |T^*|^{1-\lambda}U|T^*|^\lambda \\ &= U|T|^{1-\lambda}|T^*|^\lambda.\end{aligned}$$

Since T is binormal, then $|T||T^*| = |T^*||T|$. Also by functional calculus, we get $|T|^{1-\lambda}|T^*|^\lambda = |T^*|^\lambda|T|^{1-\lambda}$ for $\lambda \in]0,1[$. Then, by using this equality and (1), we deduce that

$$\begin{aligned}\left(\Delta_\lambda(T^{-1})\right)^{-1} &= U|T^*|^\lambda|T|^{1-\lambda} \\ &= |T|^\lambda U|T|^{1-\lambda} \\ &= \Delta_\lambda(T).\end{aligned}$$

Hence, $(\Delta_\lambda(T))^{-1} = \Delta_\lambda(T^{-1})$, for all $\lambda \in]0,1[$.
(2) $\Longrightarrow$ (3). Trivial.
(3) $\Longrightarrow$ (1). Assume that $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$ for some $\lambda \in]0,1[$. From (1), we obtain

$$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda} = U|T^*|^\lambda|T|^{1-\lambda}.$$

On the other hand, by $P(2)$ we have

$$(\Delta_\lambda(T^{-1}))^{-1} = U|T|^{1-\lambda}|T^*|^\lambda.$$

Using our assumption, we get that

$$U|T^*|^\lambda|T|^{1-\lambda} = U|T|^{1-\lambda}|T^*|^\lambda, \quad for\ some\ \lambda \in]0,1[.$$

Since U is unitary, we conclude that

$$|T^*|^\lambda|T|^{1-\lambda} = |T|^{1-\lambda}|T^*|^\lambda, \quad for\ some\ \lambda \in]0,1[.$$

By the continuous functional calculus, we obtain $|T^*||T| = |T||T^*|$. So T is binormal.
□

The following corollary generalizes one implication of [17, Theorem 3.7] to infinite-dimensional Hilbert space.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. Then*

$$\Delta(T) = T \implies \Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}, \text{ for all } \lambda \in]0,1[.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Since $\Delta(T) = T$, then T is normal. It follows that $U|T| = |T|U$ and so $U^2|T| = |T|U^2$. Hence, $T \in \delta(\mathcal{H})$. Moreover, since T is normal, T is binormal and by Theorem 2.4, we deduce that $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, for all $\lambda \in]0,1[$.
□

Remark 2.6. *In Corollary 2.5, the reverse implication is false in infinite-dimensional Hilbert space as shown by the following example.*

Example 2.7. *Let $T = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where $P \geq 0$ and $P \neq I$ is invertible. The polar decomposition of T is $T = U|T|$, where*

$$|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad U = T|T|^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

For any $\lambda \in]0,1[$, we have

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U |T|^{1-\lambda} \\ &= \begin{pmatrix} P^\lambda & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} P^{1-\lambda} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^\lambda \\ P^{1-\lambda} & 0 \end{pmatrix}. \end{aligned}$$

It follows that

$$(\Delta_\lambda(T))^{-1} = \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}.$$

Also we have

$$\begin{aligned} \Delta_\lambda(T^{-1}) &= |T^*|^{-\lambda} U^* |T^*|^{-(1-\lambda)} \\ &= \begin{pmatrix} I & 0 \\ 0 & P^{-\lambda} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^{-(1-\lambda)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & P^{-(1-\lambda)} \\ P^{-\lambda} & 0 \end{pmatrix}. \end{aligned}$$

Hence, $\Delta_\lambda(T^{-1}) = (\Delta_\lambda(T))^{-1}$, while $\Delta_\lambda(T) \neq T$.

The following is an example of a binormal operator which is not in $\delta(\mathcal{H})$.

