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Warped Product Pointwise Hemi-Slant Submanifolds of a Para-Kaehler Manifold

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Abstract. In this paper, we introduce pointwise hemi-slant submanifolds of para-Kaehler manifolds. Using this notion, we investigate the geometry of warped product pointwise hemi-slant submanifolds. We provide some non-trivial examples of such submanifolds.

1. Introduction

The notion of slant submanifolds was introduced by Chen in [10], and the first results on slant submanifolds were collected in his book [11]. Since then, this subject has been studied extensively by many geometers during the last two and half decades. Also, the study of slant submanifolds in a pseudo-Riemannian manifold has been initiated:Chen and Mihai classified slant surfaces in Lorentzian complex space forms in [12]. Arslan et. al defined slant submanifols of a neutral Kaehler manifold in [6], while Alegre studied slant submanifolds of Lorentzian Sasakian and para-Sasakian manifolds in [1]. Recently, slant, bi-slant and quasi bi-slant submanifolds of (para)-Hermitian manifolds have been defined in [2, 3, 5]. As an extension of slant submanifolds, Etayo [17] defined the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds.
On the other hand, Bishop and O'Neill started the concept of warped product which is one of the most effective generalizations of semi-Riemannian manifold. The notion of warped product has recognized various significant contributions in differential geometry as well as in physics, particularly in general theory of relativity [13, 26]. Since then, the study of warped product submanifolds has been investigated by many geometers (see, e.g.,[4, 9, 15, 16, 18–21, 24, 25, 27–35] among many others, and for the most up-to-date overview of this subject, see [14]).

In this paper, we introduce pointwise hemi-slant submanifolds of para-Kaehler manifolds and using this notion, we investigate the geometry of warped product pointwise hemi-slant submanifolds of the form $N_{1\varphi} \times N_{2\perp}$ in a para-Kaehler manifold $\tilde{N}$, where $N_{2\perp}$ is a totally real submanifold and $N_{1\varphi}$ is a neutral proper poitwise slant submanifold of $\tilde{N}$ with slant function φ.
In the present paper, in section 2, we give preliminaries and definitions needed for this paper. In section 3 we define and study pointwise hemi-slant submanifolds of para-Kaehler manifolds. Then, we give some non-trivial examples of pointwise hemi-slant submanifolds and investigate the geometry of the leaves

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of distributions. In section 4, we prove some preparatory results and obtain a necessary and sufficient condition for the existence of a submanifold of the form $N_{1\varphi} \times N_{2\perp}$ to be locally warped product and locally semi-Riemannian product. Also, we give some examples illustrating such submanifolds. In section 5 we describe the warped product submanifolds $\tilde{N}$ by giving geometric inequalities in term of second fundamental form and warping function φ for the $N_{1\varphi} \times N_{2\perp}$ of a para-Kaehler manifold.

2. Preliminaries

Let $(\tilde{N}, g)$ be an almost para-Hermitian manifold with almost para-complex structure P and a semi-Riemannian metric g such that

$$P^2 Y_1 = Y_1, \quad g(PY_1, PY_2) + g(Y_1, Y_2) = 0, \tag{1}$$

for all $Y_1, Y_2 \in \Gamma(T\tilde{N})$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{N}$ of the semi-Riemannian metric g. If the para-complex structure P satisfies

$$(\tilde{\nabla}_{Y_1} P) Y_2 = 0, \tag{2}$$

for all $Y_1, Y_2 \in \Gamma(T\tilde{N})$, then $\tilde{N}$ is called a para-Kaehler manifold([23]).

Now, let N be a semi-Riemannian submanifold of $(\tilde{N}, P, g)$ and we denote by the same symbol g the semi-Riemannian metric induced on N. Let $\Gamma(TN)$ be the Lie algebra of vector fields in N and $\Gamma(T^\perp N)$, the set of all vector fields normal to N. If ∇ be the induced Levi-Civita connection on N, then the Gauss and Weingarten formulas are given by:

$$\tilde{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \sigma(Y_1, Y_2), \tag{3}$$

$$\tilde{\nabla}_{Y_1} Y_3 = -\mathcal{A}_{Y_3} Y_1 + \nabla^\perp_{Y_1} Y_3, \tag{4}$$

for any $Y_1, Y_2 \in \Gamma(TN)$ and $Y_3 \in \Gamma(T^\perp N)$, where $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp N$ and $\mathcal{A}_{Y_3}$ is the shape operator of N with respect to the normal vector Y_3. Also, $\sigma : TN \times TN \to T^\perp N$ is the second fundamental form of N in $\tilde{N}$. Moreover $\mathcal{A}_{Y_3}$ and σ are related by:

$$g(\sigma(Y_1, Y_2), Y_3) = g(\mathcal{A}_{Y_3} Y_1, Y_2) \tag{5}$$

for any $Y_1, Y_2 \in \Gamma(TN)$ and $Y_3 \in \Gamma(T^\perp N)$.
For any Y_1 tangent to N we write

$$PY_1 = \alpha Y_1 + \beta Y_1, \tag{6}$$

where αY_1 and βY_1 are the tangential and normal parts of PY_1, respectively.
Also, for any $Y_3 \in \Gamma(T^\perp N)$, we get

$$PY_3 = \acute{\alpha} Y_3 + \acute{\beta} Y_3, \tag{7}$$

here $\acute{\alpha} Y_3$ and $\acute{\beta} Y_3$ are the tangential and normal parts of PY_3, respectively.

In [8], Chen and Garay introduced pointwise slant submanifold in a Kaehler manifold. Let N be a submanifold of a Kaehler manifold $(\tilde{N}, P, g)$. Then the submanifold N is called pointwise slant submanifold if at each point $p \in N$, the slant angle $\varphi(Y_1)$ between PY_1 and T_pN is independent of the choice of the non-zero vector $Y_1 \in T_pN$. In this case, the slant angle gives rise to a real-valued function $\varphi : TN - \{0\} \to R$ which is called the slant function of the pointwise slant submanifold. If αY is the projection of PY_1 over N, they can be characterized as $\alpha^2 = \mu Id$.

We say that a semi-Riemannian submanifold N of a para-Hermitian manifold $(\tilde{N}, P, g)$ is called a pointwise slant if for every non-lightlike $Y_1 \in \Gamma(TN)$, the quotient $g(\alpha Y_1, \alpha Y_1)/g(PY_1, PY_1)$ is non-constant. A submanifold is called invariant if it is a pointwise slant with slant function zero, that is if $g(\alpha Y_1, \alpha Y_1)/g(PY_1, PY_1) = 1$ for all non-lightlike $Y_1 \in \Gamma(TN)$. It is called anti-invariant if $\alpha Y_1 = 0$ for all $Y_1 \in \Gamma(TN)$. In other cases, it is called a proper pointwise slant submanifolds.

