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# Note on Weakly 1-Absorbing Primary Ideals

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**Abstract.** An ideal $I$ of a commutative ring $R$ is called a weakly primary ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. An ideal $I$ of $R$ is called weakly 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. In this paper, we characterize rings over which every ideal is weakly 1-absorbing primary (resp. weakly primary). We also prove that, over a non-local reduced ring, every weakly 1-absorbing primary ideals is weakly primary.

## 1. Introduction

Throughout this paper, $R$ denotes a commutative ring with $1 \neq 0$. An ideal $I$ of a ring $R$ is said to be proper if $I \neq R$. Let $R$ be a ring and $I$ be an ideal of $R$. The radical of $I$ is denoted by $\sqrt{I} := \{x \in R \mid x^n \in I$ for some integer $n \geq 1\}$ and the nilradical of $R$ is denoted by $\sqrt{0} := \sqrt{(0)}$. Let Spec($R$) denotes the set of all prime ideals of $R$ and let $Z(R)$ denotes the set of zero-divisors of $R$.
An ideal $\mathfrak{q}$ of $R$ is said to be primary if, whenever $a, b \in R$ with $ab \in \mathfrak{q}$, then $a \in \mathfrak{q}$ or $b \in \sqrt{\mathfrak{q}}$. In this case $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is a prime ideal of $R$ and $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary.
Since prime and primary ideals have key roles in commutative ring theory, many authors have studied generalizations of these ideals. In [2], Anderson and Smith introduced the notion of weakly prime ideals. A proper ideal $I$ of $R$ is called weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in I$. In [3], Atani and Farzalipour introduced the concept of weakly primary ideals. A proper ideal $I$ of $R$ is called a weakly primary ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. Recent generalizations of primary ideals and weakly primary ideals are, respectively, the notions of 1-absorbing primary ideals and weakly 1-absorbing primary ideals introduced by Badawi and Yetkin in [4, 6]. A proper ideal $I$ of $R$ is called 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. A proper ideal $I$ of $R$ is called weakly 1-absorbing primary if whenever nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. It is proved that

$$\text{primary} \longrightarrow \text{weakly primary} \longrightarrow \text{weakly 1-absorbing primary},$$

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and

$$\text{primary} \longrightarrow \text{1-absorbing primary} \longrightarrow \text{weakly 1-absorbing primary}$$

and all arrows are irreversible. For other generalizations of prime and primary ideals please see for example [5, 10, 11].
In Section 2, we characterize (reduced) rings over which every ideal is weakly 1-absorbing primary (resp. weakly primary) (Theorem 2.5 and Corollaries 2.8 and 2.9). In Section 3, among other results, we give some details about weakly 1-absorbing primary ideals which are not weakly primary over a non-local ring (Theorem 3.1). Hence, we prove that, over a non-local reduced ring, every weakly 1-absorbing primary ideal is weakly primary (Theorem 3.2). We also characterize rings over which every nonzero weakly 1-absorbing primary ideal is prime and local Noetherian rings over which every (nonzero) weakly 1-absorbing primary ideal is (weakly) primary (Theorem 3.3, Corollary 3.4, and Theorem 3.7).

## 2. Rings over which all ideals are weakly 1-absorbing primary

The main goal of this section is to characterize rings over which every ideal is weakly 1-absorbing primary. To do so, we need the following lemmas.

**Lemma 2.1.** *Let $R$ be a domain. Then, the following are equivalent:*

1. *Every ideal of $R$ is weakly 1-absorbing primary.*
2. *Every ideal of $R$ is 1-absorbing primary.*
3. *$R$ is a field or $R$ is a domain with a unique nonzero prime ideal.*

*Proof.* The equivalence (1) $\Leftrightarrow$ (2) follows from the fact that over a domain weakly 1-absorbing primary ideals coincide with 1-absorbing primary ideals. For the equivalence (2) $\Leftrightarrow$ (3), it follows from [1, Corollary 2.14]. □

**Lemma 2.2.** *If every ideal of $R$ is weakly 1-absorbing primary, then every ideal of $R/I$ is weakly 1-absorbing primary for each proper ideal $I$ of $R$.*

*Proof.* By [6, Theorem 16]. □

**Lemma 2.3.** *Let $R$ be a ring over which every ideal is weakly 1-absorbing primary. Then, the following hold:*

1. *Every prime ideal of $R$ is either minimal or maximal.*
2. *Every non-maximal prime ideal is contained strictly in a unique prime ideal.*