Example 2.8. *Consider $T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^4$. Then T is invertible and binormal since*

$$TT^*T^*T = T^*TTT^* = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

By a direct calculation, we have

$$|T| = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ |T^*| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \ \text{and} \ U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $U^2|T| \neq |T|U^2$, *then* $T \notin \delta(\mathcal{H})$. *Moreover, since*

$$\Delta_\lambda(T) = \begin{pmatrix} 0 & 0 & 2^\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2^{1-\lambda} & 0 & 0 & 0 \end{pmatrix}, \ \text{then} \ (\Delta_\lambda(T))^{-1} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 2^{-\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Also, we have

$$\Delta_\lambda(T^{-1}) = |T^*|^{-\lambda} U^* |T^*|^{-(1-\lambda)} = \begin{pmatrix} 0 & 0 & 0 & 2^{-(1-\lambda)} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2^{-\lambda} & 0 & 0 \end{pmatrix}.$$

Hence, $\Delta_\lambda(T^{-1}) \neq (\Delta_\lambda(T))^{-1}$ *for any* $\lambda \in]0,1[$.

Now, we provide equivalent conditions under which an invertible binormal operator belongs to $\delta(\mathcal{H})$.

Theorem 2.9. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be an invertible binormal operator and* $T = U|T|$ *be its polar decomposition. Then the following statements are equivalent.*

1. $T \in \delta(\mathcal{H})$.
2. $\Delta(T^{-1}) = (\Delta(T))^{-1}$.
3. $U\Delta(T) = \Delta(T)U$.

Proof. (1) $\Rightarrow$ (2). The proof follows from Theorem 2.4.
(2) $\Rightarrow$ (3). Since T is invertible, U is unitary. Using $P(3)$ (iii), we get

$$\begin{aligned} \Delta(T^{-1})U\Delta(T) &= |T^{-1}|^{\frac{1}{2}} U^* |T^{-1}|^{\frac{1}{2}} U|T|^{\frac{1}{2}} U|T|^{\frac{1}{2}} \\ &= |T^{-1}|^{\frac{1}{2}} |T|^{-\frac{1}{2}} |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}} \\ &= |T^{-1}|^{\frac{1}{2}} U|T|^{\frac{1}{2}} \\ &= UU^*|T^{-1}|^{\frac{1}{2}} U|T|^{\frac{1}{2}} \\ &= U|T|^{-\frac{1}{2}} |T|^{\frac{1}{2}} \\ &= U. \end{aligned}$$

Thus, the condition $\Delta(T^{-1}) = (\Delta(T))^{-1}$ implies that $U\Delta(T) = \Delta(T)U$.
(3) $\Rightarrow$ (1). Assume that $U\Delta(T) = \Delta(T)U$. Then we have

$$\begin{aligned} U|T|^{\frac{1}{2}} U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}} U &\Longrightarrow U|T|^{\frac{1}{2}} |T^*|^{\frac{1}{2}} U = |T|^{\frac{1}{2}} |T^*|^{\frac{1}{2}} U^2 && \text{by } P(2) \\ &\Longrightarrow |T^*|^{\frac{1}{2}} U|T^*|^{\frac{1}{2}} U = |T^*|^{\frac{1}{2}} |T|^{\frac{1}{2}} U^2 && \text{since } T \text{ is binormal} \\ &\Longrightarrow U|T^*|^{\frac{1}{2}} U = |T|^{\frac{1}{2}} U^2 \\ &\Longrightarrow U^2|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} U^2 && \text{by } P(2) \\ &\Longrightarrow U^2|T| = |T|U^2. \end{aligned}$$

Hence, $T \in \delta(\mathcal{H})$.
□

Finally, we focus on the binormalilty of $\Delta_\lambda(T)$ when T is binormal. In [9], Ito, Yamazaki and Yanagida gave an example of a binormal invertible operator T such that its Aluthge transform $\Delta(T)$ is not binormal. However, it was proved in [13] that if T is a binormal invertible operator, then its Duggal transform is binormal.