Definition 2.1. *Let N be a proper pointwise slant submanifold of a para-Hermitian manifold* $(\tilde{N}, P, g)$. *We say that it is of*
type 1 if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, αY_1 *is timelike (spacelike), and* $\frac{\|\alpha Y_1\|}{\|PY_1\|} > 1$,
type 2 if for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$, αY_1 *is timelike (spacelike), and* $\frac{\|\alpha Y_1\|}{\|PY_1\|} < 1$.

The proof of the following result is the same as slant submanifolds (see [2]and [3]), therefore we omit its proof.

Theorem 2.2. *Let N be a semi-Riemannian submanifold of a para-Hermitian manifold* $(\tilde{N}, P, g)$. *Then,*
(i) N is a pointwise slant submanifold of type 1 *if and only if for any spacelike (timelike) vector field* $Y_1 \in \Gamma(TN)$, αY_1 *is timelike (spacelike), and there exists a function* $\mu \in (1, \infty)$ *such that*

$$\alpha^2 Y_1 = \mu Y_1. \tag{8}$$

If φ *denotes the slant function of N then* $\mu = \cosh^2 \varphi$.
(ii) N is a pointwise slant submanifold of type 2 *if and only if for any spacelike (timelike) vector field* $Y_1 \in \Gamma(TN)$, αY_1 *is timelike (spacelike), and there exists a function* $\mu \in (0, 1)$ *such that*

$$\alpha^2 Y_1 = \mu Y_1. \tag{9}$$

If φ *denotes the slant function of N then* $\mu = \cos^2 \varphi$.

In every case, a real-valued function φ is called the slant function of the proper pointwise slant submanifold. From the Theorem 2.2, we have:

Corollary 2.3. *Let* $\mathcal{D}$ *be a distribution on N. Then,*
(i) $\mathcal{D}$ *is a proper pointwise slant of type* 1 *if and only if for any spacelike (timelike) vector field* $Y_1 \in \Gamma(\mathcal{D})$, αY_1 *is timelike (spacelike), and there exists a function* $\mu \in (1, \infty)$ *such that*

$$(\alpha Q_\varphi)^2 Y_1 = \mu Y_1 \tag{10}$$

where Q_φ *denotes the orthogonal projection on* $\mathcal{D}$. *Also, in this case* $\mu = \cosh^2 \varphi$.
(ii) $\mathcal{D}$ *is a proper pointwise slant of type* 2 *if and only if for any spacelike (timelike) vector field* $Y_1 \in \Gamma(\mathcal{D})$, αY_1 *is timelike (spacelike), and there exists a function* $\mu \in (0, 1)$ *such that*

$$(\alpha Q_\varphi)^2 Y_1 = \mu Y_1 \tag{11}$$

where Q_φ *denotes the orthogonal projection on* $\mathcal{D}$. *Also, in this case* $\mu = \cos^2 \varphi$.

In every case, a real-valued function φ is called the slant function of the proper pointwise slant distribution.

Let us point out that for both proper pointwise slant distributions of type 1 and 2, if Y_1 is a spacelike tangent vector field, then αY_1 is a timelike tangent vector field. So, all type 1, and type 2 proper pointwise slant distributions are neutral.

Remember that a para-holomorphic distribution satisfies $P\mathcal{D} = \mathcal{D}$, so every para-holomorphic distribution is a pointwise slant distribution with slant function zero. It is called a totally real distribution if $P\mathcal{D} \subseteq T^{\perp}N$, therefore every totaly distribution is anti-invariant.
If $\mathcal{D}$ is a para-holomorphic distribution, then $\|\alpha Y_1\| = \|PY_1\|$, for all $Y_1 \in \Gamma(\mathcal{D})$. If $\mathcal{D}$ is a totally real distribution, then $\|\alpha Y_1\| = 0$, for all $Y_1 \in \Gamma(\mathcal{D})$.

3. Proper pointwise hemi-slant submanifolds

In this section we define and study proper pointwise hemi-slant submanifold of a para-Kaehler manifold $(\tilde{N}, P, g)$.

Definition 3.1. *A semi-Riemannian submanifold N of a para-Hermitian $(\tilde{N}, P, g)$ is called a pointwise bi-slant submanifold if the tangent space admits a decomposition $TN = \mathcal{D}_\varphi \oplus \mathcal{D}_\omega$ with both $\mathcal{D}_\varphi$ and $\mathcal{D}_\omega$ pointwise slant distributions with slant functions φ and ω.*
It is called a pointwise semi-slant submanifold if the tangent space admits a decomposition $TN = \mathcal{D}_\top \oplus \mathcal{D}_\varphi$ with $\mathcal{D}_\top$ a para-holomorphic distribution and $\mathcal{D}_\varphi$ a proper pointwise slant distribution with slant function φ.
It is called a pointwise hemi-slant submanifold if the tangent space admits a decomposition $TN = \mathcal{D}_\perp \oplus \mathcal{D}_\varphi$ with $\mathcal{D}_\perp$ a totally real distribution and $\mathcal{D}_\varphi$ a proper pointwise slant distribution with slant function φ.

Note that given a pseudo-Euclidean space R^{2n}_n with coordinates $(x_1, ..., x_{2n})$ on R^{2n}_n, we can naturally choose an almost paracomplex structure P on R^{2n}_n as follows:

$$P(\frac{\partial}{\partial x_{2i}}) = \frac{\partial}{\partial x_{2i-1}}, \;\; P(\frac{\partial}{\partial x_{2i-1}}) = \frac{\partial}{\partial x_{2i}},$$

where $i = 1, ..., n$. Let R^{2n}_n be a pseudo-Euclidean space of signature (+,-,+,-,...) with respect to the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{2n}})$.

Now, we can present some examples of proper pointwise hemi-slant submanifolds.

Example 3.2. *Let N be a semi-Riemannian submanifold of R^8_4 defined by the immersion $\phi : N \to R^8_4$:*

$$\phi(u, v, t, s) = (\sin u, \sin v, \cos u, \cos v, t, k_1, k_2, s),$$

such that $u \neq v \neq 0$, for non-vanishing functions u and v on N. Then N is a neutral pointwise hemi-slant submanifold of type 2 with neutral anti-invariant distribution $\mathcal{D}_\perp = Span\{Y_3 = \frac{\partial}{\partial x_5}, Y_4 = \frac{\partial}{\partial x_8}\}$ and the neutral pointwise slant distribution of type 2 $\mathcal{D}_\varphi = Span\{Y_1 = \cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_3}, Y_2 = \cos v \frac{\partial}{\partial x_2} - \sin v \frac{\partial}{\partial x_4}\}$ with slant function $\varphi = u - v$.

Example 3.3. *Let N be a semi-Riemannian submanifold of R^8_4 defined by the immersion $\phi : N \to R^8_4$:*

$$\phi(u, v, t, s) = (v, \sinh u, \cosh u, u, t, k_1, k_2, s),$$

such that $u > 2$ and $v \neq 0$, for non-vanishing function u on N. Then N is a neutral pointwise hemi-slant submanifold of type 1 with neutral anti-invariant distribution $\mathcal{D}_\perp = Span\{Y_3 = \frac{\partial}{\partial x_5}, Y_4 = \frac{\partial}{\partial x_8}\}$ and the neutral pointwise slant distribution of type 1 $\mathcal{D}_\varphi = Span\{Y_1 = \cosh u \frac{\partial}{\partial x_2} + \sinh u \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, Y_2 = \frac{\partial}{\partial x_1}\}$ with slant function $\varphi = \cosh^{-1}(\frac{\cosh u}{\sqrt{2}})$.

Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold $(\tilde{N}, P, g)$, and set the projections on the distributions $\mathcal{D}_\perp$ and $\mathcal{D}_\varphi$ by $Q_\perp$ and Q_φ, respectively. Then we can write

$$Y_1 = Q_\perp Y_1 + Q_\varphi Y_1 \tag{12}$$

for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$. Applying P to equation (12) and using (6), we get

$$PY_1 = \beta Q_\perp Y_1 + \alpha Q_\varphi Y_1 + \beta Q_\varphi Y_1. \tag{13}$$

From (13), we have

$$\beta Q_\perp Y_1 \in \Gamma(\mathcal{D}_\perp), \;\; \alpha Q_\perp Y_1 = 0, \tag{14}$$

$$\alpha Q_{\varphi} Y_1 \in \Gamma(\mathcal{D}_{\varphi}), \ \ \beta Q_{\varphi} Y_1 \in \Gamma(TN_{\perp}). \tag{15}$$

Using (6) in (13), we obtain

$$\alpha Y_1 = \alpha Q_{\varphi} Y_1, \ \ \beta Y_1 = \beta Q_{\perp} Y_1 + \beta Q_{\varphi} Y_1 \tag{16}$$

for any spacelike (timelike) vector field $Y_1 \in \Gamma(TN)$. Since $\Gamma(\mathcal{D}_{\varphi})$ is a proper pointwise slant distribution, from Theorem 2.2 and Corollary 2.3, we conclude that

$$type1 : \alpha^2 Y_1 = (\cosh^2 \varphi) Y_1, \ \ type2 : \alpha^2 Y_1 = (\cos^2 \varphi) Y_1 \tag{17}$$

for any spacelike (timelike) vector field $Y_1 \in \Gamma(\mathcal{D}_{\varphi})$ and a real-valued function φ defined on N. We give the following results for the characterization of proper pointwise hemi-slant submanifold.

Proposition 3.4. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$. *Then N is a proper pointwise hemi-slant submanifold if and only if there exists a function* $\mu \in (1, \infty)$ *and a distribution of type 1* $\mathcal{D}$ *on N such that*
(a) $\mathcal{D} = \{Y_1 \in \Gamma(TN) : (\alpha_{\mathcal{D}})^2 Y_1 = \mu Y_1\}$,
(b) $\alpha Y_2 = 0$ *for any spacelike(timelike) vector field* $Y_2 \in \Gamma(TN)$ *orthogonal to* $\mathcal{D}$.
Moreover, if φ *denotes the slant function of N then* $\mu = \cosh^2 \varphi$.

Proof. Let N be a proper pointwise hemi-slant submanifold of $(\tilde{N}, P, g)$. By setting $\mu = \cosh^2 \varphi$ and using (14) and (15), we obtain that $\mathcal{D} = \mathcal{D}_{\varphi}$, which follows *a* and *b*. Conversely, (a) and (b) imply that $TN = \mathcal{D} \oplus \mathcal{D}_{\perp}$. Since $\alpha(\mathcal{D}) \subseteq \mathcal{D}$, we received from (b) that $\mathcal{D}_{\perp}$ is a totally real distribution. □

In a similar way, we obtain:

Proposition 3.5. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$. *Then N is a proper pointwise hemi-slant submanifold if and only if there exists a function* $\mu \in (0, 1)$ *and a distribution of type 2* $\mathcal{D}$ *on N such that*
(a) $\mathcal{D} = \{Y_1 \in \Gamma(TN) : (\alpha_{\mathcal{D}})^2 Y_1 = \mu Y_1\}$,
(b) $\alpha Y_2 = 0$ *for any spacelike(timelike) vector field* $Y_2 \in \Gamma(TN)$ *orthogonal to* $\mathcal{D}$.
Moreover, if φ *denotes the slant function of N then* $\mu = \cos^2 \varphi$.

From the above Propositions, we have:

Corollary 3.6. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$. *Then* $\mathcal{D}_{\varphi}$ *is a proper pointwise slant distribution of*
type 1 if and only if $g(\alpha Y_1, \alpha Y_2) = -\cosh^2 \varphi g(Y_1, Y_2), \ \ g(\beta Y_1, \beta Y_2) = \sinh^2 \varphi g(Y_1, Y_2)$,
type 2 if and only if $g(\alpha Y_1, \alpha Y_2) = -\cos^2 \varphi g(Y_1, Y_2), \ \ g(\beta Y_1, \beta Y_2) = -\sin^2 \varphi g(Y_1, Y_2)$
for all spacelike (timelike) vector fields $Y_1, Y_2 \in \Gamma(\mathcal{D}_{\varphi})$.

Using (1), (6) and (7), the Propositions 3.4 and 3.5, we get:

Lemma 3.7. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$. *Then* $\mathcal{D}_{\varphi}$ *is a proper pointwise slant distribution of*
type 1 if and only if (a) $\acute{\alpha}\beta Y_1 = (-\sinh^2 \varphi) Y_1$, *(b)* $\acute{\beta}\beta Y_1 = -\beta\alpha Y_1$,
type 2 if and only if (*a*) $\acute{\alpha}\beta Y_1 = (\sin^2 \varphi) Y_1)$, *(b)* $\acute{\beta}\beta Y_1 = -\beta\alpha Y_1$,
for all spacelike (timelike) vector field $Y_1 \in \Gamma(\mathcal{D}_{\varphi})$.

Now we examine the conditions for integrability and totally geodesic foliation of distributions associated with the definition of proper pointwise hemi-slant submanifolds of a para-Kaehler manifold.

