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Positive Solutions for Some Asymptotically Linear and Superlinear Weighted Problems

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Abstract. In this paper, we study the following nonlinear elliptic problem

$$-\text{div}(a(x)\nabla u) = f(x,u), \quad x \in \Omega \;\; u \in H_0^1(\Omega) \tag{P}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 2$, $a(x)$ a bounded positive function and the nonlinear reaction source is strongly asymptotically linear in the following sense

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = q(x)$$

uniformly in $x \in \Omega$.
We use a variant version of Mountain Pass Theorem to prove that the problem (P) has a positive solution for a large class of $f(x,t)$ and $q(x)$. Here, the existence of solution is proved without use neither the Ambrosetti-Rabionowitz condition nor one of its refinements. As a second result, we use the same techniques to prove the existence of solutions when $f(x,t)$ is superlinear and subcritical on t at infinity.

1. Introduction and Main Results

In this paper, let Ω be a regular bounded domain in $\mathbb{R}^N$, $N \geq 2$ and consider the following quasi-linear elliptic weighted problem:

$$\begin{cases} -\text{div}(a(x)\nabla u) = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $a(x)$ is a continuous function on $\overline{\Omega}$ and $f(x,t)$ is strongly asymptotically linear function:

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = q(x),$$

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where $q(x)$ is a bounded function. When modeling morphogenesis phenomena in Biology and the population dynamic, Turing (1952) induce this type of equations for the interaction of species or chemicals:

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x)\nabla u) = f(x,u),$$

u is the density and $\operatorname{div}(a(x)\nabla u)$ represents the substance of diffusion through the system and finally f models the interaction of substances. In the stationary case and when $f(x,t) = f(t)$ depends only on t and f is asymptotically linear at $+\infty$, that is $q(x) \equiv l = const.$, the problem (1) was studied by Sâanouni and Trabelsi in [21]. In fact the results in [21] was a generalisation of those founded by Mironescu and Rădulescu in [10, 15, 16, 18] where a is constant with the same conditions on the nonlinearity $f(t)$. Their proof of the existence of positive solutions is based on the condition $f(0) > 0$, since they take a positive, C^1, convex increasing real values function f. Also, with this conditions on the reaction function f and when f is super-linear ($l = +\infty$) the problem was studied in [5, 13, 14]. After that, the same problem (1) (and the same conditions on f) with $a(x) = const.$ was generated to the p-Laplace operator in [9, 20]. The problem with the Bi-Laplacian operator was treated in [1, 2, 22, 25].
In order to study the problem for more large class of functions, Zhou in [26] consider the case when the asymptotically nonlinear term $f(x,s)$ depends on x and s and $f(x,0) = 0$. More precisely, he considered the following conditions:

$(F1)$ $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,t) \geq 0$ for all $t > 0$ and $x \in \overline{\Omega}$ and $f(x,t) \equiv 0$ for $t \leq 0$ and $x \in \overline{\Omega}$.

$(F2)$ $\lim_{t\to 0} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to+\infty} \frac{f(x,t)}{t} = q(x)$ uniformly in a.e.$x \in \Omega$, where $p(x)$ and $q(x)$ are bounded functions and $\|p(x)\|_\infty < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $(-\operatorname{div}(a(x)\nabla.), H_0^1(\Omega))$.

$(F3)$ The function $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$, for $a.e.x \in \Omega$.

In this paper, we will study the solvability of the problem (1) when the function a is not constant and $f(x,s)$ is strongly asymptotically linear.
We start by giving the definition of solution (weak solution) for the problem (1).

Definition 1.1. *A function $u \in H_0^1(\Omega)$ is called solution of the problem* (1) *if*

$$\int_\Omega a(x)\nabla u \nabla \varphi dx = \int_\Omega f(x,u)\varphi dx, \tag{2}$$

for all $\varphi \in H_0^1(\Omega)$.

Our approach is variational and we consider the following functional I defined on $H_0^1(\Omega)$ by

$$I(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 dx - \int_\Omega F(x,u)dx \tag{3}$$

where

$$F(x,s) = \int_0^s f(x,t)dt.$$

To prove the existence of nonzero critical point of I, we use a different version of the Mountain Pass Theorem given in [7].

