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On τ-Bounded Spaces and Hyperspaces

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Abstract. In this paper, we prove facts and properties on τ-bounded spaces, which are introduced in [10]. More precisely, we prove that an arbitrary product of τ-bounded spaces is τ bounded and vice versa, and that the τ-bounded property is preserved by τ-continuous maps. In particular, continuous maps preserve τ-bounded spaces. Moreover, we investigate the behavior of the minimal tightness and functional tightness of topological spaces under the influence of an exponential functor of finite degree. It is proved that this functor preserves the functional tightness and the minimal tightness of compact sets.

1. Introduction

In recent researches an interest in the theory of cardinal invariants and their behavior under the influence of various covariant functors is increasing fast. In [2–5, 9] the authors investigated several cardinal invariants under the influence of some weakly normal and normal functors and hyperspaces.

The current paper is devoted to the investigation of cardinal invariants such as the functional tightness, the minimal tightness and some other topological properties of hyperspaces of elements with finite degree. Also, some basic properties of τ-bounded spaces are studied.

The concept of functional tightness of a topological space was first introduced by A. Arhangel'skii in [1]. As it turned out, cardinal invariants such as the minimal tightness and the functional tightness are in many ways similar to each other, and for many natural and classical cases they coincide. However, there is an example of a topological space, the minimum tightness of which is countable, and the functional tightness is uncountable (see [12]).

In [10], the action of closed and R-quotient maps on functional tightness is investigated. It is proved that R-quotient mapping does not increase the functional tightness. As well as, in [11] it is proved that the functional tightness of the product of two locally compact spaces does not exceed the product of the functional tightnesses of those spaces.

In this paper, we study the behavior of the minimal tightness and functional tightness of topological spaces under the influence of an exponential functor of finite degree. It is proved that this functor preserves the functional tightness and the minimal tightness of compact sets. Also, the τ-continuity of the mapping $\exp_n f\colon \exp_n(X) \to \exp_n(Y)$ is proved for any τ-continuous mapping $f : X \to Y$. Besides, we prove some facts and properties on τ-bounded spaces, which are introduced in [10]. More precisely, we prove that an

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arbitrary product of τ-bounded spaces is τ-bounded and vice versa, and that the τ-bounded property is preserved by τ-continuous maps. In particular, continuous maps preserve τ-bounded spaces.

Throughout the paper all spaces are assumed to be Hausdorff and τ means an infinite cardinal number.

2. Some properties of τ-bounded spaces

Definition 2.1. ([1]) Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be τ-continuous if for every subspace A of X such that $|A| \le \tau$, the restriction $f|_A$ is continuous.

Definition 2.2. ([1]) Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be strictly τ-continuous if for every subspace A of X such that $|A| \le \tau$, the restriction of f to A coincides with the restriction to A of some continuous function $g\colon X \to Y$.

Definition 2.3. ([10]) A subset F of a space X is said to be τ-closed if for every $B \subset F$ with $|B| \le \tau$, the closure $\overline{B}$ in X of the set B is contained in F.
The τ-closure of a set A is defined as $[A]_\tau = \bigcup\{\overline{B}\colon B \subset A, |B| \le \tau\}$ and a set A is said to be τ-dense in X if $[A]_\tau = X$.

Proposition 2.4. ([11]) *A mapping $f\colon X \to Y$ is τ-continuous if and only if for every closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*

The following result generalizes Proposition 2.4.

Proposition 2.5. *For a mapping $f\colon X \to Y$ of arbitrary topological spaces X and Y the following conditions are equivalent:*

(1) $f\colon X \to Y$ is τ-continuous;
(2) for every closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X;
(3) for every τ-closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X;
(4) $f([A]_\tau) \subset [f(A)]_\tau$ for an arbitrary subset $A \subset X$;
(5) $[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$ for an arbitrary subset $B \subset Y$.

Proof. (1) $\Longrightarrow$ (2) follows from Proposition 2.4.

(2) $\Longrightarrow$ (3) Let F be an arbitrary τ-closed set in Y. Consider an arbitrary subset $M \subset f^{-1}(F)$ with $|M| \le \tau$. Clearly, $f(M) \subset F$ and $|f(M)| \le \tau$. Since F is τ-closed, we have $\overline{f(M)} \subset F$. By (2) (see also Proposition 2.4), it is clear that $f^{-1}(\overline{f(M)})$ is a τ-closed set in X. Therefore, the inclusion $M \subset f^{-1}(\overline{f(M)})$ implies that $\overline{M} \subset f^{-1}(\overline{f(M)}) \subset f^{-1}(F)$. This proves that $f^{-1}(F)$ is τ-closed.