*Proof.* If every prime ideal of $R$ is maximal (that is the krull dimension of $R$ is zero), then the desired result follows trivially. Otherwise, let $P$ be a prime non-maximal ideal of $R$. Let $P_1$ and $Q$ be prime ideals of $R$ such that $P_1 \subseteq P \subsetneq Q$. By Lemma 2.2, every ideal of $R/P_1$ is weakly 1-absorbing primary. Moreover, $R/P_1$ is not a field, otherwise $P_1$ is maximal and then $P_1 = P$ is maximal, a contradiction. Hence, by Lemma 2.1, $R/P_1$ has a unique nonzero prime ideal. Thus, $P/P_1 = (0)$ and $Q/P_1$ is the unique nonzero prime ideal of $R/P_1$. Thus, $P = P_1$ is a minimal prime ideal of $R$ and $Q$ is the unique prime ideal which contains strictly $P$. Consequently, (1) and (2) hold. □

**Lemma 2.4.** *Let $I$ be a radical ideal of $R$ (that is $\sqrt{I} = I$). If $I$ is weakly 1-absorbing primary, then $I$ is prime or $x^3 = 0$ for all $x \in I$. In this last case, we get $I = \sqrt{0}$.*

*Proof.* Suppose that there exists $x_0 \in I$ such that $x_0^3 \neq 0$.
Let $a, b \in R$ such that $ab \in I$. We have to prove that $a \in I$ or $b \in I$. We may assume that $a$ and $b$ are nonunit. If $a^2b \neq 0$, then $a^2 \in I$ or $b \in \sqrt{I} = I$. Hence, $a \in I$ or $b \in I$. Similarly, if $ab^2 \neq 0$ we get the same conclusion.
Now, suppose that $a^2b = ab^2 = 0$. If $a^2I \neq (0)$, then there exists $x \in I$ such that $a^2x \neq 0$, and so $0 \neq a^2x = a^2(b + x) \in I$. If $b + x$ is a unit, then $a \in I$. Otherwise, $a^2 \in I$ or $b + x \in I$. Thus, $a \in I$ or $b \in I$. Similarly, if $b^2I \neq (0)$, then $a \in I$ or $b \in I$.
Suppose now that $a^2I = b^2I = (0)$. We have $(a^2 + x_0)^2(b^2 + x_0) = x_0^3 \in I$. If $a^2 + x_0$ (resp. $b^2 + x_0$) is a unit, then $b \in I$ (resp. $a \in I$). Otherwise, $(a^2 + x_0)^2 \in I$ or $b^2 + x_0 \in I$. Thus, $a \in I$ or $b \in I$.
We conclude so that $I$ is a prime ideal of $R$. □

Recall from [7], that a ring $R$ is said to be an *UN*-ring if every nonunit element $a$ of $R$ is a product a unit and a nilpotent elements, or equivalently every element of $R$ is either nilpotent or unit ([12, Proposition 2.25]). That is $R$ is local with maximal ideal $\sqrt{0}$. A simple example of *UN*-rings is $\mathbb{Z}/9\mathbb{Z}$.

The main result of this section is stated as follows:

**Theorem 2.5.** *Let R be a ring. Then, every ideal of R is weakly 1-absorbing primary if and only if one of the following holds:*

1. $R \cong k_1 \times k_2$ *where* $k_1$ *and* $k_2$ *are fields, or*
2. *R is an UN-ring, or*
3. *R is local with maximal ideal M and* $\mathrm{Spec}(R) = \{\sqrt{0}, M\}$ *such that* $\sqrt{0}M\big(M \setminus \sqrt{0}\big) = \{0\}$.

