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Existence of Solutions for Weighted p(t)-Laplacian Mixed Caputo Fractional Differential Equations at Resonance

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Abstract. Using Mawhin's coincidence degree theory, we investigate the existence of solutions for a class of weighted $p(t)-$Laplacian boundary value problems at resonance and involving left and right Caputo fractional derivatives. An example is provided to illustrate the main existence results.

1. Introduction

The mathematical modeling of several physical processes leads to a class of boundary value problems at resonance, that have recently received a lot of attention since any works are devoted to the study of the existence of solutions for this type of problem, see [5,9,11,13,14,18,19,21,23,31]. For some interesting results on boundary value problems in literature see [1,8,10,12,26,27,28,29].

Moreover, considerable attention is paid to p-Laplacian differential equations due to their importance in theory and application of mathematics and physics. Recently, the existence, uniqueness and the stability of solutions for differential equations with $p(t)$-Laplacian operator is studied in some papers [30-32], which is an interesting subject for investigation.

This work is devoted to the study of the existence of solutions for a class of $p(t)$-Laplacian differential equations involving left and right Caputo fractional derivatives:

$$(P)\begin{cases} D_{1^-}^{\theta}\left(\omega(t)\,\phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)\right)=f\left(t,x(t)\right),\ 0<t<1,\\ x(0)=0,\phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)_{|t=0}=\phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)_{|t=1}, \end{cases}$$

where $0<\theta,v<1$, $\theta+v>1$, $\phi_{p(t)}(u)=|u|^{p(t)-2}u$, for $(t,u)\in[0,1]\times\mathbb{R}$, $p(t)>1$, $0\le t\le 1$, $p\in C^1[0,1]$, $p(0)=p(1)$, $\min_{0\le t\le 1}p(t)=p_*$, $\max_{0\le t\le 1}p(t)=p^*$, $\omega\in C[0,1]$, $\omega(t)>0$, $\omega(0)=\omega(1)$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$,

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$D_{1^-}^{\theta}$ and $D_{0^+}^{v}$ refer to the left and right Caputo fractional derivatives respectively. Note that the problem (P) is at resonance, i.e., the corresponding homogeneous fractional boundary value problem

$$\begin{aligned} D_{1^-}^{\theta}\left(\omega(t)\,\phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)\right) &= 0,\ 0<t<1, \\ x(0) &= 0, \phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)_{|t=0} = \phi_{p(t)}\left(D_{0^+}^{v}x(t)\right)_{|t=1}, \end{aligned}$$

has $x(t) = I_{0^+}^{v}\left(\phi_{p(t)}(a)\right)$, $a \in \mathbb{R}$, as nontrivial solutions. We establish some sufficient conditions for the existence of at least one solution for problem (P). Since the operator $D_{1^-}^{\theta}\phi_{p(t)}\left(D_{0^+}^{v}x\right)$ is nonlinear and in order to apply Mawhin's coincidence degree, we transform the problem (P) into an equivalent system of two differential equations

$$(S)\begin{cases} D_{0^+}^{v}x(t) = \phi_{p(t)}^{-1}\left((\omega(t))^{-1}z(t)\right), \\ D_{1^-}^{\theta}z(t) = f(t,x(t)), \\ x(0) = 0, z(0) = z(1), \end{cases}$$

that permits to write the linear operator as

$$L(x,z) = \left(D_{0^+}^{v}x(t), D_{1^-}^{\theta}z(t)\right),$$

see Section 3. Let us mention that the study of resonant boundary value problems involving mixed fractional-order derivatives have not been extensively studied, we can expose some existing works:

In [30], the authors investigated the existence and uniqueness of solution by the use of some fixed point theorems and Mawhin's coincidence degree, in resonance and non resonance cases, for the following Riemann-Liouville fractional boundary value problem:

$$\begin{aligned} &D_{0^+}^{p}\phi_{p(t)}\left(D_{0^+}^{q}x(t)\right) + f(t,x(t)) = 0,\ 0<t<1, \\ &x(0) = 0, D_{0^+}^{q}x(0) = 0, D_{0^+}^{q-1}x(1) = \gamma I_{0^+}^{q-1}x(\eta), \end{aligned}$$

where $1 < q \le 2, 0 < p < 1, 0 < \eta < 1, \gamma > 0$.