Theorem 1.2. *[7] Let H be a real Banach space and suppose that $I \in C^1(H,\mathbb{R})$ satisfies the condition*

$$\max\{I(0), I(e)\} \leq \alpha < \beta \leq \inf_{\|u\|=\rho} I(u)$$

for some $\alpha < \beta$, $\rho > 0$, and $e \in H$ with $\|e\| \geq \rho$. Let $c \geq \beta$ be characterized by

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],H);\ \gamma(0) = 0 \ \text{ and }\ \gamma(1) = e\}$ the set of continuous paths joining 0 and e. Then, there exists a sequence (u_n) in H satisfying the Cerami conditions:

$$I(u_n) \to c \quad as \ \ n \to +\infty \tag{4}$$

and

$$(1 + \|u_n\|)\|I'(u_n)\|_* \to 0 \quad as \ \ n \to +\infty. \tag{5}$$

In order to prove that the Cerami sequence given by Theorem 1.2 has a convergent subsequence and so the functional I has a nontrivial critical point, it is often assumed that the nonlinearity satisfies the following Ambrosetti-Rabionovitz condition introduced in [3, 17]:
there exist some constants $\theta > 2$ and $M > 0$ such that

$$(AR) \qquad\qquad 0 < \theta F(x,t) \leq f(x,t)t,$$

for all $|t| \geq M$ and $x \in \Omega$.
But here, for the asymptotically nonlinearities, we can not suppose such condition since the condition (AR) implies that $\lim_{t\to+\infty} \dfrac{F(x,t)}{t^2} = +\infty$ and as consequence
$\lim_{t\to+\infty} \dfrac{f(x,t)}{t} = +\infty$ which contradicts $(F2)$. There are many other conditions imposed to solve the compactness problem, we can refer to [6, 7, 11, 20, 23, 24, 26] and the references therein. In this paper we will prove the compactness property for the Cerami sequence without any additive assumption or hypothesis on the function f.
Before introducing our results, we remark that the asymptotically nonlinearities attract more and more attention: In [12] Li and Huang consider a generalized quasilinear Schrödinger equations with asymptotically linear nonlinearities. They supposed that the nonlinearities f depend only on t and they proved the existence of positive solutions using variational methods.
In this paper, we suppose that $a(x)$ is positive and bounded:

$$(A) \qquad\qquad 0 < a_1 \leq a(x) \leq a_2,$$

for some positive constants a_1 and a_2 a.e. $x \in \Omega$.
Let $\|u\|_p = \left(\int_\Omega |u|^p\right)^{1/p}$ denotes the $L^p(\Omega)$-norm. Consider the inner product in $H_0^1(\Omega)$ given by

$$< u,v >= \int_\Omega a(x)\nabla u.\nabla v\, dx,$$

and the induced norm will be denoted

$$\|u\| = \Big(\int_\Omega a(x)|\nabla u|^2\, dx\Big)^{\frac{1}{2}}.$$

Set φ_1 a normalised positive eigenfunction associated to λ_1 the first eigenvalue of the operator $-\text{div}(a(x)\nabla u)$ with Dirichlet boundary condition on the open domain Ω.

$$\begin{cases} -\text{div}(a(x)\nabla\varphi_1) = \lambda_1\varphi_1 & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega \\ \int_\Omega \varphi_1^2 dx = 1. \end{cases} \tag{6}$$

In this paper, we define

$$\Lambda_1 = \inf_{u\in H_0^1(\Omega), u\neq 0} \frac{\int_\Omega a(x)|\nabla u|^2 dx}{\int_\Omega q(x)u^2 dx} \tag{7}$$

and we will see in Lemma 2.1, in the next section, that $\Lambda_1 > 0$ and it is achieved by some positive function ϕ_1 in $H_0^1(\Omega)$.
Before stating our main results, let us recall the assumptions on the nonlinearity.

$(F1)$ $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,t) \geq 0$ for all $t > 0$ and $x \in \overline{\Omega}$ and $f(x,t) \equiv 0$ for $t \leq 0$ and $x \in \overline{\Omega}$.