*Proof.* ($\Rightarrow$) Suppose that $R$ is non-local. Let $M_1$ and $M_2$ be two distinct maximal ideals of $R$. The ideal $M_1 \cap M_2$ is a non-prime weakly 1-absorbing primary radical ideal. By Lemma 2.4, $M_1 \cap M_2 = \sqrt{0}$.
Let $m_1 \in M_1$ and $m_2 \in M_2$ such that $m_1 + m_2 = 1$. Clearly, $m_1 \notin \sqrt{0}$, otherwise $m_2 = 1 - m_1$ is a unit, a contradiction. Similarly, $m_2 \notin \sqrt{0}$. We claim that $m_1^2$ is an idempotent element of $R$. We have $m_1^2m_2 + m_1^3 = m_1^2$. Suppose that $m_1^2m_2 \neq 0$. We have $m_1^2m_2 \in M_1 \cap M_2 = \sqrt{0}$. Thus, $\sqrt{(m_1^2m_2)} = \sqrt{0}$. Now, since $(m_1^2m_2)$ is weakly 1-absorbing primary and $0 \neq m_1^2m_2 \in (m_1^2m_2)$ and $m_2 \notin \sqrt{0}$, we get $m_1^2 \in (m_1^2m_2) \subseteq \sqrt{0}$. Thus, $m_1 \in \sqrt{0}$, a contradiction. Consequently, $m_1^2m_2 = 0$, and so $m_1^2 = m_1^3 = m_1^4$. Thus, $m_1^2$ is an idempotent element. Set $m_1^2 = e \in M_1$.
Let $0 \neq x \in \sqrt{0}$. We have either $ex \neq 0$ or $(1-e)x \neq 0$. Suppose for example that $ex \neq 0$. Assume that $a = ex + 1 - e$ is a unit. Then, $1 - e = a - ex$ is unit since $ex \in \sqrt{0}$, and so since $1 - e$ is an idempotent element we get $1 - e = 1$, which means that $e = m_1^2 = 0$, a contradiction since $m_1 \notin \sqrt{0}$. Hence, $a$ is nonunit. Now, $e^2a = e^2(ex + 1 - e) = ex \neq 0$. Clearly, $e = e^2 \notin (ex) \subseteq \sqrt{0}$ and $a = ex + 1 - e \notin \sqrt{0} = \sqrt{(ex)}$, otherwise $1 - e \in \sqrt{0}$, and then $m_1^2 = e = 1$, a contradiction. Hence, the ideal $(ex)$ is not weakly 1-absorbing primary, a contradiction. Accordingly, $\sqrt{0} = (0)$. Hence, $R \cong R/M_1 \times R/M_2$. Thus, $R$ is isomorphic to a product of two fields.
Suppose that $R$ is local with maximal ideal $M$. Suppose that $R$ is not an *UN*-ring. So, $R$ admits non-maximal prime ideals. Let $P$ and $Q$ be two non-maximal prime ideals of $R$.
Let $\{I_\alpha\}_\alpha$ be the set of all $M$-primary ideals of $R$ and set $J := \bigcap_\alpha I_\alpha$. Consider $x \in J$. Suppose that $x \notin P$. Then, $P \subsetneq P + (x^2) \subseteq M$. By Lemma 2.3, $M$ is the unique prime ideal wich contains $P + (x^2)$. Thus, $\sqrt{P + (x^2)} = M$. Hence, $x \in P + (x^2)$ since $P + (x^2) \in \{I_\alpha\}_\alpha$. Thus, there exist $r \in R$ such that $x(1 - rx) \in P$. Thus, $1 - rx \in P \subsetneq M$, a contradiction. Consequently, $J \subseteq P$, and similarly $J \subseteq Q$.
Let $x \in P$ and $y \in M \setminus Q$. If $xy\big(M \setminus P\big) \neq 0$, then there exists $z \in M \setminus P$ such that $xyz \neq 0$. For each $\alpha$, we have $\sqrt{I_\alpha P} = \sqrt{I_\alpha P + (xyz)} = P$. Thus, since $I_\alpha P + (xyz)$ is weakly 1-absorbing primary, $0 \neq xyz \in I_\alpha P + (xyz)$, and $z \notin P$, we get $xy \in I_\alpha P + (xyz)$. Thus, for some $r \in R$, we have $xy(1 - zr) \in I_\alpha P \subseteq I_\alpha$. Thus, $xy \in I_\alpha$ since $1 - zr$ is a unit. Thus, $xy \in \bigcap_\alpha I_\alpha = J \subseteq Q$. Hence, $x \in Q$.
Now, assume that $xy\big(M \setminus P\big) = 0$. If $M = P \cup Q$, then $P$ or $Q$ is maximal, a contradiction. Thus, $M \neq P \cup Q$, and so there exists $z \in M \setminus P \cup Q$. By assumption, we have $xyz = 0 \in Q$. Thus, since $y$ and $z$ are not elements of $Q$, we get that $x \in Q$.
From the both cases, we conclude that $P \subseteq Q$. Similarly, $Q \subseteq P$, and so $P = Q$.
Consequently, $R$ admits a unique non-maximal prime ideal which is necessarily $\sqrt{0}$, and then $\sqrt{0}$ and $M$ are the only prime ideals of $R$.
Let $x \in \sqrt{0}$, $y \in M$, and $z \in M \setminus \sqrt{0}$. Suppose that $xyz \neq 0$. Then, since the principal ideal $(xyz)$ is weakly 1-absorbing primary and $z \notin \sqrt{(xyz)} = \sqrt{0}$, we obtain that $xy \in (xyz)$, and then $xy(1 - rz) = 0$ for some $r \in R$. Thus, $xy = 0$, a contradiction with the assumption $xyz \neq 0$. Hence, we conclude that $\sqrt{0}M\big(M \setminus \sqrt{0}\big) = 0$.
($\Leftarrow$) If $R$ is a product of two field, then the result follows from [6, Theorem 14] and if $R$ is a *UN*-ring, then