In [31], the authors studied, by means of Mawhin's coincidence degree, the existence of solutions for the following Caputo-Riemann-Liouville fractional boundary value problem at resonance:

$$\begin{aligned} &{}^{C}D_{0^+}^{p}\phi_{p(t)}\left(D_{0^+}^{q}x(t)\right) = f\left(t,x(t),D_{0^+}^{q}x(t)\right),\ 0<t<T, \\ &t^{1-q}x(t)_{|t=0} = 0, D_{0^+}^{q}x(t)_{|t=0} = D_{0^+}^{q}x(t)_{|t=T}, \\ &0<p,q<1, 1<p+q<2. \end{aligned}$$

Fractional differential equations containing mixed type fractional derivatives have been studied in some works, by different methods such, Lower and upper solutions method, fixed point theorems..., see [2,3,4,7,15,16,17].

Next, we start with some necessary background. In Section 3, we prove some lemmas that will used in the proof of the main results. In Section 4, we give the existence result, then we end this paper by an illustrative example.

2. Preliminaries

We began by defining Riemann-Liouville fractional integrals and Caputo fractional derivatives, then we state some of their properties, that can be found in details in [6,20,24,25].

The left and right Riemann-Liouville fractional integrals of order $\theta > 0$, on $[a,b]$ of a function y are defined respectively by

$$\begin{aligned} I_{a^+}^{\theta}y(t) &= \frac{1}{\Gamma(\theta)}\int_a^t (t-s)^{\theta-1}y(s)ds, t>a, \\ I_{b^-}^{\theta}y(t) &= \frac{1}{\Gamma(\theta)}\int_t^b (s-t)^{\theta-1}y(s)ds,\ t<b. \end{aligned}$$

The left and the right Caputo derivatives $D_{a^+}^{\theta}$ and $D_{b^-}^{\theta}$ of order $\theta > 0$, on $[a,b]$ of the function $y \in AC^n[a,b]$, are defined

$$\begin{aligned} D_{a^+}^{\theta} y(t) &= \frac{1}{\Gamma(n-\theta)} \int_a^t (t-s)^{n-\theta-1} y^{(n)}(s)\, ds, t > a, \\ D_{b^-}^{\theta} y(t) &= \frac{(-1)^n}{\Gamma(n-\theta)} \int_t^b (s-t)^{n-\theta-1} y^{(n)}(s)\, ds, t < b \end{aligned}$$

respectively, where $n = [\theta] + 1$, $[\theta]$ is the integer part of θ.

We present some properties of fractional integrals and Caputo derivatives.

1- The homogeneous fractional differential equations $D_{a^+}^{q} g(t) = 0$ and $D_{b^-}^{q} g(t) = 0$ have respectively the following solutions

$$g(t) = \sum_{i=0}^{n-1} c_i (t-a)^i \text{ and } g(t) = \sum_{i=0}^{n-1} a_i (b-t)^i,$$

where, $a_i, c_i \in \mathbb{R}$, $i = 1, ..., n$ and $n = [q] + 1$, if $q \notin \mathbb{N}$, $n = q$, if $q \in \mathbb{N}$.

2- $D_{a^+}^{\theta} I_{a^+}^{\theta} y(t) = y(t)$, $D_{b^-}^{\theta} I_{b^-}^{\theta} y(t) = y(t)$.

3- $D_{a^+}^{\theta} (t-a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} (t-a)^{\gamma-\theta-1}$,

$D_{b^-}^{\theta} (b-t)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\theta)} (b-t)^{\gamma-\theta-1}$, $\gamma > [\theta] + 1$.

Now, we present some definitions and notations which will be used later.

Lemma 2.1. *[32] For any $(t,u) \in [0,T] \times \mathbb{R}$, $\phi_{p(t)}(u) = |u|^{p(t)-2} u$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and strictly monotone increasing for any fixed t. Moreover, its inverse operator is defined by $\phi_{p(t)}^{-1} = \phi_{q(t)}(u) = |u|^{\frac{2-p(t)}{p(t)-1}} u$, where $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, and is continuous and sends bounded sets to bounded sets.*

Let X and Y be two real Banach spaces and let $L : domL \subset X \to Y$ be a linear operator.