$(F2)$ $\lim_{t\to 0} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to +\infty} \frac{f(x,t)}{t} = q(x)$ uniformly in a.e.$x \in \Omega$, where $p(x)$ and $q(x)$ are bounded functions and $\|p(x)\|_\infty < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $(-\mathrm{div}(a(x)\nabla .), H_0^1(\Omega))$.

$(F3)$ The function $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$, for $a.e.x \in \Omega$.

Theorem 1.3. *Suppose that* $(F1)$ *and* $(F2)$ *hold, then we have.*

(i) If $\Lambda_1 > 1$ *and* $(F3)$ *holds, then the problem* (1) *does not have a positive solution.*
(ii) If $\Lambda_1 < 1$, *then the problem* (1) *has a non-trivial positive solution.*
(iii) If $\Lambda_1 = 1$ *and* $(F3)$ *holds, then* (1) *has a non-trivial positive solution* $u \in H_0^1(\Omega)$ *if and only if there exists a constant* $c > 0$ *such that* $u = c\phi_1$ *and* $f(x,u) = q(x)u$ *a.e. in* Ω.

For the case when $q(x) = +\infty$ a.e. in Ω, let

$$r^* = \begin{cases} \frac{2N}{N-2} & if \quad N > 2 \\ +\infty & if \quad N = 2 \end{cases} \tag{8}$$

be the critical Sobolev exponent. We prove the following result.

Theorem 1.4. *Suppose that* $(F1)$, $(F2)$ *and* $(F3)$ *hold,* $q(x) = +\infty$ *a.e. in* Ω *and* $f(x,t)$ *is subcritical:* $\lim_{t\to+\infty} \frac{f(x,t)}{t^{r-1}} = 0$ *uniformly in* $x \in \Omega$, *for some real* r *with* $r \in (2, r^*)$. *Then the problem* (1) *has a non-trivial positive solution.*

2. Preliminaries

Lemma 2.1. *Let* $q(x)$ *be a bounded non-negative function and* $a(x)$ *be a positive function on* Ω. *The following eigenvalue-eigenfunction problem:*

$$\begin{cases} -div(a(x)\nabla u) & = \Lambda \quad q(x)u \quad in \quad \Omega, \\ u & = \quad 0 \quad on \quad \partial\Omega, \end{cases} \tag{9}$$

has a solution (Λ_1, ϕ_1) satisfying $\phi_1 > 0$ a.e. in Ω, $\phi_1 \in H_0^1(\Omega)$, $\Lambda_1 > 0$ and

$$\Lambda_1 = \inf\{\int_\Omega a(x)|\nabla u|^2 dx, \ u \in H_0^1(\Omega) \text{ and } \int_\Omega q(x)u^2 dx = 1\}.$$

Proof. Since the function a is continuous and satisfies the condition (A), By the same scheme of [Lemma 2.1, 26] we get $\Lambda_1 > 0$ and there exists ϕ_1 solution to the equation (9). If ϕ_1 is not non-negative, we can take $|\phi_1|$ and using maximum principle, we get a solution, still denoted ϕ_1, satisfying $\phi_1 > 0$ a.e. in Ω. □

In the proof of the mains results, we need that $(H_0^1(\Omega), \|.\|)$ as a Hilbert space. In fact, we have the following result.

Lemma 2.2. $(H_0^1(\Omega), \|.\|)$ *is a Hilbert space and the norm* $\|u\|$ *is equivalent to the norm*

$$\|u\|_{W^{1,2}(\Omega)}.$$

Proof. $(H_0^1(\Omega), \|.\|_{W^{1,2}(\Omega)})$ is a Banach space and the equivalence between the two norms is due to the condition (A). □

The next lemma assures the two geometric properties for the functional I induced by (3) when $\Lambda_1 < 1$.

Lemma 2.3. *Suppose that the function* f *satisfies* $(F1)$ *and* $(F2)$*, then the following results hold.*

(i) There exist $\rho, \beta > 0$ *such that* $I(u) \geq \beta$ *for all* $u \in H_0^1(\Omega)$ *with* $\|u\| = \rho$.

(ii) If $\Lambda_1 < 1$*, then* $I(t\phi_1) \to -\infty$ *as* $t \to +\infty$.