Theorem 3.8. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$. *Then the totally real distribution* $\mathcal{D}_{\perp}$ *is integrable.*

Proof. It is known that is a para-Kaehler manifold, then $d\mathcal{F} = 0$, where d is exterior derivative and $\mathcal{F}$ is the fundamental 2−form defined $\mathcal{F}(Y_1, Y_2) = g(Y_1, PY_2)$ for any spacelike (timelike) vector fields $Y_1, Y_2 \in \Gamma(T\tilde{N})$ (see [23]). Since $\mathcal{F}$ is closed ($d\mathcal{F} = 0$), for any spacelike (timelike) vector fields $Y_1 \in \Gamma(\mathcal{D}_\varphi)$ and $Y_2, Y_3 \in \Gamma(\mathcal{D}_\perp)$ we have

$$\begin{aligned} 3d\mathcal{F}(\alpha Y_1, Y_2, Y_3) &= \alpha Y_1 \mathcal{F}(Y_2, Y_3) - Y_2\mathcal{F}(\alpha Y_1, Y_3) + Y_3\mathcal{F}(\alpha Y_1, Y_2) \\ &= -\mathcal{F}([\alpha Y_1, Y_2], Y_3) + \mathcal{F}([\alpha Y_1, Y_3], Y_2) - \mathcal{F}([Y_2, Y_3], \alpha Y_1) = 0. \end{aligned}$$

Since $\mathcal{D}_\perp$ and $\mathcal{D}_\varphi$ are orthogonal and $\mathcal{D}_\perp$ is anti-invariant, using Proposition 3.4 and (6) we get

$$Y_2 g(\beta\alpha Y_1, Y_3) - \cosh^2\varphi g([Y_2, Y_3], Y_1) - g([Y_2, Y_3], \beta\alpha Y_1) = 0.$$

Since $[Y_2, Y_3] \in \Gamma(TN)$ and $\beta\alpha Y_1 \in \Gamma(TN_\perp)$ we obtain

$$\cosh^2\varphi g([Y_2, Y_3], Y_1) = 0.$$

Since N is a proper pointwise hemi-slant submanifold and Y_1, Y_2, Y_3 are all non-zero, we have $[Y_2, Y_3] \in \Gamma(\mathcal{D}_\perp)$. □

Note that the Theorem 3.8 holds for proper pointwise slant submanifold $N_{1\varphi}$ of type 2.
From the Theorem 3.8, we have:

Corollary 3.9. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$*. Then the totally real distribution* $\mathcal{D}_\perp$ *is integrable if and only if for any spacelike (timelike) vector fields* $Y_1, Y_2 \in \Gamma(\mathcal{D}_\perp)$ *the shape operator satisfies* $\mathcal{A}_{PY_2}Y_1 = \mathcal{A}_{PY_1}Y_2$.

Theorem 3.10. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$*. Then the totally real distribution* $\mathcal{D}_\perp$ *defines a totally geodesic foliation if and only if for every spacelike (timelike) vector fields* $Y_1 \in \Gamma(\mathcal{D}_\perp)$ *and* $Y_3 \in \Gamma(\mathcal{D}_\varphi)$, $\mathcal{A}_{PY_1}\alpha Y_3 = \mathcal{A}_{\beta\alpha Y_3}Y_1$.

Proof. For any spacelike (timelike) vector fields $Y_1, Y_2 \in \Gamma(\mathcal{D}_\perp)$ and $Y_3 \in \Gamma(\mathcal{D}_\varphi)$, using (1)-(7) we get

$$\begin{aligned} g(\nabla_{Y_1}Y_2, Y_3) &= -g(\tilde{\nabla}_{Y_1}PY_2, PY_3) \\ &= -g(\tilde{\nabla}_{Y_1}PY_2, \alpha Y_3) + g(\tilde{\nabla}_{Y_1}Y_2, \acute{\alpha}\beta Y_3) \\ &+ g(\tilde{\nabla}_{Y_1}Y_2, \acute{\beta}\beta Y_3). \end{aligned} \tag{18}$$

From (4), (5) and Lemma 3.7(type 1), we obtain

$$\begin{aligned} g(\nabla_{Y_1}Y_2, Y_3) &= g(\mathcal{A}_{PY_2}Y_1, \alpha Y_3) - \sinh^2\varphi g(\tilde{\nabla}_{Y_1}Y_2, Y_3) \\ &= -g(\mathcal{A}_{\beta\alpha Y_3}Y_1, Y_2). \end{aligned}$$

Using (3), we get

$$\cosh^2\varphi g(\nabla_{Y_1}Y_2, Y_3) = g(\mathcal{A}_{PY_2}\alpha Y_3, Y_1) - g(\mathcal{A}_{\beta\alpha Y_3}Y_2, Y_1).$$

□

Now, analogous to the proof of the Theorems 3.8 and 3.10 we give the following results for proper pointwise hemi-slant submanifolds.

Theorem 3.11. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$*. Then the proper pointwise slant distribution* $\mathcal{D}_\varphi$ *is integrable if and only if*

$$g(\mathcal{A}_{\beta\alpha Y_2}Y_1 - \mathcal{A}_{PY_1}\alpha Y_2, Y_3) = g(\mathcal{A}_{\beta\alpha Y_3}Y_1 - \mathcal{A}_{PY_1}\alpha Y_3, Y_2)$$

for every spacelike (timelike) vector fields $Y_1 \in \Gamma(\mathcal{D}_\perp)$ *and* $Y_2, Y_3 \in \Gamma(\mathcal{D}_\varphi)$.

Theorem 3.12. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$*. Then the proper pointwise slant distribution* $\mathcal{D}_\varphi$ *defines a totally geodesic foliation if and only if* $\mathcal{A}_{\beta\alpha Y_2}Y_1 - \mathcal{A}_{PY_1}\alpha Y_2 = 0$ *for any spacelike (timelike) vector fields* $Y_1 \in \Gamma(\mathcal{D}_\perp)$ *and* $Y_2 \in \Gamma(\mathcal{D}_\varphi)$*.*

From the Theorems 3.10 and 3.12 we have:

Corollary 3.13. *Let N be a proper pointwise hemi-slant submanifold of a para-Kaehler manifold* $(\tilde{N}, P, g)$*. Then a necessary and sufficient condition for N to be locally semi-Riemannian product of the form* $N = N_1 \times N_{2\varphi}$ *is that the Weingarten operator satisfies* $\mathcal{A}_{\beta\alpha Y_2}Y_1 - \mathcal{A}_{PY_1}\alpha Y_2 = 0$ *for any spacelike (timelike) vector fields* $Y_1 \in \Gamma(\mathcal{D}_\perp)$ *and* $Y_2 \in \Gamma(\mathcal{D}_\varphi)$*, where* N_1 *is a totally real submanifold and* $N_{2\varphi}$ *is a proper pointwise slant submanifold of* $\tilde{N}$*.*

4. Warped products $N_{1\varphi} \times_h N_{2\perp}$ in para-Kaehler manifolds

Let (N_1, g_1) and (N_2, g_2) be two semi-Riemannian manifolds, let $h : N_1 \to R_+$, and let $\eta_1 : N_1 \times N_2 \to N_1$ and $\eta_2 : N_1 \times N_2 \to N_2$ the projection maps given by $\eta_1(r, s) = r$ and $\eta_2(r, s) = s$ for all $(r, s) \in N_1 \times N_2$. The warped product([7]) $N = N_1 \times N_2$ is the manifold $N_1 \times N_2$ equipped with the semi-Riemannian structure such that

$$g(Y_1, Y_2) = g_1(\eta_{1*}Y_1, \eta_{1*}Y_2) + (h \circ \eta_1)^2 g_2(\eta_{2*}Y_1, \eta_{2*}Y_2)$$

for every spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(TN)$, here $*$ denotes the tangent map. The function h is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the manifold N is said to be trivial.