Theorem 2.10. *Let $T \in \delta(\mathcal{H})$ and $T = U|T|$ be its polar decomposition. If U is unitary, then for $\lambda \in]0,1[$, we have*

$$T \text{ is binormal} \implies \Delta_\lambda(T) \text{ is binormal}.$$

Proof. Since U is unitary, we obtain that

$$\begin{aligned}
|\Delta_\lambda(T)^*|^2|\Delta_\lambda(T)|^2 &= |T|^\lambda U|T|^{2(1-\lambda)}U^*|T|U^*|T|^{2\lambda}U|T|^{1-\lambda} \\
&= |T|^\lambda|T^*|^{2(1-\lambda)}|T|U^*|T|^{2\lambda}U|T|^{1-\lambda} \quad by \ P(2) \\
&= |T|^\lambda|T^*|^{2(1-\lambda)}|T||T^*|^{2\lambda}U^*U|T|^{1-\lambda} \quad by \ (1) \\
&= |T|^2|T^*|^2 \quad since \ T \ is \ binormal.
\end{aligned}$$

And

$$\begin{aligned}
|\Delta_\lambda(T)|^2|\Delta_\lambda(T)^*|^2 &= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T|U|T|^{2(1-\lambda)}U^*|T|^\lambda \\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T||T^*|^{2(1-\lambda)}UU^*|T|^\lambda \quad by \ P(2) \\
&= |T|^{1-\lambda}U^*|T|^{2\lambda}U|T||T^*|^{2(1-\lambda)}|T|^\lambda \\
&= |T|^{1-\lambda}U^*U|T^*|^{2\lambda}|T||T^*|^{2(1-\lambda)}|T|^\lambda \quad by\,(1) \\
&= |T|^2|T^*|^2 \quad since \ T \ is \ binormal.
\end{aligned}$$

Hence, $\Delta_\lambda(T)$ is binormal. □

Remark 2.11. (*i*) *The reverse implication of the previous Theorem is false. Indeed if we take the example 2.3, we obtain $T \in \delta(\mathcal{H})$ and $\Delta(T)$ is binormal but T is not binormal.*

(*ii*) *If $T \in \mathcal{B}(\mathcal{H})$ is a binormal invertible operator such that $\Delta_\lambda(T)$ is binormal then T need not be in $\delta(\mathcal{H})$. To see this, consider the example 2.8. Then T is binormal and $\Delta_\lambda(T)$ is also binormal as*

$$|\Delta_\lambda(T)^*|^2|\Delta_\lambda(T)|^2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4^\lambda & 0 \\ 0 & 0 & 0 & 4^{1-\lambda} \end{pmatrix} = |\Delta_\lambda(T)|^2|\Delta_\lambda(T)^*|^2$$

but $T \notin \delta(\mathcal{H})$.

3. The λ-Aluthge transform of closed range operators

We start this section by giving a new proof to the following lemma from [15].

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be positive and $\alpha > 0$. Then $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^\alpha)$ is closed. In this case $\mathcal{R}(T) = \mathcal{R}(T^\alpha)$.*

Proof. First, recall that the reduced minimum modulus of an operator $S \in \mathcal{B}(\mathcal{H})$ is defined by

$$\gamma(S) := \begin{cases} \inf\{\|Sx\|; \|x\| = 1, \ x \in \mathcal{N}(S)^\perp\} & if \ S \neq 0 \\ +\infty & if \ S = 0. \end{cases}$$

It is well known that $\gamma(S) > 0$ if and only if S has a closed range [8].
($\Rightarrow$). Suppose that $\mathcal{R}(T)$ is closed and $\mathcal{R}(T^\alpha)$ is not closed, for some $\alpha > 0$. Then $\gamma(T^\alpha) = 0$ and so there existe a sequence of unit vectors $x_n \in \mathcal{N}(T^\alpha)^\perp$ such that $T^\alpha x_n \longrightarrow 0$. Since $\mathcal{N}(T^\alpha) = \mathcal{N}(T)$, $x_n \in \mathcal{N}(T)^\perp$, for all n. In case $\alpha \in]0,1]$, we have

$$Tx_n = T^{1-\alpha}T^{\alpha}x_n \longrightarrow 0,$$

Now, in case $\alpha > 1$, by Hölder-McCarthy inequality, we have

$$\|T^{\frac{1}{2}}x_n\|^{2\alpha} = \langle Tx_n, x_n\rangle^{\alpha} \leq \langle T^{\alpha}x_n, x_n\rangle \leq \|T^{\alpha}x_n\|,$$

for all n. Hence $T^{\frac{1}{2}}x_n \longrightarrow 0$, so $Tx_n \longrightarrow 0$. Therefore, in both cases the sequence $(Tx_n)_n$ converges to 0, which is a contradiction with the fact that $\mathcal{R}(T)$ is closed.
($\Leftarrow$). Suppose that $\mathcal{R}(T^{\alpha})$ is closed, for $\alpha > 0$, by the above implication we obtain $R((T^{\alpha})^{\frac{1}{\alpha}}) = R(T)$ is also closed and then $R(T) = R(T^{\alpha})$.
□