Definition 2.2. *A linear operator L is said to be Fredholm operator of index zero, if ImL is closed subset in Y and* $\dim \ker L = co \dim ImL < \infty$.

Define the continuous projections P and Q respectively by $P : X \to X$, $Q : Y \to Y$ such that $ImP = \ker L$, $\ker Q = ImL$. Then $X = \ker L \oplus \ker P$, $Y = ImL \oplus ImQ$, thus $L\mid_{domL \cap \ker P} : domL \cap \ker P \to ImL$ is invertible, denote its inverse by K_P.

Definition 2.3. *Let Ω be an open bounded subset of X such that $domL \cap \Omega \neq \emptyset$. The map $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if the map $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.*

Since ImQ is isomorphic to $\ker L$, there exists an isomorphism $J : ImQ \to \ker L$. It is known that the coincidence equation $Lx = Nx$ is equivalent to $x = (P + JQN)x + K_P(I-Q)Nx$.

Theorem 2.4. *([22]) Let L be a Fredholm operator of index zero and N be L− compact on $\overline{\Omega}$. Assume that the following conditions are satisfied :*

(1) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(domL \setminus \ker L) \cap \partial\Omega] \times (0,1)$.*

(2) *$Nx \notin ImL$ for every $x \in \ker L \cap \partial\Omega$.*

(3) $\deg(QN\mid_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *where* $Q : Y \to Y$ is a projection such that $ImL = \ker Q$.

Then the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

3. Some Lemmas

We rewrite the problem (P) as an equivalent system (S):

$$(S)\begin{cases} D_{0^+}^{\upsilon}x(t)=\phi_{p(t)}^{-1}\left((\omega(t))^{-1}z(t)\right), \\ D_{1^-}^{\theta}z(t)=f(t,x(t)), \\ x(0)=0, z(0)=z(1). \end{cases}$$

It is clear that if (x,z) is a solution for system (S), then x is a solution for problem (P). Let X be the Banach product space $X=C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$, with the norm $\|(x,z)\|_X=\max(\|x\|,\|z\|)$, where $\|.\|$ is the uniform norm in $C([0,1],\mathbb{R})$.

Define the linear operator $L: domL\subset X\to X$ by

$$L(x,z)=\left(D_{0^+}^{\upsilon}x(t), D_{1^-}^{\theta}z(t)\right), \tag{3.1}$$

where

$$domL=\left\{(x,z)\in X, D_{0^+}^{\upsilon}x, D_{1^-}^{\theta}z(t)\in C([0,1],\mathbb{R}), x(0)=0, z(0)=z(1)\right\}.$$

Let $N: X\to X$ be the operator

$$N(x(t),z(t))=\left(\phi_{p(t)}^{-1}\left((\omega(t))^{-1}z(t)\right), f(t,x(t))\right),\ t\in[0,1],$$

then the system (S) can be written as $L(x,z)=N(x,z)$.

Lemma 3.1. *We have*

$$\begin{aligned} \ker L &= \{(x,z)\in domL,\ (x(t),z(t))=(0,a), a\in\mathbb{R}, t\in[0,1]\}, \\ ImL &= \left\{(y_1,y_2)\in X, \int_0^1 s^{\theta-1}y_2(s)ds=0\right\}. \end{aligned}$$

Proof. Let $U=(x,z)\in\ker L$, then $LU=0$, i.e $D_{0^+}^{\upsilon}x(t)=0$ and $D_{1^-}^{\theta}z(t)=0$. From the properties of fractional derivatives we get $x(t)=b$ and $z(t)=a$, $a,b\in\mathbb{R}$, thus $b=0$ by condition $x(0)=0$ and consequently $U=(0,a)$, $a\in\mathbb{R}$.