(3) $\Longrightarrow$ (4) Consider an arbitrary subset $A \subset X$. Clearly, $[f(A)]_\tau$ is a τ-closed set in Y. By assumption we have that $f^{-1}([f(A)]_\tau)$ is τ-closed in X. Clearly, we have $A \subset f^{-1}([f(A)]_\tau)$ and therefore, $[A]_\tau \subset f^{-1}([f(A)]_\tau)$. Hence, we have $f([A]_\tau) \subset [f(A)]_\tau$.

(4) $\Longrightarrow$ (5) Let B be an arbitrary subset in Y. Taking $A = f^{-1}(B)$ in (4) we obtain

$$f([f^{-1}(B)]_\tau) \subset [f(f^{-1}(B))]_\tau \subset [B]_\tau.$$

Hence, $[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$.

(5) $\Longrightarrow$ (1) Assume that the inclusion $[f^{-1}(B)]_\tau \subset f^{-1}([B]_\tau)$ holds for any $B \subset Y$. Consider an arbitrary closed subset F of Y. We have to show that the preimage $f^{-1}(F)$ is τ-closed in X. By the condition we have that $[f^{-1}(F)]_\tau \subset f^{-1}([F]_\tau) = f^{-1}(F)$, since F is closed in Y. Hence, we have that $[f^{-1}(F)]_\tau = f^{-1}(F)$, i.e. $f^{-1}(F)$ is τ- closed in X. By Proposition 2.4 f is τ-continuous. □

Proposition 2.6. ([10]) *If a mapping $f\colon X \to Y$ is strictly τ-continuous, then for every closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*

The following result is a generalization of Proposition 2.6.

Proposition 2.7. *If a mapping $f\colon X \to Y$ is strictly τ-continuous, then for every τ-closed set F in Y, the preimage $f^{-1}(F)$ is τ-closed in X.*

Proof. Suppose $f\colon X \to Y$ is a strictly τ-continuous map and F is a τ-closed subset of Y. Then f is τ-continuous. Thus, the result follows by Proposition 2.5 (3). □

O. Okunev in [10] introduced the notion of τ-bounded space. A space X is called *τ-bounded*, if the closure in X of every subset of cardinality at most τ is compact. In [10] it is proved that the minitightness of a space coincides with the minitightness of its τ-bounded extension. Direct verification shows that every closed subset of a τ-bounded space is τ-bounded. The following results describe some properties of τ-bounded spaces.

Proposition 2.8. *Let $f\colon X \to Y$ be a strictly τ-continuous map from X onto Y. If X is τ-bounded, then so is Y.*

Proof. Get an arbitrary subset $M \subset Y$ with $|M| \le \tau$. Fix a point x_y in $f^{-1}(\{y\})$ for every $y \in M$ and denote by B the set of all these points, i.e. $B = \{x_y : y \in M\}$. It is clear that $|B| \le \tau$. Since X is τ-bounded, we have that the closure of B in X is compact. There is a continuous mapping $g\colon X \to Y$ such that $g|_{\overline{B}} = f|_{\overline{B}}$ (by strictly τ-continuity of f). Hence, $f(\overline{B})$ is a compact subset of Y, and therefore, is closed in Y. Since $M = f(B)$, we obtain that $\overline{M} \subset f(\overline{B})$. This implies that $\overline{M}$ is compact, which proves the τ-boundedness of Y. □

Proposition 2.8 immediately implies the following result.

Corollary 2.9. *Continuous image of a τ-bounded space is τ-bounded.*

Theorem 2.10. *The Cartesian product $\prod_{\alpha\in A} X_\alpha$ of nonempty spaces is τ-bounded if and only if all spaces X_α, $\alpha \in A$, are τ-bounded.*

Proof. If $X = \prod_{\alpha\in A} X_\alpha$ is τ-bounded, then by Corollary 2.9 every X_α is τ-bounded, since projection maps $p_\alpha\colon X \to X_\alpha$ are continuous.