The λ-Aluthge transform preserves many properties of the original operator. However, an operator $T \in \mathcal{B}(\mathcal{H})$ may have a closed range without $\Delta_\lambda(T)$ having a closed range as shown by the following example.

Example 3.2. *Let* $T = \begin{pmatrix} A & 0 \\ (I - A^*A)^{\frac{1}{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, *where* A *is a contraction and* $\mathcal{R}(A)$ *is not closed. Then*

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

is an orthogonal projection. Hence T *is a partial isometry. This implies that* $\mathcal{R}(T)$ *is closed and* $T = T|T| = TT^*T$ *is the polar decomposition of* T. *Therefore, for* $\lambda \in]0, 1]$, *we have*

$$\Delta_\lambda(T) = (T^*T)^{\lambda}\, T(T^*T)^{1-\lambda} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

So $\mathcal{R}(\Delta_\lambda(T))$ *is not closed.*

The next result provide a necessary and sufficient condition for the range of $\Delta_\lambda(T)$ to be closed .

Proposition 3.3. *Let* $\lambda \in]0, 1]$ *and* $T \in \mathcal{B}(\mathcal{H})$ *with closed range. Let* P *be an idempotent with range* $\mathcal{R}(T)$ *and* Q *be an idempotent with kernel* $\mathcal{N}(T)$. *Then*

$$\mathcal{R}(\Delta_\lambda(T)) \text{ is closed if and only if } \mathcal{R}(QP) \text{ is closed.}$$

Proof. Assume that $R(T)$ is closed. Since $\mathcal{R}(P) = \mathcal{R}(T)$ and $\mathcal{N}(Q) = \mathcal{N}(T)$, then for $\lambda \in]0, 1]$, we get

$$\begin{aligned}
\mathcal{R}(\Delta_\lambda(T)) \text{ is closed} &\iff \mathcal{R}(|T|^{\lambda}U|T|^{1-\lambda}) \text{ is closed} \\
&\iff \mathcal{R}(|T|^{\lambda}|T^*|^{1-\lambda}U) \text{ is closed} \qquad \text{by } P(2) \\
&\iff |T|^{\lambda}|T^*|^{1-\lambda}\mathcal{R}(|T^*|) \text{ is closed} \\
&\iff |T|^{\lambda}|T^*|^{1-\lambda}\mathcal{R}(|T^*|^{\lambda}) \text{ is closed} \qquad \text{by lemma } 3.1 \\
&\iff |T|^{\lambda}\mathcal{R}(|T^*|) \text{ is closed} \\
&\iff |T|^{\lambda}\mathcal{R}(T) \text{ is closed} \\
&\iff |T|^{\lambda}\mathcal{R}(P) \text{ is closed} \\
&\iff \mathcal{R}(P^*|T|^{\lambda}) \text{ is closed} \\
&\iff P^*\mathcal{R}(|T|) \text{ is closed} \\
&\iff P^*\mathcal{R}(T^*) \text{ is closed} \\
&\iff \mathcal{R}(P^*Q^*) \text{ is closed} \qquad \text{since } \mathcal{R}(Q^*) = \mathcal{N}(Q)^{\perp} = \mathcal{R}(T^*) \\
&\iff \mathcal{R}(QP) \text{ is closed.}
\end{aligned}$$

□

The following result, which is one of the main results of this section, generalizes Theorems 3.3 and 3.15 obtained for complex matrices in [17] to the closed range operators on an arbitrary Hilbert space.