Let $Y=(y_1,y_2)\in ImL$, then there exists $U=(x,z)\in domL$, such $LU=Y$. Thus $y_1(t)=D_{0^+}^{\upsilon}x(t)$ and $y_2(t)=D_{1^-}^{\theta}z(t)$. Taking into account the properties of fractional integrals and derivatives and the boundary conditions in the system (S), it yields $x(t)=I_{0^+}^{\upsilon}y_1(t)$ and

$$\int_0^1 s^{\theta-1}y_2(s)\,ds=0. \tag{3.2}$$

Conversely if (3.2) holds, set $(x(t),z(t))=\left(I_{0^+}^{\upsilon}y_1(t), I_{1^-}^{\theta}y_2(t)\right)$, then $x(0)=0$ and $z(0)=z(1)$, thus $(x,z)\in domL$ and $L(x,z)=(y_1,y_2)$, that is $Y=(y_1,y_2)\in ImL$. The proof is complete. □

Lemma 3.2. *The operator $L: domL\subset X\to X$ is a Fredholm operator of index zero. The linear projector operators P, $Q: X\to X$ satisfy*

$$\begin{aligned} P(x(t),z(t)) &= (0,z(0)), \\ Q(y_1(t),y_2(t)) &= \left(0,\theta\int_0^1 s^{\theta-1}y_2(s)ds\right). \end{aligned}$$

Furthermore, the operator $K_p: ImL\to domL\cap\ker P$ defined by

$$K_p(y_1(t),y_2(t))=\left(I_{0^+}^{\upsilon}y_1(t), I_{1^-}^{\theta}y_2(t)\right)$$

is the inverse of $L\,|_{domL\cap\ker P}$ *and satisfies*

$$\left\|K_pY\right\|_X \leq \max\left(\frac{1}{\Gamma(\upsilon)}, \frac{1}{\Gamma(\theta)}\right)\|Y\|_X . \tag{3.3}$$

Proof. Let $Y = (y_1, y_2)$, we claim that the continuous operator Q is a projector. In fact

$$Q^2(Y(t)) = Q\left(\left(0, \theta\int_0^1 s^{\theta-1}y_2(s)ds\right)\right) = Q(Y(t)),$$

and $ImL = \ker Q$.

We claim that L is a Fredholm operator of index zero.
Indeed, since $Y = (Y - QY) + QY$, then $Y - QY \in \ker Q = ImL$, $QY \in ImQ$ and $ImQ \cap ImL = \{0\}$, hence $X = ImL \oplus ImQ$. Thus, $1 = \dim\ker L = \dim ImQ = co\dim ImL$.
Now, Let $U = (x, z)$,

$$P^2U(t) = P(0, z(0)) = (0, z(0)) = PU(t),$$

thus P is a projection and $ImP = \ker L$.
In view of $U = (U - PU) + PU$, then $X = \ker P + \ker L$. Moreover we have $\ker L \cap \ker P = \{0\}$, thus $X = \ker L \oplus \ker P$. We claim that the generalized inverse of L is K_P. In fact, let $Y = (y_1, y_2) \in ImL$, then

$$\left(LK_p\right)Y(t) = \left(D_{0^+}^{\upsilon}I_{0^+}^{\upsilon}y_1(t), D_{1^-}^{\theta}I_{1^-}^{\theta}y_2(t)\right) = Y(t).$$

Furthermore, if $U = (x, z) \in domL \cap \ker P$, it yields

$$\begin{aligned}\left(K_pL\right)U(t) &= K_p\left(D_{0^+}^{\upsilon}x(t), D_{1^-}^{\theta}z(t)\right) = \left(I_{0^+}^{\upsilon}D_{0^+}^{\upsilon}x(t), I_{1^-}^{\theta}D_{1^-}^{\theta}z(t)\right)\\ &= (x(t) + x(0), z(t) + z(1)).\end{aligned}$$

Since $P(x(t), z(t)) = 0$, then $z(0) = z(1) = 0$, consequently $\left(K_pL\right)U(t) = U(t)$, that implies $K_p = \left(L|_{domL\cap\ker P}\right)^{-1}$.
By definition of K_p, we get

$$\begin{aligned}\left\|K_pY\right\|_X &= \max\left(\left\|I_{0^+}^{\upsilon}y_1\right\|, \left\|I_{1^-}^{\theta}y_2\right\|\right) \leq \max\left(\frac{1}{\Gamma(\upsilon)}\left\|y_1\right\|, \frac{1}{\Gamma(\theta)}\left\|y_2\right\|\right)\\ &\leq \max\left(\frac{1}{\Gamma(\upsilon)}, \frac{1}{\Gamma(\theta)}\right)\|Y\|_X,\end{aligned}$$