Proof. (i) Let $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ and $t_0 \geq 1$ such that for all $t \geq t_0$, $f(x,t) \leq At$. For $r \geq 1$, we get $f(x,t) \leq At^r$ and then

$$F(x,t) \leq \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)t^2 + \frac{A}{r+1}|t|^{r+1}, \tag{10}$$

for all $(x,t) \in \Omega \times \mathbb{R}$.
By choosing r such that $2 < r+1 < r^*$, r^* given by (8), we obtain $\|u\|_{r+1}^{r+1} \leq C\|u\|^{r+1}$ and then

$$\begin{array}{rcl} I(u) & \geq & \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx \\ & \geq & \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - \frac{A}{r+1}\|u\|_{r+1}^{r+1} \\ & \geq & \frac{1}{2}\|u\|^2 - \frac{1}{2}(\|p(x)\|_\infty + \varepsilon)\|u\|_2^2 - \frac{A}{r+1}C\|u\|^{r+1}. \end{array}$$

By definition of λ_1, we have

$$I(u) \geq \frac{1}{2}(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\lambda_1})\|u\|^2 - \frac{A}{r+1}C\|u\|^{r+1}.$$

From $(F2)$, for $\varepsilon > 0$ small enough such that $\|p(x)\|_\infty + \varepsilon < \lambda_1$, we can choose $\|u\| = \rho$ very small in order to get $I(u) \geq \beta$ for a given $\beta > 0$ sufficiently small.
(ii) Suppose that $\Lambda_1 < 1$. For $t > 0$, we have

$$I(t\phi_1) = \frac{t^2}{2}\int_\Omega a(x)|\nabla\phi_1|^2 dx - \int_\Omega F(x, t\phi_1)\,dx. \tag{11}$$

By the condition $(F2)$ and the definition of the function $F(x,t)$, we have

$$\lim_{t\to\infty}\frac{F(x,t)}{t^2} = \frac{q(x)}{2}.$$

Then, using the Fatou's Lemma and the fact that ϕ_1 is a solution for the minimization problem (7), we get

$$\begin{aligned} \lim_{t\to\infty}\frac{I(t\phi_1)}{t^2} &= \frac{1}{2}\int_\Omega a(x)|\nabla\phi_1|^2 dx - \lim_{t\to\infty}\int_\Omega \frac{F(x,t\phi_1)}{t^2}\,dx \\ &\leq \frac{1}{2}\int_\Omega a(x)|\nabla\phi_1|^2 dx - \int_\Omega \lim_{t\to\infty}\frac{F(x,t\phi_1)}{(t\phi_1)^2}\phi_1^2\,dx \\ &\leq \frac{1}{2}\int_\Omega a(x)|\nabla\phi_1|^2 dx - \frac{1}{2}\int_\Omega q(x)\phi_1^2 dx \\ &\leq \frac{1}{2}\int_\Omega a(x)|\nabla\phi_1|^2 dx - \frac{1}{2\Lambda_1}\int_\Omega a(x)|\nabla\phi_1|^2 dx \\ &\leq \frac{1}{2\Lambda_1}(\Lambda_1 - 1)\int_\Omega a(x)|\nabla\phi_1|^2 dx < 0, \end{aligned}$$

for $\Lambda_1 < 1$. So $I(t\phi_1) \to -\infty$ as $t \to +\infty$. □

Remark 2.4. *Assume the conditions* $(F1)-(F3)$ *and suppose that the function* f *is subcritical with respect to* t *and* $q(x)=+\infty$ *a.e. in* Ω*. Then, the results of the Lemma 2.3 hold.*

Lemma 2.5. *If* (u_n) *is a convergent sequence to* u *in* $L^p(\Omega)$*, for some* $1 \leq p < +\infty$*, then* (u_n^+) *converges to* u^+ *in* $L^p(\Omega)$*, where* $u_n^+ = \max(0, u_n)$ *and* $u^+ = \max(0, u)$*.*

Proof.

$$\begin{aligned}
\|u_n^+ - u^+\|_p^p &= \int_\Omega |u_n^+ - u^+|^p dx \\
&= \frac{1}{2^p}\int_\Omega \Big|(u_n - u) + (|u_n| - |u|)\Big|^p dx \\
&\leq \frac{1}{2^p}\int_\Omega \Big(|u_n - u| + \big||u_n| - |u|\big|\Big)^p dx \\
&\leq \frac{1}{2^p}\int_\Omega \Big(|u_n - u| + |u_n - u|\Big)^p dx \\
&\leq \frac{1}{2^p}\int_\Omega \Big(2|u_n - u|\Big)^p dx \\
&\leq \|u_n - u\|_p^p.
\end{aligned}$$

Then (u_n^+) converges to u^+ in $L^p(\Omega)$. □

We end this section by the following elementary result that will be used in the proof of Theorem 1.4 (the superlinear linearities cases).