Lemma 4.1. *([7]) For spacelike(timelike) vector fields* $Y_1, Y_2 \in \Gamma(TN_1)$ *and* $Y_3, Y_4 \in \Gamma(TN_2)$*, we get on warped product manifold* $N = N_1 \times_h N_2$ *that*
$(a)\nabla_{Y_1}Y_2 \in \Gamma(TN_1)$,
$(b)\nabla_{Y_1}Y_3 = \nabla_{Y_3}Y_1 = (\frac{Y_1 h}{h})Y_3$,
$(c)\nabla_{Y_3}Y_4 = \frac{-g(Y_3,Y_4)}{h}\nabla h$,
where ∇ *denotes the Levi-Civita connection on N and* ∇h *is the gradient of h defined by* $g(\nabla h, Y_1) = Y_1 h$*.*

It is also important to note that for a warped product $N = N_1 \times_h N_2$, N_1 is totally geodesic and N_2 is totally umbilical in N([7]).
In this section, we investigate the existence of warped product submanifolds $N_{1\varphi} \times_h N_{2\perp}$ of para-Kaehler manifolds such that $N_{1\varphi}$ is a proper pointwise slant submanifold and $N_{2\perp}$ is a totally real submanifold of $\tilde{N}$. First, we are going to give some examples of a warped product pointwise hemi-slant submanifold of the form $N_{1\varphi} \times_h N_{2\perp}$.

Example 4.2. *Consider a semi-Riemannian submanifold of* R^8_4 *with the cartesian coordinates* $(x_1, ..., x_8)$ *and the almost para-complex structure*

$$P(\frac{\partial}{\partial x_{2i}}) = \frac{\partial}{\partial x_{2i-1}}, \ \ P(\frac{\partial}{\partial x_{2i-1}}) = \frac{\partial}{\partial x_{2i}}, \ \ 1 \leq i \leq 4.$$

Let R^8_4 *be a semi-Euclidean space of signature (+,-,+,-,+,-,+,-) with respect to the canonical basis* $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_8})$*.*
Let N be defined by the immersion ψ *as follows*

$$\psi(u, v, t) = (\sinh u, v, u, \cosh u, \cosh(t^3), a, \sinh(t^3), b)$$

for any non-vanishing function u on N, where a, b are constants and $u > 1$*. Then the tangent space TN of N is spanned by the following vectors*

$$\psi_u = \cosh u\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sinh u\frac{\partial}{\partial x_4}, \ \ \psi_v = \frac{\partial}{\partial x_2},$$

$$\psi_t = 3t^2\sinh(t^3)\frac{\partial}{\partial x_5} + 3t^2\cosh(t^3)\frac{\partial}{\partial x_7}.$$

Then we obtain

$$P\psi_u = \cosh u\frac{\partial}{\partial x_2} + \sinh u\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},\;\; P\psi_v = \frac{\partial}{\partial x_1},$$
$$P\psi_t = 3t^2\sinh(t^3)\frac{\partial}{\partial x_6} + 3t^2\cosh(t^3)\frac{\partial}{\partial x_8}.$$

It is easy to see that $P\psi_t \perp TN = span\{\psi_u, \psi_v, \psi_t\}$ and thus, we consider $D_\perp = span\{\psi_t\}$ is a spacelike totally real distribution and $D_\varphi = span\{\psi_u, \psi_v\}$ is a neutral proper poitwise slant distribution of type 1 with slant function $\varphi = \cosh^{-1}(\frac{\cosh u}{\sqrt{2}})$. It is easy to observe that D_φ and $D_\perp$ are integrable. If we denote the integral manifolds of D_φ and $D_\perp$ by $N_{1\varphi}$ and $N_{2\perp}$, respectively, then the metric tensor of N is given by

$$ds^2 = 2du^2 - dv^2 + 9t^4\cosh(2t^3)dt^2.$$

Thus, N is a warped product submanifold of the form $N = N_{1\varphi} \times_h N_{2\perp}$ in R^8_4 with the warping function $h = 3t^2\sqrt{\cosh(2t^3)}$.

Example 4.3. *Let N be an immersed semi-Riemannian submanifold of a para-Kaehler manifold $\tilde{N}$ (as given in Example 4.2) defined by*

$$\psi(x, y, z) = (\sin x, \sin y, \cos x, \cos y, \cos e^z, a, \sin e^z, b),$$

such that $u \neq v \neq 0$, for non-vanishing functions u and v on N. Then the tangent space TN of N is spanned by the following vectors:

$$\psi_x = \cos x\frac{\partial}{\partial x_1} - \sin x\frac{\partial}{\partial x_3},\;\; \psi_x = \cos y\frac{\partial}{\partial x_2} - \sin y\frac{\partial}{\partial x_4},$$
$$\psi_z = -e^z\sin(e^z)\frac{\partial}{\partial x_5} + e^z\cos(e^z)\frac{\partial}{\partial x_7}.$$

Thus, we consider $D_\perp = span\{\psi_z\}$ is a spacelike totally real distribution and $D_\varphi = span\{\psi_x, \psi_y\}$ is a neutral proper poitwise slant distribution of type 2 with slant function $\varphi = u - v$. It is easy to observe that D_φ and $D_\perp$ are integrable. If we denote the integral manifolds of D_φ and $D_\perp$ by $N_{1\varphi}$ and $N_{2\perp}$, respectively, then the metric tensor of N is given by

$$ds^2 = dx^2 - dy^2 + e^{2z}dz^2.$$

Hence, N is a 3-dimensional pointwise hemi-slant warped product submanifold of R^8_4 with the warping function $h = e^z$.

Example 4.4. *Let N be defined by the immersion ψ as follows*

$$\psi(x, y, z, t) = (\sinh x, \sinh y, \cosh y, \cosh x, \sinh(z + t), z, \sinh(z + t), t)$$

for any non-vanishing functions x and y on N. Then the tangent space TN of N is spanned by the following vectors

$$\psi_x = \cosh x\frac{\partial}{\partial x_1} + \sinh x\frac{\partial}{\partial x_4},\;\; \psi_y = \cosh y\frac{\partial}{\partial x_2} + \sinh y\frac{\partial}{\partial x_3},$$

$$\psi_z = \cosh(z + t)\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} + \cosh(z + t)\frac{\partial}{\partial x_7},\;\; \psi_t = \cosh(z + t)\frac{\partial}{\partial x_5} + \cosh(z + t)\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}.$$

It is easy to see that $P\psi_z$ and $P\psi_t \perp TN = span\{\psi_x, \psi_y, \psi_z, \psi_t\}$ and thus, we consider $D_\perp = span\{\psi_z, \psi_t\}$ is a spacelike totally real distribution and $D_\varphi = span\{\psi_x, \psi_y\}$ is a neutral proper poitwise slant distribution of type 1 with slant function $\alpha^2 = \cosh^2(x - y)$. It is easy to observe that D_φ and $D_\perp$ are integrable. If we denote the integral manifolds of D_φ and $D_\perp$ by $N_{1\varphi}$ and $N_{2\perp}$, respectively, then the metric tensor of N is given by

$$ds^2 = dx^2 - dy^2 + \cosh(2(z + t))(dz^2 + dt^2).$$

Thus, N is a pointwise hemi-slant warped product submanifold of the form $N = N_{1\varphi} \times_h N_{2\perp}$ in R^8_4 with the warping function $h = \sqrt{\cosh(2(z + t))}$.