The proof is complete. □

4. Existence of solutions

We make the following hypotheses:
(H1) There exist functions $\alpha, \beta \in C[0, 1]$, such that for all $x \in \mathbb{R}$, $t \in [0, 1]$, we have

$$\left|f(t, x)\right| \leq \alpha(t)|x| + \beta(t). \tag{4.1}$$

(H2) There exists a constant $M > 0$, such that if $\left|D_{0^+}^{\upsilon}x(t)\right| > M$, for all $t \in [0, 1]$, then

$$\int_0^1 s^{\theta-1}f(s, x(s))\,ds \neq 0. \tag{4.2}$$

(H3) There exists a constant $M^* > 0$, such that for any $(0, a) \in \ker L$ with $|a| > M^*$, either

$$a\int_0^1 s^{\theta-1}f(s, 0)\,ds < 0, \tag{4.3}$$

or

$$a\int_0^1 s^{\theta-1} f(s,0)\,ds > 0. \tag{4.4}$$

Lemma 4.1. *Let Ω is be an open bounded subset of X such that $domL \cap \overline{\Omega} \neq \emptyset$. Under hypothesis (H1), N is $L-$compact on $\overline{\Omega}$.*

Proof. To prove that N is $L-$compact on $\overline{\Omega}$, it suffices to prove that $QN\left(\overline{\Omega}\right)$ is bounded and $K_P(I-QN)\left(\overline{\Omega}\right)$ is compact. Let $U=(x,z)\in\overline{\Omega}$, then there exists $r>0$, such that $\|U\|_X=\max(\|x\|,\|z\|)\le r$. We have

$$\begin{aligned} QNU(t) &= Q\left(\phi_{p(t)}^{-1}\left((\omega(t))^{-1} z(t)\right), f(t,x(t))\right),\\ &= \left(0, \theta\int_0^1 s^{\theta-1} f(s,x(s))\,ds\right),\ t\in[0,1]. \end{aligned}$$

Thanks to hypothesis (H1), we get

$$\left|\theta\int_0^1 s^{\theta-1} f(s,x(s))\,ds\right| \le \|\alpha\|\|x\| + \left\|\beta\right\| \le r\|\alpha\| + \left\|\beta\right\|,$$

thus

$$\|QNU\|_X \le r\|\alpha\| + \left\|\beta\right\|, \tag{4.5}$$

from which we conclude $QN\left(\overline{\Omega}\right)$ is bounded.

Next, we claim that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is compact. In fact, set

$$\varpi = \max_{0\le t\le 1}\left((\omega(t))^{\frac{-1}{p(t)-1}}\right),$$

so,

$$\left|\phi_{p(t)}^{-1}\left(\omega^{-1}(t)z(t)\right)\right| = \left|(\omega(t))^{-1} z(t)\right|^{\frac{1}{p(t)-1}} \le \varpi\|z\|^{\frac{1}{p_*-1}} \le \varpi r^{\frac{1}{p_*-1}}, \tag{4.6}$$

and (4.1) gives

$$\left|f(t,x(t))\right| \le r\|\alpha\| + \left\|\beta\right\|. \tag{4.7}$$

Since

$$NU(t) = \left(\phi_{p(t)}^{-1}\left((\omega(t))^{-1} z(t)\right), f(t,x(t))\right),$$

then, taking (4.6) and (4.7) into account, it yields

$$\|NU\|_X \le \max\left(r\|\alpha\| + \left\|\beta\right\|, \varpi r^{\frac{1}{p_*-1}}\right). \tag{4.8}$$

Now, from the definition of K_P and inequalities (3.3), (4.5) and (4.8), we obtain

$$\begin{aligned} \|K_P(I-Q)NU\|_X &\le \max\left(\frac{1}{\Gamma(\upsilon)}, \frac{1}{\Gamma(\theta)}\right)\|(I-Q)NU\|_X,\\ &\le \max\left(\frac{1}{\Gamma(\upsilon)}, \frac{1}{\Gamma(\theta)}\right)\left[\|NU\|_X + \|QNU\|_X\right], \end{aligned}$$

$$\leq \max\left(\frac{1}{\Gamma(\upsilon)}, \frac{1}{\Gamma(\theta)}\right)\left[\max\left(\left(r\|\alpha\| + \|\beta\|\right), \varpi r^{\frac{1}{p_*-1}}\right) + \left(r\|\alpha\| + \|\beta\|\right)\right]. \tag{4.9}$$

that implies $K_P(I-Q)N\left(\overline{\Omega}\right)$ is uniformly bounded.