Lemma 2.6. *Suppose that* $(F3)$ *holds and* (u_n) *a sequence in* $H_0^1(\Omega)$ *such that*

$$\|I'(u_n)\|_\star \to 0.$$

Then, up to a subsequence, for all $t > 0$ *we have*

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n). \tag{12}$$

Proof. $\|I'(u_n)\|_\star \to 0$. So for all $\varphi \in H_0^1(\Omega)$, $\langle I'(u_n), \varphi\rangle \to 0$, in particular

$$\langle I'(u_n), u_n\rangle \to 0.$$

Up to subsequence, for all $n \geq 1$,

$$|\langle I'(u_n), u_n\rangle| \leq \frac{1}{n} \tag{13}$$

and then

$$-\frac{1}{n} \leq \|u_n\| - \int_\Omega f(x, u_n)u_n dx \leq \frac{1}{n}, \quad \forall n \geq 1. \tag{14}$$

We have

$$I(tu_n) = \frac{1}{2}t^2\|u_n\| - \int_\Omega F(x, tu_n)dx$$

and from (14) we get

$$I(tu_n) \leq \frac{1}{2}\frac{t^2}{n} + \int_\Omega \Big[\frac{1}{2}t^2 f(x, u_n)u_n - F(x, tu_n)\Big]dx. \tag{15}$$

If we study the write hand site in (15) as a real function on t, we find that

$$I(tu_n) \leq \frac{1}{2}\frac{t^2}{n} + \int_\Omega \Big[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big]dx. \tag{16}$$

From (14), we have

$$I(u_n) \geq -\frac{1}{2n} + \int_\Omega \Big[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big]dx. \tag{17}$$

By combining (15) and (17), we deduce that

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n),$$

for all $n \geq 1$ and $t > 0$. □

3. Proof of the Main Results

Proof of the Theorem 1.3 (*i*) By contradiction. Suppose that $\Lambda_1 > 1$ and $u \in H_0^1(\Omega)$ is a positive solution of the problem (1). Then u satisfies the equation (3) for all $\varphi \in H_0^1(\Omega)$. If we take $\varphi = u$, we obtain by using ($F2$) and ($F3$) that

$$\int_\Omega a(x)|\nabla u|^2 dx = \int_\Omega f(x,u)u dx \leq \int_\Omega q(x)u^2 dx. \tag{18}$$

So $\Lambda_1 \leq 1$. Theorem 1.3 (*i*) follows.

(*ii*) Suppose that $\Lambda_1 < 1$ and the conditions ($F1$) − ($F'2$) hold. By lemma 2.2, there exists t_0 such that the function $e = t_0\phi_{\Lambda_1} \in H_0^1(\Omega)$, $\|e\| > \rho$ and $I(e) < 0$ for some $\beta, \rho > 0$, where $I(u) \geq \beta$ for all $u \in \partial B(O,\rho)$ in $H_0^1(\Omega)$. Since the space $(H_0^1(\Omega), \|.\|)$ is a Banach space and the functional I is C^1, we have a Cerami sequence $(u_n) \subset H_0^1(\Omega)$ satisfying

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x,u_n)\, dx \to c \quad \text{as } n \to +\infty \tag{19}$$

and

$$(1 + \|u_n\|)\|I'(u_n)\|_* \to 0 \quad \text{as } n \to +\infty. \tag{20}$$

The idea is to prove that (u_n) has a convergent subsequence in $H_0^1(\Omega)$ and to prove that the limit will be a positive solution to the equation (1).