Now let X_α be τ-bounded for every $\alpha \in A$. Consider arbitrary subset $M \subset X$ with $|M| \le \tau$. It is clear that $M \subset \prod_{\alpha\in A} \overline{p_\alpha(M)}$. We have that $\overline{p_\alpha(M)}$ is a compact subset of X_α for every $\alpha \in A$, since all spaces X_α are τ-bounded. By the Tychonoff theorem (see [6], Theorem 3.2.4) $\prod_{\alpha\in A} \overline{p_\alpha(M)}$ is compact, and consequently, is closed in X. Therefore, $\overline{M}$ is compact. This proves that X is τ-bounded. □

It is said that a family γ of subsets of a topological space *has the finite intersection property* if every finitely many elements of γ has nonempty intersection. The following statement is true.

Theorem 2.11. *If a space X is a τ-bounded space, then every family of closed sets of power $\le \tau$ with the finite intersection property has nonempty intersection.*

Proof. Let X be a τ-bounded space and $\gamma = \{F_\alpha\colon \alpha \in A\}$ a family of closed sets with $|A| \le \tau$. Denote by γ_1 the family of all finite intersections of elements of γ, i.e.

$$\gamma_1 = \{E\colon E \text{ can be represented as the finite intersection of elements of } \gamma\}.$$

Clearly, $|\gamma_1| \le \tau$. Choose a point x_E from each element $E \in \gamma_1$ and denote by M the set of all such points, i.e. $M = \{x_E\colon E \in \gamma_1\}$. In this case, $\overline{M}$ is compact, since X is τ-bounded and $|M| \le \tau$. Consider the family $\gamma_2 = \{\overline{M} \cap F_\alpha\colon F_\alpha \in \gamma\}$. Then by the construction of the set M the family γ_2 has the finite intersection property and consists of closed subsets of $\overline{M}$. Since $\overline{M}$ is compact, we obtain that $\cap\gamma_2 \ne \emptyset$, which implies that $\cap\gamma \ne \emptyset$. □

The following example shows that the restriction in Theorem 2.11 for the cardinality of a family with finite intersection property is essential. We note that the intersection of a family of closed sets with finite intersection property in a τ-bounded space may be empty, in general.

Example 2.12. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal ω_1 with the natural order $<$. Consider on W the topology generated by the base β consisting of all segments

$$(y, x] = \{z \in W : y < z \leq x\},$$

where $y < x \leq \omega_1$, and one point set $\{0\}$, where 0 is the order type of the empty set.

Let us consider the subspace $X = W \setminus \{\omega_1\}$. In this space consider the following closed sets: $F_\alpha = (\alpha, \omega_1)$, where $(\alpha, \omega_1) = \{x \in X : \alpha < x < \omega_1\}$ for every $\alpha < \omega_1$. Since the closure of every countable subspace of X is compact as a closed subspace of the compact W, the space X is ω-bounded, where ω denotes the countable cardinal. Consider the following family of closed subsets in X: $\sigma = \{F_\alpha : \alpha \in X\}$. Then it is clear that the family σ has the finite intersection property but its intersection is empty: $\bigcap \sigma = \bigcap_{\alpha \in X} F_\alpha = \emptyset$. Note that σ is uncountable.

A partially ordered set A is directed if for any $\alpha, \beta \in A$ there exists a $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. Let A be a directed partially ordered set and $\{X_\alpha, \pi^\alpha_\beta, A\}$ be an inverse system, consisting of Tychonoff spaces. By $\varprojlim X_\alpha$ we will denote the limit of this system and by $\pi_\alpha : \varprojlim X_\alpha \to X_\alpha, \alpha \in A$ its bonding mappings (limit projections).

Theorem 2.13. *The limit of an inverse system $S = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of nonempty τ-bounded spaces with $|\Sigma| \leq \tau$ is τ-bounded and nonempty.*

Proof. For each $\rho \in \Sigma$ let

$$Z_\rho = \{\{x_\sigma\} \in \prod_{\sigma \in \Sigma} X_\sigma : \pi^\rho_\gamma(x_\rho) = x_\gamma \text{ for } \gamma \leq \rho\}.$$