Let us prove that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equicontinuous. Define the operators $T_i : X \to C[0,1], i = 1,2,$ by

$$\begin{aligned} T_1(x(t), z(t)) &= I_{0^+}^{\upsilon}\phi_{p(t)}^{-1}\left((\omega(t))^{-1} z(t)\right), \\ T_2(x(t), z(t)) &= I_{1^-}^{\theta} f(t, x(t)) - \frac{(1-t)^{\theta}}{\Gamma(\theta)}\int_0^1 s^{\theta-1} f(s, x(s))\, ds, \end{aligned}$$

then

$$(K_P(I-Q)NU)(t) = (T_1(x(t), z(t)), T_2(x(t), z(t))).$$

To prove that $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equicontinuous it suffices to prove that $T_i\left(\overline{\Omega}\right), i = 1,2$ are equicontinuous. For any $U = (x,z) \in \overline{\Omega}$, and $t_1, t_2 \in [0,1], t_1 < t_2$, we get by the help of (4.6),

$$\begin{aligned} & |T_1(x(t_1), z(t_1)) - T_1(x(t_2), z(t_2))| \\ =\ & \left|I_{0^+}^{\upsilon}\phi_{p(t)}^{-1}\left((\omega(t_1))^{-1} z(t_1)\right) - I_{0^+}^{\upsilon}\phi_{p(t)}^{-1}\left((\omega(t_2))^{-1} z(t_2)\right)\right|, \\ \leq\ & \frac{1}{\Gamma(\theta)}\int_0^{t_1}\left((t_1-s)^{\nu-1} - (t_2-s)^{\nu-1}\right)\left|\phi_{p(s)}^{-1}\left((\omega(s))^{-1} z(s)\right)\right| ds \\ & + \frac{1}{\Gamma(\theta)}\int_{t_1}^{t_2}(t_2-s)^{\nu-1}\left|\phi_{p(s)}^{-1}\left((\omega(s))^{-1} z(s)\right)\right| ds, \\ \leq\ & \frac{\varpi r^{\frac{1}{p_*-1}}}{\Gamma(\theta+1)}\left(t_1^{\upsilon} + 2(t_2-t_1)^{\upsilon}) - t_2^{\upsilon}\right) \to 0, \quad \text{as } t_1 \to t_2. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & |T_2(x(t_1), z(t_1)) - T_2(x(t_2), z(t_2))| \\ \leq\ & \left|I_{1^-}^{\theta} f(t_1, x(t_1)) - I_{1^-}^{\theta} f(t_2, x(t_2))\right| \\ & + \frac{(1-t_2)^{\theta} - (1-t_1)^{\theta}}{\Gamma(\theta)}\int_0^1 s^{\theta-1}\left|f(s, x(s))\right| ds, \\ \leq\ & \frac{1}{\Gamma(\theta)}\int_{t_1}^{t_2}(s-t_1)^{\theta-1}\left|f(s, x(s))\right| ds \\ & + \frac{1}{\Gamma(\theta)}\int_{t_2}^{1}\left((s-t_2)^{\theta-1} - (s-t_1)^{\theta-1}\right)\left|f(s, x(s))\right| ds \\ & + \frac{(1-t_1)^{\theta} - (1-t_2)^{\theta}}{\Gamma(\theta)}\int_0^1 s^{\theta-1}\left|f(s, x(s))\right| ds, \\ \leq\ & \frac{2\left(r\|\alpha\| + \|\beta\|\right)}{\Gamma(\theta+1)}(t_2-t_1)^{\theta} \to 0, \quad \text{as } t_1 \to t_2. \end{aligned}$$