Theorem 3.4. *For $T \in \mathcal{B}(\mathcal{H})$ with closed range and $\lambda \in]0,1]$, we have*

$$T \text{ is an EP operator} \iff \Delta_\lambda(T) \text{ is EP and } \mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T)).$$

Proof. ($\Longrightarrow$). We assume that T is EP. Then $\mathcal{R}(T)$ is closed and $\mathcal{R}(T) = \mathcal{R}(T^*)$. This implies that $P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)} = P_{\mathcal{R}(T)}$. Then $\mathcal{R}(P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)})$ is closed and by Proposition 3.3, we deduce that $\mathcal{R}(\Delta_\lambda(T))$ is closed. Now we show that $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*)$. Since $\mathcal{R}(T) = \mathcal{R}(T^*)$, it follows that

$$\mathcal{N}(|T|) = \mathcal{N}(T) = \mathcal{N}(T^*) = \mathcal{N}(|T^*|).$$

Since, for $\lambda \in]0,1]$, $\mathcal{N}(|T|) = \mathcal{N}(|T|^\lambda)$ and $\mathcal{N}(|T^*|) = \mathcal{N}(|T^*|^\lambda)$, then we get $\mathcal{N}(|T|^\lambda) = \mathcal{N}(|T^*|^\lambda)$. Let $x \in \mathcal{H}$. Hence, for $\lambda \in]0,1]$ we have

$$\begin{aligned} |T|^\lambda U|T|^{1-\lambda}x = 0 &\iff |T^*|^\lambda U|T|^{1-\lambda}x = 0 \\ &\iff U|T|^\lambda|T|^{1-\lambda}x = 0 \qquad by\ P(2) \\ &\iff Tx = 0 \\ &\iff |T|x = 0 \\ &\iff |T|^\lambda x = 0 \\ &\iff |T|^{1-\lambda}U^*|T|^\lambda x = 0 \\ &\iff \Delta_\lambda(T)^*x = 0. \end{aligned}$$

Therefore, $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*) = \mathcal{N}(T)$. Consequently, $\Delta_\lambda(T)$ is also EP. By taking the orthogonal complements in the relation $\mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(T)$ and since T and $\Delta_\lambda(T)$ are EP, we conclude that $\mathcal{R}(\Delta_\lambda(T)) = \mathcal{R}(T)$.
($\Longleftarrow$). We suppose that $\Delta_\lambda(T)$ is EP. Since $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, then $\mathcal{R}(\Delta_\lambda(T))$ is closed and $\mathcal{R}(\Delta_\lambda(T)) = \mathcal{R}(\Delta_\lambda(T)^*)$. Thus,

$$\mathcal{N}(T^*) = \mathcal{N}(\Delta_\lambda(T)) = \mathcal{N}(\Delta_\lambda(T)^*).$$

Since $\mathcal{N}(T) \subset \mathcal{N}(\Delta_\lambda(T))$, then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. Hence, to prove T is EP, it is enough to prove that $\mathcal{N}(\Delta_\lambda(T)) \subset \mathcal{N}(T)$. Let $x \in \mathcal{N}(\Delta_\lambda(T))$. This implies that $U|T|^{1-\lambda}x \in \mathcal{N}(|T|^\lambda) = \mathcal{N}(T)$. Hence $|T^*|^\lambda U|T|^{1-\lambda}x = 0$, because $\mathcal{N}(T) \subset \mathcal{N}(T^*) = \mathcal{N}(|T^*|^\lambda)$. According to $P(2)$, we get $T(x) = 0$. Therefore $\mathcal{N}(\Delta_\lambda(T)) \subset \mathcal{N}(T)$. Finally T is EP. □

Remark 3.5. *Without the condition $\mathcal{R}(T) = \mathcal{R}(\Delta_\lambda(T))$, the reverse implication does not hold, as the following example shows.*

Example 3.6. *let $T = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\mathcal{R}(T)$ is closed and $T^+ = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. Furthermore, $T^2 = 0$. Hence $\Delta_\lambda(T) = 0$ is EP, while T is not EP because*

$$TT^+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = T^+T.$$

Now we prove a version of Theorem 2.4 for closed range operators.