Consider a point $z_\rho \in X_\rho$. Let $z_\gamma = \pi^\rho_\gamma(z_\rho)$ for $\gamma \leqslant \rho$, and get arbitrarily a $z_\sigma \in X_\sigma$ for all the other $\sigma \in \Sigma$. It is clear that $z = \{z_\sigma\} \in Z_\rho$, and therefore, $Z_\rho \neq \emptyset$. From Theorem 1.5.4 [6] it follows that Z_ρ is a closed subset of $\prod_{\sigma \in \Sigma} X_\sigma$. Since $Z_{\rho_1} \subset Z_{\rho_2}$ whenever $\rho_1 \leq \rho_2$ and the set Z_ρ is directed, the family $\{Z_\rho\}_{\rho \in \Sigma}$ of closed subsets of the product $\prod_{\sigma \in \Sigma} X_\sigma$ has the finite intersection property. In this case we have that $\varprojlim S = \cap_{\rho \in \Sigma} Z_\rho \neq \emptyset$, i.e. it is nonempty and τ-bounded as a closed subspace of a τ-bounded space. □

3. The functional tightness and minitightness of hyperspaces

The set of all non-empty closed subsets of a topological space X is denoted by $\exp(X)$. The family of all sets of the form

$$O\langle U_1, \ldots, U_n\rangle = \left\{F : F \in \exp(X), F \subset \bigcup_{i=1}^{n} U_i, F \bigcap U_i \neq \emptyset, i = 1, 2, \ldots, n\right\},$$

where $U_1, \ldots, U_n$ are open subsets of X, is a base for a topology on the set $\exp(X)$. This topology is called the *Vietoris topology*. The set $\exp(X)$ with the Vietoris topology is called *exponential space* or *the hyperspace of a space X*. Put $\exp_n(X) = \{F \in \exp(X) : |F| \leq n\}$ [7].

Let $f : X \to Y$ be a continuous mapping between topological spaces X and Y. For $F \in \exp_n(X)$ put $(\exp_n f)(F) = f(F)$. It is known that the mapping $\exp_n f : \exp_n(X) \to \exp_n(Y)$ is also continuous [7].

Theorem 3.1. *Let $f : X \to Y$ be a τ-continuous mapping of the Hausdorff space X into the Hausdorff space Y. Then the mapping $\exp_n f : \exp_n(X) \to \exp_n(Y)$ is also τ-continuous.*

Proof. Consider an arbitrary subset $\gamma \subset \exp_n(X)$ with $|\gamma| \leq \tau$. Let us prove that the restriction of the mapping $\exp_n f$ to the set γ is continuous. Take an arbitrary element F from γ with $F = \{x_1, x_2, \ldots, x_n\}$. Then

$$f(F) = \{f(x_1), f(x_2), \ldots, f(x_n)\} = \{y_1, y_2, \ldots, y_k\} \in \exp_n(Y),\ k \leq n.$$

Suppose $O\langle V_1, V_2, \ldots, V_e\rangle$ is an arbitrary neighborhood of the set $f(F)$ in $\exp_n(Y)$, where $V_1, V_2, \ldots, V_e$ are open sets in Y. Put $S = \{V_1, V_2, \ldots, V_e\}$ and $M = \cup\gamma$. Since $M \subset X$ and $|M| \leq \tau$, we have that $f\big|_M : M \to Y$ is continuous. By continuity of f on M, there exist neighborhoods W_i of the points x_i, $i = 1, 2, \ldots, n$, in M such that $W_i = U_i \cap M$, where U_i are neighborhoods of the points x_i, $i = 1, 2, \ldots, n$, in X satisfying the following conditions:

(1) The family $\{f(W_1), f(W_2), \ldots, f(W_n)\}$ is a refinement of the family S;

(2) For each set V_j there exists W_i such that $f(W_i) \subset V_j$.

Now we will prove the inclusion

$$(\exp_n f)(O\langle U_1, U_2, \ldots, U_n\rangle \cap \gamma) \subset O\langle V_1, V_2, \ldots, V_e\rangle.$$

Take an arbitrary element $E \in (\exp_n f)(O\langle U_1, \ldots, U_n\rangle \cap \gamma)$. For some $C \in O\langle U_1, \ldots, U_n\rangle \cap \gamma$ we have $f(C) = E$. Besides, $C \subset \bigcup_{i=1}^{n} U_i$ and $U_i \cap C \neq \emptyset$ for $i = 1, 2, \ldots, n$. Moreover, since $C \in \gamma$, we have $C \subseteq M$. Since the family $\{f(W_1), \ldots, f(W_n)\}$ is a refinement of S, we have

$$f(C) \subset f(\bigcup_{i=1}^{n}(U_i \cap M)) = \bigcup_{i=1}^{n} f(U_i \cap M) = \bigcup_{i=1}^{n} f(W_i) \subset \bigcup_{j=1}^{e} V_j.$$