So $K_P(I-Q)N\left(\overline{\Omega}\right)$ is equicontinuous on $[0,1]$, and then $K_P(I-QN) : \overline{\Omega} \to X$ is compact. $\square$

Lemma 4.2. *Let* $\Omega_1 = \{U \in domL \backslash \ker L : LU = \lambda NU \text{ for some } \lambda \in (0,1)\}$. *If conditions (H1)-(H2) hold, then* Ω_1 *is bounded.*

Proof. Suppose that $U = (x, z) \in \Omega_1$, then $x(0) = 0$ and $z(0) = z(1)$. Since $I_{0^+}^{v} D_{0^+}^{v} x(t) = x(t)$, then

$$\|x\| \leq \frac{\left\|D_{0^+}^{v} x\right\|}{\Gamma(v+1)}.$$

Taking into account that $NU \in ImL$, we get $\int_0^1 s^{\theta-1} \left|f(s, x(s))\right| ds = 0$, then by hypothesis (H2), we deduce that $\left|D_{0^+}^{v} x(t)\right| \leq M$, for all $t \in [0,1]$. Hence

$$\|x\| \leq \frac{M}{\Gamma(v+1)}. \tag{4.10}$$

In addition we have

$$z(t) = I_{1^-}^{\theta} D_{1^-}^{\theta} z(t) - z(1),$$

using the condition $z(0) = z(1)$, it yields

$$z(1) = \frac{1}{2\Gamma(\theta)} \int_0^1 s^{\theta-1} D_{1^-}^{\theta} z(s)\, ds,$$

that implies

$$\|z\| \leq \frac{3\left\|D_{1^-}^{\theta} z\right\|}{2\Gamma(\theta+1)}. \tag{4.11}$$

Now, since $D_{1^-}^{\theta} z(t) = \lambda f(t, x(t))$ for some $\lambda \in (0,1)$, then in view of hypothesis (H1) and (4.10), we obtain

$$\left\|D_{1^-}^{\theta} z\right\| \leq \|x\| \|\alpha\| + \left\|\beta\right\| \leq \frac{M\|\alpha\|}{\Gamma(v+1)} + \left\|\beta\right\|, \tag{4.12}$$

combining (4.11) and (4.12) it yields

$$\|z\| \leq \frac{3}{2\Gamma(\theta+1)} \left(\frac{M\|\alpha\|}{\Gamma(v+1)} + \left\|\beta\right\| \right). \tag{4.13}$$

From (4.10) and (4.13) we obtain

$$\begin{aligned} \|U\|_X &= \max(\|x\|, \|z\|), \\ &\leq \max\left(\frac{M}{\Gamma(v+1)}, \frac{3}{2\Gamma(\theta+1)} \left(\frac{M\|\alpha\|}{\Gamma(v+1)} + \left\|\beta\right\| \right) \right) < \infty, \end{aligned}$$

which shows that Ω_1 is bounded. □

Lemma 4.3. *Assume that (H3) holds. Then the set*

$$\Omega_2 = \{U \in \ker L : \ NU \in ImL\}$$

is bounded.

Proof. Let $U = (x, z) \in \Omega_2$, then $x(t) = 0$ and $z(t) = a$, $a \in \mathbb{R}$, $t \in [0,1]$. Now since $ImL = \ker Q$, we get $QNU = 0$, thus $\int_0^1 s^{\theta-1} f(s, 0) ds = 0$, which implies by hypothesis (H3) that $|a| \leq M^*$, so $\|U\|_X \leq M^*$, hence Ω_2 is bounded. □

Lemma 4.4. *Assume that conditions (H2) and (H3) hold. Then the set*

$$\Omega_3 = \{U \in \ker L \ : -\lambda JU + (1-\lambda)\, QNU = 0, \ \ \lambda \in [0,1]\}$$

is bounded, where $J : \ker L \to ImQ$ *is the linear isomorphism given by*
$J(0,c) = (0,c)\,, \forall c \in \mathbb{R}.$