Corollary 3.7. *Let $T \in \delta(\mathcal{H})$ with closed range. If T is EP, then the following statements are equivalent.*

(1) *T is binormal.*
(2) $\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+$ *for all* $\lambda \in]0,1[$.
(3) $\Delta_\lambda(T^+) = (\Delta_\lambda(T))^+$ *for some* $\lambda \in]0,1[$.

Proof. Since T is an EP operator, then $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ and T has the following matrix form

$$T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where the operator $A : \mathcal{R}(T) \longrightarrow \mathcal{R}(T)$ is invertible. Now it is known that

$$U = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \text{ and } |T| = \begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix},$$

where $A = V|A|$ is the polar decomposition of A. Then for $\lambda \in]0, 1[$

$$\Delta_\lambda(T) = \begin{pmatrix} \Delta_\lambda(A) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } (\Delta(T))^+ = \begin{pmatrix} (\Delta_\lambda(A))^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

also we have

$$T^+ = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \Delta(T^+) = \begin{pmatrix} \Delta_\lambda(A^{-1}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$(\Delta_\lambda(T))^+ = \Delta_\lambda(T^+) \iff (\Delta_\lambda(A))^{-1} = \Delta_\lambda(A^{-1}).$$

Hence, the implications (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), and (3) $\Longrightarrow$ (1) holds by using Theorem 2.4. □

The assumption T is an *EP* operator is necessary in the previous theorem as shown by the following example.

Example 3.8. *Consider the right shift operator S, defined on the Hilbert space $\ell^2(\mathbb{N})$ by $S(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Then $S^*(x_1, x_2, ...) = (x_2, x_3, ...)$ and so $S^*S = I$. Hence S is an isometry, which implies that $S^* = S^+$ and S is not EP because $S^*S \neq SS^*$. Since $|S| = I$, it follows that $S = S|S|$ is the polar decomposition of S and $S \in \delta(\mathcal{H})$. On the other hand a simple calculation shows that*

$$(\Delta(S))^+ = S^+ = S^* \neq SS^*S^* = \Delta(S^+).$$

The next proposition was etablished by Jabbarzadeh and Bakhshkandi in the case $\lambda = \frac{1}{2}$, (see [10, Theorem 2.5]).

Proposition 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$ be binormal with closed range and $T = U|T|$ be its polar decomposition. Then $\mathcal{R}(\Delta_\lambda(T))$ is closed and $(\Delta_\lambda(T))^+ = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$, for all $\lambda \in]0, 1]$.*

In order to prove proposition 3.9, we need the following two lemmas.

Lemma 3.10. *[7, Theorem 2] Let $A, B \geq 0$ and $[A, B] = 0$. Then*

$$[P_{\mathcal{N}(A)^\perp}, P_{\mathcal{N}(B)^\perp}] = [P_{\mathcal{N}(A)^\perp}, B] = [A, P_{\mathcal{N}(B)^\perp}] = 0,$$

where $[T, S] = TS - ST$ for T and S in $\mathcal{B}(\mathcal{H})$.

Lemma 3.11. *[6] Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(i) $(AB)^+ = B^+A^+$.

(ii) $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$.

Proof. (Proposition 3.9) First we show that $\mathcal{R}(\Delta_\lambda(T))$ is closed for $\lambda \in]0, 1]$.
Since $\mathcal{R}(T)$ is closed and T is binormal, then $P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)} = P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}$, by Lemma 3.10. Therefore $P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)}$ is a projection, it follows that $\mathcal{R}(P_{\mathcal{R}(T^*)}P_{\mathcal{R}(T)})$ is closed. So by using Proposition 3.3, we have $\mathcal{R}(\Delta_\lambda(T))$ is closed.
For $\lambda \in]0, 1[$, we Put $S = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$. By [14, Lemma 3.1], $(|T|^+)^\alpha = (|T|^\alpha)^+$, for all $\alpha > 0$, then

$$(|T|^+)^\lambda|T|^\lambda = (|T|^\lambda)^+|T|^\lambda = P_{\mathcal{R}(|T|^\lambda)},$$

and

$$|T|^{1-\lambda}(|T|^+)^{1-\lambda} = |T|^{1-\lambda}(|T|^{1-\lambda})^+ = P_{\mathcal{R}(|T|^{1-\lambda})}.$$