Now we will consider warped product pointwise hemi-slant submanifolds $N = N_{1\varphi} \times_h N_{2\perp}$ such that $N_{1\varphi}$ is a neutral proper pointwise slant submanifold and $N_{2\perp}$ is a totally real submanifold of a para-Kaehler manifold $\tilde{N}$.

Lemma 4.5. *Let $N = N_{1\varphi} \times_h N_{2\perp}$ be a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold $\tilde{N}$. Then*

$$g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) = (-\cosh^2\varphi)(Y_1 \ln h)g(Y_3, Y_4) + g(\mathcal{A}_{\beta\alpha Y_1} Y_3, Y_4) \tag{19}$$

for any spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(TN_{1\varphi})$ and $Y_3, Y_4 \in \Gamma(TN_{2\perp})$.

Proof. From (1)-(6) we obtain

$$g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) = -g(\nabla_{Y_3} Y_1, Y_4) + g(\tilde{\nabla}_{Y_3} P\beta Y_1, Y_4). \tag{20}$$

Using Lemma 4.1, Lemma 3.7(type 1) and (7) we get

$$\begin{aligned} g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) &= -(Y_1 \ln h)g(Y_3, Y_4) + g(\tilde{\nabla}_{Y_3}(-\sinh^2\varphi)Y_1, Y_4 \\ &+ g(\tilde{\nabla}_{Y_3}(-\beta\alpha Y_1), Y_4). \end{aligned}$$

From the fact that φ is slant function and using (4) we obtain

$$\begin{aligned} g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) &= -(Y_1 \ln h)g(Y_3, Y_4) - (\sinh^2\varphi)g(\tilde{\nabla}_{Y_3} Y_1, Y_4) \\ &+ g(\mathcal{A}_{\beta\alpha Y_1} Y_3, Y_4), \end{aligned}$$

since $g(Y_1, Y_4) = 0$. Using (3) we have

$$g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) = (-\cosh^2)(Y_1 \ln h)g(Y_3, Y_4) + g(\mathcal{A}_{\beta\alpha Y_1} Y_3, Y_4).$$

□

Theorem 4.6. *Let N be a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold $\tilde{N}$. Then N is locally isometric to pointwise hemi-slant warped product submanifold of the form $N = N_{1\varphi} \times_h N_{2\perp}$ if and only if the shape operator of N satisfies*

$$\mathcal{A}_{PY_4}\alpha Y_1 - \mathcal{A}_{\beta\alpha Y_1} Y_4 = (-\cosh^2\varphi)(Y_1 \ln h)Y_4, \tag{21}$$

for some function τ on N such that $Y_3(\tau) = 0$, where spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(\mathcal{D}_\varphi)$ and $Y_3, Y_4 \in \Gamma(\mathcal{D}_\perp)$.

Proof. Let us consider that N is a pointwise hemi-slant warped product submanifold of a para-Kaehler manifold $\tilde{N}$. Then, Lemma 4.5, we have (21). We know that h is a function on N_2 , therefore setting $\tau = \ln h$ implies that $Y_3(\tau) = 0$. Conversely we assume that N is a pointwise hemi-slant submanifold of $\tilde{N}$ such that (21) holds. Taking the inner product of (21) with Y_2, we can say from Theorem 3.12 that the integral manifold $N_{1\varphi}$ of $\mathcal{D}_\varphi$ is totally geodesic foliation in N. Thus, by Corollary 3.9 the distribution $\mathcal{D}_\perp$ is integrable if and only if

$$g(\mathcal{A}_{PY_3} Y_4, \alpha Y_1) = g(\mathcal{A}_{PY_4}\alpha Y_1, Y_3) \tag{22}$$

for any spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(\mathcal{D}_\varphi)$ and $Y_3, Y_4 \in \Gamma(\mathcal{D}_\perp)$. From (19) and (22) we obtain

$$g(\mathcal{A}_{PY_3} Y_4, \alpha Y_1) = (-\cosh^2\varphi)g(\nabla_{Y_3} Y_1, Y_4) + g(\mathcal{A}_{\beta\alpha Y_1} Y_3, Y_4) \tag{23}$$

on the other hand, taking the inner product of (21) with Y_3 we obtain

$$g(\mathcal{A}_{PY_4}\alpha Y_1 - \mathcal{A}_{\beta\alpha Y_1} Y_4, Y_3) = g((-\cosh^2\varphi)(Y_1 \ln h)Y_4, Y_3). \tag{24}$$

From (23), (24) and Lemma 4.1, we have

$$-g(\sigma_\perp(Y_3, Y_4), Y_1) = g(Y_1(\tau)Y_3, Y_4) = g(Y_3, Y_4)g(\nabla\tau, Y_1).$$

Thus $\sigma_\perp(Y_3, Y_4) = g(Y_3, Y_4)(-\nabla\tau)$, here $\sigma_\perp$ is a second fundamental form of $\mathcal{D}_\perp$ in N and $\nabla\tau$ is a gradient of $\tau = \ln h$. Hence the integrable manifold N_2 of $\mathcal{D}_\perp$ is totally umbilical submanifold in N and its mean curvature is non-zero and parallel and $Y_3(\tau) = 0$ for every spacelike (timelike) vector field $Y_3 \in \Gamma(\mathcal{D}_\perp)$. Therefore, from Theorem 1.2 ([22], page 211), we deduce that N is a pointwise hemi-slant warped product submanifold of $\tilde{N}$. □

Now we maintain a necessary and sufficient condition for a warped product submanifold of the form $N = N_{1\varphi} \times_h N_{2\perp}$ to be a semi-Riemannian product.