Step1 ((u_n) is bounded in $H_0^1(\Omega)$, up to subsequence)
We argue by contradiction and we suppose that the Cerami sequence (u_n) is not bounded in $H_0^1(\Omega)$. So , up to subsequence, $\|u_n\| \to +\infty$. Consider

$$w_n = \frac{1}{\|u_n\|}u_n = k_n u_n; \quad k_n = \frac{1}{\|u_n\|} \tag{21}$$

The sequence (w_n) is bounded in $H_0^1(\Omega)$. By using the compactness of Sobelev embedding Theorem, there exists w in $H_0^1(\Omega)$, such that

$$w_n \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega),$$

$$w_n \to w \quad \text{strongly in} \quad L^2(\Omega),$$
$$w_n(x) \to w(x) \quad \text{a.e. in} \quad \Omega.$$

Claim 1: $w \not\equiv 0$. Indeed, the second Cerami condition (20) gives

$$\langle I'(u_n), u_n\rangle \to 0 \quad \text{as} \quad n \to +\infty, \tag{22}$$

then

$$\|u_n\| - \int_\Omega f(x, u_n)u_n dx \to 0 \quad \text{as} \quad n \to +\infty. \tag{23}$$

So,

$$\|u_n\| - \int_\Omega f(x, u_n)u_n dx \to 0 \quad \text{as} \quad n \to +\infty. \tag{24}$$

We obtain

$$1 = \lim_{n\to+\infty} \int_\Omega \frac{f(x, u_n(x))}{u_n(x)} w_n^2 dx. \tag{25}$$

From $(F1)$ and $(F2)$ there exists $M > 0$ such that for all $t > 0$ and $x \in \Omega$, we have

$$\frac{f(x,t)}{t} \leq M. \tag{26}$$

Then,

$$\int_\Omega \frac{f(x, u_n(x))}{u_n(x)} w_n^2 dx \leq M \int_\Omega w_n^2 dx. \tag{27}$$

If we suppose that $w \equiv 0$, from (25) and (27), we get $1 \leq 0$ and this is impossible. The claim 1 is proved.
Claim 2: w satisfies the following equation

$$\int_\Omega a(x)\nabla w \nabla\varphi dx = \int_\Omega q(x)w\, \varphi dx, \tag{28}$$

for all $\varphi \in H_0^1(\Omega)$. For the proof of this claim, We use the condition $(F1)$ to define the function $g_n(x)$ on Ω as $g_n(x) = \frac{f(x,u_n(x))}{u_n(x)}$ if $u_n(x) > 0$ and $g_n(x) = 0$ if $u_n(x) < 0$. By (26), we get

$$0 \leq g_n(x) \leq M.$$

Since the sequence g_n is bounded in $L^2(\Omega)$, there exists a function g in $L^2(\Omega)$ such that,up to subsequence

$$g_n \rightharpoonup g \quad \text{weakly in} \quad L^2(\Omega),$$
$$g_n(x) \to g(x) \quad \text{a.e. in} \quad \Omega,$$
$$0 \leq g(x) \leq M \quad \text{for all} \quad x \in \Omega.$$

Consider $\Omega_+ = \{x \in \Omega;\ w(x) > 0\}$. We have $u_n(x) = \|u_n\|w_n(x) \to +\infty$ for all $x \in \Omega_+$ and so

$$g(x) = q(x), \quad \text{for all} \quad x \in \Omega_+. \tag{29}$$

Also,we have $w_n \to w$ in $L^2(\Omega)$. By Lemma 2.5, we have $w_n^+ \to w^+$ in $L^2(\Omega)$ and so, we get for all $\varphi \in L^2(\Omega)$:

$$\int_\Omega g_n(x)w_n(x)\varphi(x)dx = \int_\Omega g_n(x)w_n^+(x)\varphi(x)dx \to \int_\Omega g(x)w^+(x)\varphi(x)dx. \tag{30}$$

Since $H_0^1(\Omega)$ is a subspace of $L^2(\Omega)$, (30) is true for all $\varphi \in H_0^1(\Omega)$.
From (20) we have $\|I'(u_n)\|_* \to 0$ and then

$$\lim_{n\to+\infty} \int_\Omega a(x)\, \nabla u_n.\nabla\varphi dx - \int_\Omega f(x,u_n)\varphi dx = 0, \text{ for all } \varphi \in H_0^1(\Omega) \tag{31}$$