In addition, for each $V_j \in S$ we see that $f(C) \cap V_j \neq \emptyset$, since $C \cap U_i \neq \emptyset$ for any $i = 1, 2, \ldots, n$. Hence, $E = f(C) \in O\langle V_1, V_2, \ldots, V_e\rangle$. Thus, we have proved the continuity of the mapping $\exp_n f\big|_\gamma : \gamma \to \exp_n(Y)$. □

In [1] A.V. Arhangel'skii introduced cardinal invariants so called the *functional tightness* and the *minimal tightness* of a topological space as follows:

Definition 3.2. ([1]) The *functional tightness* of a space X is

$t_0(X)$=min{τ: τ is an infinite cardinal and every τ-continuous real-valued function on X is continuous}.

Definition 3.3. ([1]) The *minimal tightness* of a space X is

$t_m(X)$=min{τ: τ is an infinite cardinal and every strictly τ-continuous real-valued function on X is continuous}.

Note that we always have that $t_m(X) \leq t_0(X)$ for an arbitrary topological space, since every strictly τ-continuous function is τ-continuous. Besides, in [1] it was shown that $t_m(X) = t_0(X)$ for an arbitrary normal space X.

For the function $f : X \to R$, where R is the set of real numbers, the operation $f_{\exp} : \exp_n(X) \to R$ is defined as follows: set $F \in \exp_n(X)$ is associated with the maximum value $f(x)$, $x \in F$, i.e.

$$f_{\exp}(F) = \max\{f(x) : x \in F\}.$$

This operation is defined correctly, since F is a finite set.

Lemma 3.4. *For any τ-continuous function $f : X \to R$, the function $f_{\exp} : \exp_n(X) \to R$ is τ-continuous.*

Proof. Let $\gamma \subset \exp_n(X)$ and $|\gamma| \leq \tau$. Take also any element $F = \{x_1, x_2, ..., x_n\} \in \gamma$. Let

$$f_{\exp}(F) = \max_{i=1,\ldots,n}\{f(x_i)\} = a$$

and $\varepsilon > 0$ be an arbitrary positive number. Put

$$\delta = \min_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \{|f(x_i) - f(x_j)| : f(x_i) \neq f(x_j)\} \text{ and } \varepsilon_1 = \frac{\min\{\varepsilon, \delta\}}{2}.$$

We denote by M the union of the family γ, i.e. $M = \cup\gamma$. Then $|M| \leq \tau$, since γ consists of finite sets. Since the function f is τ-continuous, it is continuous on the set M. Therefore, there exist neighborhoods W_{x_i} of the points x_i, $i = 1, 2, \dots, n$, in M such that $W_{x_i} = U_{x_i} \cap M$, where U_{x_i} are neighborhoods of the points x_i, $i = 1, 2, \dots, n$, in X satisfying the following condition:

$$f(W_{x_i}) \subset (f(x_i) - \varepsilon_1, f(x_i) + \varepsilon_1) \text{ for } i = 1, 2, \dots, n. \tag{1}$$

In addition, we have

$$(f(x_i) - \varepsilon_1, f(x_i) + \varepsilon_1) \cap (f(x_j) - \varepsilon_1, f(x_j) + \varepsilon_1) = \emptyset, \tag{2}$$

for $f(x_i), f(x_j)$ such that $f(x_i) \neq f(x_j)$.