Theorem 4.7. *A pointwise hemi-slant warped product submanifold of the form $N = N_{1\varphi} \times_h N_{2\perp}$ of a para-Kaehler manifold $\tilde{N}$ is simply a locally semi-Riemannian product if and only if the shape operator satisfies $\mathcal{A}_{\beta\alpha Y_3} Y_1 = 0$, for every spacelike(timelike) vector fields $Y_3 \in \Gamma(N_{1\varphi})$ and $Y_1 \in \Gamma(N_{2\perp})$.*

Proof. **Proof.** For all spacelike(timelike) vector fields $Y_3 \in \Gamma(N_{1\varphi})$ and $Y_1, Y_2 \in \Gamma(N_{2\perp})$, using (1)-(3) we have $g(\nabla_{Y_1} Y_3, Y_2) = -g(\tilde{\nabla}_{Y_1} PY_3, PY_2)$. From (1),(2) and (6) we get

$$g(\nabla_{Y_1} Y_3, Y_2) = g(\tilde{\nabla}_{Y_1} \alpha^2 Y_3, Y_2) + g(\tilde{\nabla}_{Y_1} \beta\alpha Y_3, Y_2) - g(\tilde{\nabla}_{Y_1} \beta Y_3, PY_2).$$

Using type 1 (17), (3)-(5) and the fact that $g(Y_2, Y_3) = 0$, we obtain

$$g(\nabla_{Y_1} Y_3, Y_2) = (\cosh^2\varphi) g(\tilde{\nabla}_{Y_1} Y_3, Y_2) - g(\sigma(Y_1, Y_2), \beta\alpha Y_3) - g(\nabla^\perp_{Y_1} \beta Y_3, PY_2). \tag{25}$$

Hence, from Lemma 4.1, we get

$$(\sinh^2\varphi)(Y_3 \ln h) g Y_1, Y_2) = g(\sigma(Y_1, Y_2), \beta\alpha Y_3) + g(\nabla^\perp_{Y_1} \beta Y_3, PY_2). \tag{26}$$

Interchanging Y_1 and Y_2 in (26) and then, subtracting from (26), we have:

$$g(\nabla^\perp_{Y_1} \beta Y_3, PY_2) = g(\nabla^\perp_{Y_2} \beta Y_3, PY_1). \tag{27}$$

Furthermore from (1),(3),(4) and (6) we obtain

$$g(\nabla^\perp_{Y_1} \beta Y_3, PY_2) = -(Y_3 \ln h) g(Y_1, Y_2) - g(\tilde{\nabla}_{Y_1} \alpha Y_3, PY_2). \tag{28}$$

Again by interchanging Y_1 and Y_2 in (28) we conclude that (27) holds if and only if

$$g(\tilde{\nabla}_{Y_1} \alpha Y_3, PY_2) = -g(\tilde{\nabla}_{Y_1} PY_2, \alpha Y_3) = 0. \tag{29}$$

Using type 1,(17) and (1)-(6) we have

$$(-\cosh^2\varphi)(Y_3 \ln h) g(Y_1, Y_2) + g(\sigma(Y_1, Y_2), \beta\alpha Y_3) = 0. \tag{30}$$

Hence, from (30) we can say that h is constant if and only if $g(\sigma(Y_1, Y_2), \beta\alpha Y_3) = 0$, since $N_{1\varphi}$ is proper pointwise slant submanifold and Y_3 is non-zero spacelike(timelike)vector field. □

We say that a hemi-slant submanifold is mixed geodesic if

$$\sigma(Y_1, Y_3) = 0 \tag{31}$$

for all spacelike(timelike) vector fields $Y_1 \in \Gamma(\mathcal{D}\varphi)$ and $Y_3 \in \Gamma(\mathcal{D}\perp)$.

Lemma 4.8. *For a mixed geodesic pointwise hemi-slant warped product submanifold $N = N_{1\varphi} \times_h N_{2\perp}$ of a para-Kaehler manifold $\tilde{N}$ Then, we obtain*

$$g(\sigma(Y_1, Y_2), PY_3) = 0 \tag{32}$$

$$(\alpha Y_1 \ln h) g(\sigma(Y_3, Y_4)) = g(\sigma(Y_3, Y_4), \beta Y_1) \tag{33}$$

for spacelike(timelike) vector fields $Y_1, Y_2 \in \Gamma(N_{1\varphi})$ and $Y_3, Y_4 \in \Gamma(N_{2\perp})$.

Proof. From (1) and (2) we obtain $g(\sigma(Y_1, Y_2), PY_3) = -g(\tilde{\nabla}_{Y_1} PY_2, Y_3)$. From here, $g(\sigma(Y_1, Y_2), PY_3) = g(\tilde{\nabla}_{Y_1} Y_3, PY_2)$. Using (6), we have $g(\sigma(Y_1, Y_2), PY_3) = g(\nabla_{Y_1} Y_3, \alpha Y_2) + g(\sigma(Y_1, Y_3), \beta Y_2)$. Thus from Lemma 4.1 we get (32). In a similar way, we have (33). □

Note that the Lemma 4.8 holds for proper pointwise slant submanifold $N_{1\varphi}$ of type 2.

5. An optimal inequality

We establish general sharp geometric inequality for proper pointwise hemi-slant warped product submanifolds of the form $N_{1\varphi} \times_h N_{2\perp}$ of a para-Kaehler manifold $(\tilde{N}, P, g)$.

Let $x \in N$ and $\{E_1, ..., E_m, \hat{E}_1, ..., \hat{E}_n, PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ be an orthonormal basis of the tangent space $T_x\tilde{N}$ such that $\{E_1, ..., E_m, \hat{E}_1, ..., \hat{E}_n\}$ are tangent to N at x and $\{PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ are normal to N, and thus $T_x\tilde{N} = T_xN \oplus T_x^{\perp}N$. Now, we can take $\{E_1, ..., E_m, \hat{E}_1, ..., \hat{E}_n\}$ in such a way that $\{E_1, ..., E_m\}$ form an orthonormal basis of $\mathcal{D}_\perp$ and $\{\hat{E}_1, ..., \hat{E}_n\}$ form an orthonormal basis of $\mathcal{D}_\varphi$, where $dim\mathcal{D}_\perp = m$ and $dim\mathcal{D}_\varphi = n$. We can take $\{PE_1, ..., PE_m, \tilde{E}_1, ..., \tilde{E}_n\}$ in such a way that $\{PE_1, ..., PE_m\}$ form an orthonormal frame of $P(\mathcal{D}_\perp)$ and $\{\tilde{E}_1, ..., \tilde{E}_n\}$ form an orthonormal frame of $\beta(\mathcal{D}_\varphi)$. Since the metric on $N_{1\varphi}$ of a warped product $N_{1\varphi} \times_h N_{2\perp}$ is neutral, it is even-dimensional([9]). Thus $n = 2p$. Then, we can choose a orthonormal frames $\{\hat{E}_1, ..., \hat{E}_{2p}\}$ of $\mathcal{D}_\varphi$ and $\{\tilde{E}_1, ..., \tilde{E}_{2p}\}$ of $\beta(\mathcal{D}_\varphi)$ in such a way that

$$\begin{aligned} \hat{E}_1 &= sech\varphi\alpha\hat{E}_1, ..., \hat{E}_{2p} = sech\varphi\alpha\hat{E}_{2p-1}, \ \ (type1) \\ \tilde{E}_1 &= csch\varphi\beta\tilde{E}_1, ..., \tilde{E}_{2p} = csch\varphi\beta\tilde{E}_{2p}, \ \ (type1) \end{aligned}$$

where φ is the slant function. We note that such an orthonormal frame is called an adapted frame ([2]).