Since $\dfrac{1}{\|u_n\|} \to 0$, we obtain

$$\lim_{n\to+\infty} \int_\Omega a(x)\, \nabla w_n.\nabla\varphi dx - \int_\Omega g_n(x)w_n(x)\varphi(x)dx = 0, \text{ for all } \varphi \in H_0^1(\Omega). \tag{32}$$

From (32) and (30), we get

$$\int_\Omega a(x)\, \nabla w.\nabla\varphi dx = \int_\Omega g(x)w^+(x)\varphi(x)dx, \text{ for all } \varphi \in H_0^1(\Omega). \tag{33}$$

If we take $\varphi = w^-$ as a test function in (33), we get $\|w^-\| = 0$ and so $w^- \equiv 0$ a.e. in Ω.
We get $w \geq 0$ and the maximum principle yields to $w > 0$. By using (33) and (29), we finish the proof of the claim 2.
Since the function $w \in H_0^1(\Omega)$ is positive and satisfies (28), we get a contradiction with the fact that $\Lambda_1 < 1$.
As a conclusion of this step, the sequence (u_n is bounded in $(H_0^1(\Omega), \|.\|)$.

Step2. ((u_n) converge to a function u in $H_0^1(\Omega)$, up to subsequence)
Indeed, the sequence (u_n) is bounded so by compactness Sobolev embedding Theorem, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega)\\ &u_n \to u \text{ stongly in } L^2(\Omega)\\ &u_n \to u \text{ a.e in } \Omega. \end{aligned}$$

From (20) we have $\|I'(u_n)\|_* \to 0$ and so

$$\lim_{n\to+\infty} \int_\Omega a(x)\, \nabla u_n.\nabla\varphi - \int_\Omega f(x,u_n)\varphi = 0, \text{ for all } \varphi \in H_0^1(\Omega). \tag{34}$$

That is

$$\lim_{n\to+\infty} -\text{div}(a(x)\nabla u_n) - f(x,u_n) = 0 \text{ in } \mathcal{D}'(\Omega). \tag{35}$$

Also

$$\lim_{n\to+\infty} < I'(u_n), u_n >= \lim_{n\to+\infty} \|u_n\|^2 - \int_\Omega f(x,u_n)u_n = 0. \tag{36}$$

By using $(F2)$, we prove that $f(x,u_n) \to f(x,u)$ in $L^2(\Omega)$ and also

$$\int_\Omega f(x,u_n)u_n dx \to \int_\Omega f(x,u)u dx.$$

So,

$$-\text{div}(a(x)\nabla u) = f(x,u) \text{ in } \Omega, \tag{37}$$

and then by taking u as a test function in (37), we get

$$\|u\|^2 - \int_\Omega f(x,u)u = 0. \tag{38}$$

with (23) in mind, we deduce that $\|u_n\|^2 \to \|u\|^2$. Up to subsequence, (u_n) converge to u in $H_0^1(\Omega)$.

Step 3. (u is a positive solution of the equation (1))
By step 2, the sequence (u_n) converges to an element u in $H_0^1(\Omega)$. By (19) we deduce that $I(u) = c$. From (20) and (31), we get $I'(u) = 0$ and so, u is a solution of the problem (1). From the condition $(F1)$ and the maximum principle, the solution u is positive on Ω.

(*iii*) Suppose that $\Lambda_1 = 1$ and the conditions $(F1)-(F3)$ hold. First, if u is a positive solution for the problem (1) by taking ϕ_1 as test function in (2), we get

$$\int_\Omega a(x)\nabla u.\nabla\phi_1\, dx = \int_\Omega f(x,u)\phi_1 dx. \tag{39}$$

By taking u as a test function in (9), we obtain

$$\int_\Omega a(x)\nabla u.\nabla\phi_1\, dx = \int_\Omega q(x)u\phi_1 dx \tag{40}$$

and so $\int_\Omega (f(x,u) - q(x)u)\phi_1 dx = 0$. Since ϕ_1 is a positive function and the function $f(x,t)$ satisfies $(F2)$ and $(F3)$, we conclude that $f(x,u) = q(x)u$ a.e. in Ω. By a classical way introduced in the proof of Theorem 2, section 6.5 of [8], we know that there exists a positive constant $c > 0$ such that $u = c\phi_1$.
Conversely, suppose that $u = c\phi_1$, for some constant $c > 0$, and $f(x,u) = q(x)u$. We have

$$\begin{aligned} -\mathrm{div}(a(x)\nabla u) &= -c\,\mathrm{div}(a(x)\nabla\phi_1) \\ &= cq(x)\phi_1 \\ &= q(x)u \\ &= f(x,u). \end{aligned}$$