Proof. Let $U_0 = (x_0, z_0) \in \Omega_3$, then $x_0(t) = 0$, $z_0(t) = a$, $a \in \mathbb{R}$, $t \in [0,1]$. Since $\lambda JU_0 = (1-\lambda)\, QNU_0$ then

$$\lambda a = (1-\lambda)\,\theta \int_0^1 s^{\theta-1} f(s,0)ds.$$

Let $0 < \lambda < 1$, and assume that (4.3) is satisfied, then

$$\lambda a^2 = (1-\lambda)\, a\theta \int_0^1 s^{\theta-1} f(s,0)ds < 0,$$

which contradicts the fact that $\lambda a^2 \geq 0$, consequently $|a| \leq M^*$ and $\|U_0\|_X \leq M^*$, hence Ω_3 is bounded. Now $\lambda = 1$, gives $a = 0$, so $U_0 = 0$. If $\lambda = 0$, we get $\int_0^1 s^{\theta-1} f(s,0)ds = 0$, thus by the help of hypothesis (H3), it yields $|a| \leq M^*$, thus $\|U_0\|_X \leq M^*$. If we assume that (4.4) holds then we prove by analoguously that $\Omega_3 = \{U \in \ker L \ : \lambda JU + (1-\lambda)\, QNU = 0, \ \ \lambda \in [0,1]\}$ is bounded. □

Theorem 4.5. *Assume that hypotheses (H1)-(H3) hold. Then the problem (P) has at least one solution in X.*

Proof. Let Ω to be an open bounded subset of X such that $\cup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. We know by Lemmas 6 and 7, that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. From Lemma 8, we deduce that $LU \neq \lambda NU$ pour tout $(U,\lambda) \in [(domL \backslash \ker L) \cap \partial\Omega] \times (0,1)$. By Lemma 9, we see that $NU \notin ImL$ pour tout $U \in \ker L \cap \partial\Omega$. Now, let H be the homotopy joining maps $\pm J$ and QN :

$$H(U,\lambda) = \pm\lambda JU + (1-\lambda)\, QNU,$$

then $H(\cdot,0) = QN$ and $H(\cdot,1) = \pm J$. Since $\overline{\Omega}_3 \subset \Omega$, then $H(U,\lambda) \neq 0$ for every $U \in \ker L \cap \partial\Omega$. By the homotopy property of degree, we get

$$\begin{aligned} \deg\left(QN\,|_{\ker L}, \Omega \cap \ker L, 0\right) &= \deg\left(H(\cdot,0), \Omega \cap \ker L, 0\right), \\ &= \deg\left(H(\cdot,1), \Omega \cap \ker L, 0\right), \\ &= \deg\left(\pm J, \Omega \cap \ker L, 0\right) \neq 0. \end{aligned}$$

Thanks to Theorem 4, the equation $LU = NU$ has at least one solution in $domL \cap \overline{\Omega}$, thus the problem (P) has at least one solution in X. The proof is completed. □

Example 4.6. *Let us consider the problem (P) with*

$$\begin{aligned} \theta &= 0.6, \upsilon = 0.2, p(t) = 2 + t\sin(1-t)\,, \omega(t) = (1 + (e-1)\,t)\, e^{-t}, \\ f(t,x) &= e^{-t}\frac{|\sin x|}{1+x^2} + e^t \cos t, (t,x) \in [0,1] \times \mathbb{R}, \end{aligned}$$

then all hypotheses of Theorem 11 are satisfied indeed,

$$\left|f(t,x)\right| \leq \alpha(t)\,|x| + \beta(t)\,, \ \ \textit{with}\ \alpha(t) = e^{-t}, \beta(t) = e^t,$$

$$\int_0^1 s^{\theta-1} f(s, x(s))\, ds \geq \int_0^1 s^{\theta-1} e^s \cos s ds = 2.1571,$$

thus condition (H2) is satisfied for any constant $M > 0$. *Moreover for* $M^* = 1 > 0$, *such that for any* $(0, a) \in \ker L$ *with* $|a| > M^*$, *we have*

$$\int_0^1 s^{\theta-1} f(s,0)\, ds = \int_0^1 s^{\theta-1} e^s \cos s ds = 2.1571 > 0.$$

then $a \int_0^1 s^{\theta-1} f(s,0)\, ds > 0$, *if* $a > 0$ *or* $a \int_0^1 s^{\theta-1} f(s,0)\, ds < 0$, *if* $a < 0$. *Consequently, problem (P) has at least one solution in X.*

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