Let us consider
- on $\mathcal{D}_\perp$: an orthonormal basis $\{E_i\}_{i=1,...,m}$, where $m = boy\mathcal{D}_\perp$; moreover, one can suppose that $\epsilon_i = g(E_i, E_i) = 1$.
- on $P(\mathcal{D}_\perp)$: an orthonormal basis $\{PE_j\}_{j=1,...,m}$, where $m = boyP(\mathcal{D}_\perp)$ and $\epsilon_j^* = g(PE_j, PE_j) = -1$.
- on $(\mathcal{D}_\varphi)$: an orthonormal basis $\{\hat{E}_a\}_{a=1,...,n}$, where $n = boy(\mathcal{D}_\varphi)$ and $\hat{\epsilon}_a = g(\hat{E}_a, \hat{E}_a) = \mp 1$.
- on $\beta(\mathcal{D}_\varphi)$: an orthonormal basis $\{\tilde{E}_b\}_{b=1,...,n}$, where $n = boy\beta(\mathcal{D}_\varphi)$ and $\tilde{\epsilon}_b = g(\tilde{E}_b, \tilde{E}_b) = \mp 1$.

Theorem 5.1. *Let $N = N_{1\varphi} \times_h N_{2\perp}$ be a mixed geodesic warped product submanifold of a para-Kaehler manifold $\tilde{N}$ such that $N_{1\varphi}$ is a $n-$dimensional neutral proper pointwise slant submanifold and $N_{2\perp}$ is a $m-$dimensional totally real submanifold of N. Suppose that $N_{2\perp}$ is spacelike. Then, the squared norm of the second fundamental form $\|\sigma\|^2$ of N satisfies*

$$\|\sigma\|^2 \leq m \coth^2 \varphi \|\nabla(\ln h)\|^2, \tag{34}$$

where $\nabla(\ln h)$ is the gradient of $\ln h$.

Proof. Since $\|\sigma\|^2 = \|\sigma(\mathcal{D}_\varphi, \mathcal{D}_\varphi)\|^2 + 2\|\sigma(\mathcal{D}_\varphi, \mathcal{D}_\perp)\|^2 + \|\sigma(\mathcal{D}_\perp, \mathcal{D}_\perp)\|^2$, if N is mixed geodesic we obtain

$$\|\sigma\|^2 = \|\sigma(\mathcal{D}_\varphi, \mathcal{D}_\varphi)\|^2 + \|\sigma(\mathcal{D}_\perp, \mathcal{D}_\perp)\|^2. \tag{35}$$

The first factor of the right hand side of (35) can be written as

$$||\sigma(\mathcal{D}_\varphi,\mathcal{D}_\varphi)||^2=\sum_{r=1}^{2p+m}\sum_{c,d=1}^{2p}g(\sigma(\hat{E}_c,\hat{E}_d),\bar{E}_r)^2.$$

Using the adapted frame, we have

$$||\sigma(\mathcal{D}_\varphi,\mathcal{D}_\varphi)||^2=\sum_{i=1}^{m}\sum_{c,d=1}^{2p}g(\sigma(\hat{E}_c,\hat{E}_d),PE_i)^2+\sum_{a=1}^{2p}\sum_{c,d=1}^{2p}g(\sigma(\hat{E}_c,\hat{E}_d),csch\varphi\beta\hat{E}_a)^2. \tag{36}$$

From (32), we get

$$||\sigma(\mathcal{D}_\varphi,\mathcal{D}_\varphi)||^2=\sum_{a=1}^{2p}\sum_{c,d=1}^{2p}g(\sigma(\hat{E}_c,\hat{E}_d),csch\varphi\beta\hat{E}_a)^2. \tag{37}$$

On the other hand we can write the second factor of the right side of (35) as

$$||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2=\sum_{r=1}^{2p+m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),\bar{E}_r)^2.$$

Using the adapted frame we arrive at

$$||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2=\sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+\sum_{c=1}^{2p}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),csch\varphi\beta\hat{E}_c)^2. \tag{38}$$

From (33), we get

$$||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2=\sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+m\sum_{c=1}^{2p}csch^2\varphi(\alpha\hat{E}_c\ln h)^2. \tag{39}$$

Further we can write (39) as

$$\begin{aligned}||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2 &= \sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+m(csch^2\varphi(\alpha\hat{E}_1(\ln h))^2\\ &+ csch^2\varphi(\alpha\hat{E}_2(\ln h))^2+...+csch^2\varphi(\alpha\hat{E}_{2p}(\ln h))^2).\end{aligned} \tag{40}$$

From (40) and using the adapted frame, we have

$$\begin{aligned}||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2 &= \sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+m(\coth^2\varphi(sech\varphi\alpha\hat{E}_1(\ln h))^2\\ &+ csch^2\varphi(sech\varphi\alpha^2\hat{E}_1(\ln h))^2+\coth^2\varphi(sech\varphi\alpha\hat{E}_3(\ln h))^2\\ &+ csch^2\varphi(sech\varphi\alpha^2\hat{E}_3(\ln h))^2+...+\coth^2\varphi(sech\varphi\alpha\hat{E}_{2p-1}(\ln h))^2\\ &+ csch^2\varphi(sech\varphi\alpha^2\hat{E}_{2p-1}(\ln h))^2).\end{aligned}$$

Using the Proposition 3.4, we obtain

$$\begin{aligned}||\sigma(\mathcal{D}_\perp,\mathcal{D}_\perp)||^2 &= \sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+m\sum_{c=1}^{2p}(\coth^2[\tilde{E}_{2c-1}(\ln h))^2+\hat{E}_{2c-1}(\ln h))^2\\ &= \sum_{k=1}^{m}\sum_{i,j=1}^{m}g(\sigma(E_i,E_j),PE_k)^2+m\coth^2||\nabla(\ln h)||^2.\end{aligned} \tag{41}$$

From (35), (37) and (41) we obtain (34).
If the equality sign of (34) holds identically, then $N_{1\varphi}$ is totally geodesic and $N_{2\perp}$ a totally umbilical submanifold in $\tilde{N}$. □

Remark 5.2. *If the manifold $N_{2\perp}$ of Theorem 5.1 is timelike, then (34) shall be replaced by*

$$||\sigma||^2 \geq m \coth^2 \varphi ||\nabla(\ln h)||^2. \tag{42}$$

In a similar way, for proper pointwise slant submanifold $N_{1\varphi}$ of type 2, we obtain the following result:

Theorem 5.3. *Let $N = N_{1\varphi} \times_h N_{2\perp}$ be a mixed geodesic warped product submanifold of a para-Kaehler manifold $\tilde{N}$ such that $N_{1\varphi}$ is a $n-$dimensional neutral proper pointwise slant submanifold and $N_{2\perp}$ is a $m-$dimensional totally real submanifold of N. Suppose that $N_{2\perp}$ is spacelike(respectively, timelike). Then, the squared norm of the second fundamental form $||\sigma||^2$ of N satisfies*

$$||\sigma||^2 \leq m \cot^2 \varphi ||\nabla(\ln h)||^2 \ (respectively,\ ||\sigma||^2 \geq m \cot^2 \varphi ||\nabla(\ln h)||^2) \tag{43}$$

where $\nabla(\ln h)$ is the gradient of $\ln h$.

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