According to Lemma 3.1, $\mathcal{R}(|T|^\lambda) = \mathcal{R}(|T|^{1-\lambda}) = \mathcal{R}(T^*)$. So we deduce that

$$(|T|^+)^\lambda|T|^\lambda = |T|^{1-\lambda}(|T|^+)^{1-\lambda} = P_{\mathcal{R}(T^*)}.$$

Consequently

$$\begin{aligned}
S\Delta_\lambda(T)S &= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda}(|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}UP_{\mathcal{R}(T^*)}U^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}UU^*(|T|^+)^\lambda \\
&= (|T|^+)^{1-\lambda}U^*UU^*P_{\mathcal{R}(T^*)}(|T|^+)^\lambda \quad \textit{since } T \textit{ is binormal} \\
&= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda = S,
\end{aligned}$$

$$\begin{aligned}
\Delta_\lambda(T)S\Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda}(|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda} \\
&= |T|^\lambda UU^*P_{\mathcal{R}(T^*)}U|T|^{1-\lambda} \\
&= |T|^\lambda P_{\mathcal{R}(T^*)}UU^*U|T|^{1-\lambda} \qquad \textit{since } T \textit{ is binormal} \\
&= (P_{\mathcal{R}(T^*)}|T|^\lambda)^*UU^*U|T|^{1-\lambda} \\
&= |T|^\lambda U|T|^{1-\lambda} = \Delta_\lambda(T),
\end{aligned}$$

and

$$\begin{aligned}
S\Delta_\lambda(T) &= (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda|T|^\lambda U|T|^{1-\lambda} \\
&= (|T|^+)^{1-\lambda}U^*P_{\mathcal{R}(T^*)}|T^*|^{1-\lambda}U \\
&= (|T|^+)^{1-\lambda}U^*|T^*|^{1-\lambda}P_{\mathcal{R}(T^*)}U \quad \textit{by Lemma } 3.10 \\
&= (|T|^+)^{1-\lambda}|T|^{1-\lambda}U^*P_{\mathcal{R}(T^*)}U \quad \textit{by } P(1) \textit{ and } P(2) \\
&= P_{\mathcal{R}(T^*)}U^*P_{\mathcal{R}(T^*)}U \\
&= U^*P_{\mathcal{R}(T^*)}U.
\end{aligned}$$

By similar computation we have $\Delta_\lambda(T)S = P_{\mathcal{R}(T)}P_{\mathcal{R}(T^*)}$. Hence $\Delta_\lambda(T)S$ and $S\Delta_\lambda(T)$ are self-adjoint operators. From the uniqueness of Moore-Penrose inverse we conclude that $(\Delta_\lambda(T))^+ = S$.
Now, we suppose that $\lambda = 1$. Since T is binormal, we have

$$\mathcal{R}(|T||T|U) = \mathcal{R}(|T||T||T^*|) \subset \mathcal{R}(|T^*|) = \mathcal{R}(U)$$

and

$$\mathcal{R}(UU^*|T|) = \mathcal{R}(|T|UU^*) \subset \mathcal{R}(|T|).$$

So, by Lemma 3.11 we obtain $(\Delta_1(T))^+ = (|T|U)^+ = U^*|T|^+$. □

Let $T \in \mathcal{B}(\mathcal{H})$ with closed range and $d \in \mathbb{N}^*$. By using [4, Theorem 2.5], we have $(T^+)^{d+1} = 0$ if and only if $(\Delta_\lambda(T^+))^d = 0$. But, what happens if we replace $\Delta_\lambda(T^+)$ by $\Delta_\lambda(T)^+$? The following last theorem gives the answer to this question.