Now consider the basic element $O\langle U_{x_1}, U_{x_2}, ..., U_{x_n}\rangle \cap \gamma$. It is clear that $F \in O\langle U_{x_1}, U_{x_2}, ..., U_{x_n}\rangle \cap \gamma$. Let us prove the inclusion

$$f_{\exp}(O\langle U_{x_1}, U_{x_2}, ..., U_{x_n}\rangle \cap \gamma) \subset (a - \varepsilon, a + \varepsilon).$$

Take an arbitrary element $E \in O\langle U_{x_1}, U_{x_2}, \dots, U_{x_n}\rangle \cap \gamma$. Then $E \subseteq M$. Without loss of generality we can assume that $E = \{y_1, y_2, \dots, y_n\}$, where $y_i \in U_{x_i} \cap M = W_{x_i}$, $i = 1, 2, \dots, n$. Let $f_{\exp}(E) = \max\limits_{i=1,\dots,n}\{f(y_i)\} = b$. By virtue of the properties (1) and (2) we have

$$|a - b| < 2\varepsilon_1 \leq \varepsilon. \tag{3}$$

Assuming otherwise, we would have the following:

$$b = f_{\exp}(E) = \max_{i=1,\dots,n}\{f(y_i)\} = f(y_k) > a + \varepsilon = f_{\exp}(F) + \varepsilon \geq f(x_k) + \varepsilon,$$

where $k \in \{1, 2, \dots, n\}$. Hence $f(y_k) - f(x_k) > \varepsilon$. But by virtue of (1) we have

$$f(W_{x_k}) \subset (f(x_k) - \varepsilon_1, f(x_k) + \varepsilon_1)$$

and therefore

$$f(y_k) - f(x_k) < 2\varepsilon_1 \leq \varepsilon.$$

This is a contradiction. From the arbitrariness of the choice of E and the inequality (3), we obtain the inclusion

$$f_{\exp}(O\langle U_{x_1}, U_{x_2}, \dots, U_{x_n}\rangle \cap \gamma) \subset (a - \varepsilon, a + \varepsilon),$$

which proves the continuity of the function $f_{\exp}\big|_{\gamma} : \gamma \to R$. □

Lemma 3.4 immediately implies the following result.

Corollary 3.5. *The function $f : X \to R$ is continuous if and only if the function $f_{\exp} : \exp_n(X) \to R$ is continuous.*

Lemma 3.6. *Let X be an arbitrary infinite topological space. For any strictly τ-continuous function $f : X \to R$, the function $f_{\exp} : \exp_n(X) \to R$ is strictly τ-continuous.*

Proof. Let $f : X \to R$ be an arbitrary strictly τ-continuous function. We show that the function $f_{\exp} : \exp_n(X) \to R$ is also strictly τ-continuous. Consider a subset $\gamma \subset \exp_n(X)$ with $|\gamma| \leq \tau$. Denote by M the union of the family γ, i.e. $M = \cup\gamma$. Then clearly, $|M| \leq \tau$. Since f is strictly τ-continuous, there exists a continuous function $g : X \to R$ such that $g|_M = f|_M$. By Corollary 3.5 the function $g_{\exp} : \exp_n(X) \to R$ is continuous and coincides with $f_{\exp}$ on γ, since $f|_F = g|_F$ for each $F \in \gamma$. This implies that the function $f_{\exp}$ is strictly τ-continuous. □

Theorem 3.7. *For any infinite topological space X we have* $t_0(X) \leq t_0(\exp_n(X))$.

Proof. Suppose $t_0(\exp_n(X)) = \tau$. Consider an arbitrary τ-continuous function $f\colon X \to R$. By Lemma 3.4, the function $f_{\exp}\colon \exp_n(X) \to R$ is also τ-continuous. Since $t_0(\exp_n(X)) = \tau$, the function $f_{\exp} : \exp_n(X) \to R$ is continuous and the function $f\colon X \to R$ is continuous as the restriction of the continuous function $f_{\exp} : \exp_n(X) \to R$ on the subspace $\exp_1(X)$, which is homeomorphic to X. □

The following theorem can be proved similar to Theorem 3.7, that is why we give it without proof:

Theorem 3.8. *For any infinite topological space X we have* $t_m(X) \leq t_m(\exp_n(X))$.

Note that there exists a quotient map from X^n onto $\exp_n(X)$ for every topological space X and natural number n. It is known that any quotient map does not increase the functional tightness (see [11], Proposition 4.2). Moreover, we have the following result.

Corollary 3.9. *For an arbitrary infinite compact space X we have* $t_0(X) = t_m(X) = t_m(\exp_n(X)) = t_0(\exp_n(X))$.

Theorem 2.10 and Corollary 2.9 imply the following result.

Corollary 3.10. *A topological space X is* τ*-bounded if and only if* $\exp_n(X)$ *is* τ*-bounded, where n is a natural number.*

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