Then u is a positive solution for the problem (1). □

Proof of the Theorem 1.4 Suppose that $q(x) = +\infty$ and $\lim\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{r-1}} = 0$, for some real r with $r \in (2, r^*)$, uniformly in $x \in \Omega$. By Lemma 2.3, the Remark 2.4 and Theorem 1.2, there exists a Cerami sequence (u_n) (i.e. satisfying (4) and (5)).
Following the same steps as in the proof of Theorem 1.3 (*ii*), we only need here to prove that the sequence (u_n) is bounded in $H_0^1(\Omega)$.
By contradiction, suppose that (u_n) is unbounded in $H_0^1(\Omega)$, then up to subsequence
$\|u_n\| \to +\infty$. Let $d > 0$ and consider the sequence

$$w_n = \frac{u_n}{d\|u_n\|} = k_n u_n, \text{ where } k_n = \frac{1}{d\|u_n\|}. \tag{41}$$

Because w_n is bounded in $H_0^1(\Omega)$, there exists $w \in H_0^1(\Omega)$ such that, up to subsequence,

$$\begin{gathered} w_n \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega), \\ w_n \to w \quad \text{strongly in } L^2(\Omega), \\ w_n \to w \quad \text{a.e in } \Omega. \end{gathered}$$

By Lemma 2.5, we have

$$w_n^+ \to w^+ \quad \text{strongly in} \quad L^2(\Omega),$$

and

$$w_n^+(x) \to w^+(x) \quad \text{a.e. in} \quad \Omega.$$

We claim that

$$w^+(x) = 0 \quad \text{a.e. in} \quad \Omega. \tag{42}$$

Indeed, let $\Omega_1 = \{x \in \Omega; w^+(x) = 0\}$ and $\Omega_2 = \{x \in \Omega; w^+(x) > 0\}$.
By (41) and ($F2$), we have $u_n^+(x) \to +\infty$ and for a given $K > 0$ and n large enough we have

$$\frac{f(x, u_n^+(x))}{u_n^+(x)}(w_n^+(x))^2 \geq K(w^+(x))^2. \tag{43}$$

From (5) we have $\|I'(u_n)\|_\star \to 0$ and so,

$$\langle I'(u_n), u_n \rangle \to 0. \tag{44}$$

We get

$$\lim_{n\to+\infty} \left[\|u_n\|^2 - \int_\Omega \frac{f(x, u_n)}{u_n}(u_n)^2 dx\right] = 0, \tag{45}$$

then

$$\begin{array}{rcl} \frac{1}{d^2} & = & \lim_{n\to+\infty} \int_\Omega \frac{f(x,u_n)}{u_n}(w_n)^2 dx \\ & \geq & \lim_{n\to+\infty} \int_{\Omega_2} \frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2 dx \\ & \geq & \int_{\Omega_2} \lim_{n\to+\infty} \frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2 dx \\ & \geq & K \int_{\Omega_2} (w^+)^2 dx \end{array}$$

for all $K > 0$. So necessary, $|\Omega_2| = 0$ and then $w^+ \equiv 0$ in Ω. We get

$$\lim_{n\to+\infty} \int_\Omega F(x, w_n^+(x))dx = 0$$

and hence

$$\lim_{n\to+\infty} I(w_n) = \frac{1}{2d^2}. \tag{46}$$

By using Lemma 2.6, we get, up to subsequence

$$I(w_n) = I(k_n u_n) \leq \frac{1}{2n}(1 + k_n^2) + I(u_n). \tag{47}$$

$k_n = \frac{1}{d\|u_n\|}$, then from (46) and (47) we obtain

$$\frac{1}{2d^2} \leq c \tag{48}$$

this is for any $d > 0$ which is impossible and so the sequence (u_n) is bounded in
$H_0^1(\Omega)$. □

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