Theorem 3.12. *Let $\lambda \in]0,1]$. Let $T \in \mathcal{B}(\mathcal{H})$ be a binormal operator with closed range and let $d \in \mathbb{N}^*$. Then*

$$(T^+)^{d+1} = 0 \iff (\Delta_\lambda(T)^+)^d = 0.$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Let $d \in \mathbb{N}^*$. Since T is binormal with closed range, by Proposition 3.9, $(\Delta_\lambda(T))^+ = (|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda$, for $\lambda \in]0,1]$. Thus

$$(\Delta_\lambda(T)^+)^d = ((|T|^+)^{1-\lambda}U^*(|T|^+)^\lambda)^d = (|T|^+)^{1-\lambda}(U^*|T|^+)^{d-1}U^*(|T|^+)^\lambda.$$

This implies

$$(|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^* = |T|^+(U^*|T|^+)^{d-1}U^*|T|^+U^*.$$

Since $T^+ = |T|^+U^*$, it follows that

$$(|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^* = (T^+)^{d+1}.$$

Clearly, $(T^+)^{d+1} = 0$ if $(\Delta_\lambda(T)^+)^d = 0$. Conversely, for $\lambda =]0,1[$ we have

$$\begin{aligned}(T^+)^{d+1} = 0 &\Longrightarrow (|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^* = 0\\ &\Longrightarrow (|T|^+)^\lambda(\lambda(T)^+)^d(|T|^+)^{1-\lambda}U^*U = 0\\ &\Longrightarrow (|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(U^*U(|T|^+)^{1-\lambda})^* = 0\\ &\Longrightarrow (|T|^+)^\lambda(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda} = 0 \quad \textit{since } \mathcal{R}(U^*U) = \mathcal{R}(T^*) = \mathcal{R}((|T|^+)^{1-\lambda}).\end{aligned}$$

Then, $\mathcal{R}(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}) \subset \mathcal{N}((|T|^+)^\lambda)$ and since $\mathcal{N}((|T|^+)^\lambda) = \mathcal{N}((|T|^+)^{1-\lambda}) = \mathcal{N}(T)$, it follows that

$$(|T|^+)^{1-\lambda}(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda} = 0.$$

Therefore, for all $x \in \mathcal{H}$ we have

$$\left\langle (|T|^+)^{1-\lambda}(\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}x, x\right\rangle = \left\langle (\Delta_\lambda(T)^+)^d(|T|^+)^{1-\lambda}x, (|T|^+)^{1-\lambda}x\right\rangle = 0.$$

So that $(\Delta_\lambda(T)^+)^d = 0$ on $\mathcal{R}((|T|)^{1-\lambda}) = \mathcal{R}(|T|)$. On the other hand, we have $\mathcal{N}(|T|) = \mathcal{N}((|T|^+)^\lambda) \subset \mathcal{N}(\Delta_\lambda(T)^+)$. Finally, $(\Delta_\lambda(T)^+)^d = 0$ on $\mathcal{H}$.
Now, For $\lambda = 1$, we have

$$\begin{aligned}(T^+)^{d+1} = 0 &\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^* = 0\\ &\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^*U = 0\\ &\Longrightarrow |T|^+(\Delta_1(T)^+)^dU^*U|T|^+ = 0\\ &\Longrightarrow |T|^+(\Delta_1(T)^+)^d|T|^+ = 0 \quad \textit{since } \mathcal{R}(U^*U) = \mathcal{R}(T^*) = \mathcal{R}(|T|^+).\end{aligned}$$

Hence, $(\Delta_1(T)^+)^d = 0$, on $\mathcal{R}(|T|)$. Also, we have $\mathcal{N}(|T|) \subset \mathcal{N}(\Delta_1(T)^+)$. Therefore $(\Delta_1(T)^+)^d = 0$ on $\mathcal{H}$. □

Remark 3.13. *The assumption "T is binormal" is necessary in the previous theorem. Indeed, consider* $T = \begin{pmatrix}0&0&0\\1&0&0\\1&1&0\end{pmatrix}$ *acting on* $\mathbb{C}^3$*. Then T is not binormal and* $T^+ = \begin{pmatrix}0&1&0\\0&-1&1\\0&0&0\end{pmatrix}$*. An easy calculation shows that* $T^3 = 0$*. This implies* $(\Delta_\lambda(T))^2 = 0$*. Then* $(\Delta_\lambda(T)^+)^2 = 0$ *but* $(T^+)^3 \neq 0$*.*

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