the result follows from [6, Theorem 1(6)].
Now, suppose that $R$ satisfies (3). Let $I$ be an ideal of $R$. If $\sqrt{I} = M$, then $I$ is primary, and so weakly 1-absorbing primary. Hence, suppose that $\sqrt{I} = \sqrt{0}$. Let $a, b, c \in R$ nonunit elements such that $abc \in I$ and $c \notin \sqrt{0}$. Thus, $a \in \sqrt{0}$ or $b \in \sqrt{0}$. In the both cases $abc = 0$. Hence, $I$ is weakly 1-absorbing primary. □

Next, we give an example of a local ring which is not an $UN$-ring over which every ideal of $R$ is weakly 1-absorbing primary. The same ring contains an ideal which is not 1-absorbing primary.

**Example 2.6.** *Consider the ring* $A = \dfrac{k[x, y]}{(x^3, xy)}$ *with k is a field. The ideal* $P = \dfrac{(x, y)}{(x^3, xy)}$ *is a prime ideal of A. So, set* $R = A_P$. *Then, R is a local ring with the maximal ideal* $M = \left(\dfrac{\overline{x}}{1}, \dfrac{\overline{y}}{1}\right)$ *and exactly one non-maximal prime ideal which is* $P = \sqrt{0_R} = \left(\dfrac{\overline{x}}{1}\right)$. *We have* $PM^2 = \{0_R\}$. *Hence, using the Theorem 2.5, every ideal of R is weakly 1-absorbing primary.*
*Now,* $\dfrac{\overline{x^2}}{1}.\dfrac{\overline{y}}{1} = 0_R$ *but neither* $\dfrac{\overline{x^2}}{1} = 0_R$ *nor* $\dfrac{\overline{y}}{1} \in \sqrt{0_R}$. *Thus,* $\{0_R\}$ *is not a 1-absorbing primary ideal of R.*

**Corollary 2.7 ([6, Theorem 14]).** *Let* $R_1, \cdots, R_n$ *be commutative rings with* $n \geq 2$ *and set* $R = R_1 \times \cdots \times R_n$. *Then the following are equivalent:*

1. *Every ideal of R is a weakly 1-absorbing primary ideal of R.*
2. $n = 2$ *and* $R_1$ *and* $R_2$ *are fields.*

Then next result characterizes rings over which every ideal is weakly primary.

**Corollary 2.8.** *Let R be a ring. Then, every ideal of R is weakly primary if and only if one of the following holds:*

1. $R \cong k_1 \times k_2$ *where* $k_1$ *and* $k_2$ *are fields, or*
2. *R is an UN-ring, or*
3. *R is local with maximal ideal M and* $\mathrm{Spec}(R) = \{\sqrt{0}, M\}$ *such that* $\sqrt{0}\big(M \setminus \sqrt{0}\big) = \{0\}$.

*Proof.* (⇒) Since every ideal of $R$ is weakly primary, we get that every ideal of $R$ is weakly 1-absorbing primary. By Theorem 2.5, if $R$ is non-local, $R \cong k_1 \times k_2$ where $k_1$ and $k_2$ are fields.
Now, assume that $R$ is local but not an $UN$-ring. Then, following Theorem 2.5, $\sqrt{0}$ and $M$ (the maximal ideal) are the only prime ideals of $R$. Let $a \in \sqrt{0}$ and $b \in M \setminus \sqrt{0}$. Suppose that $ab \neq 0$. Then, since $(ab)$ is a weakly primary ideal and $b \notin \sqrt{(ab)} = \sqrt{0}$, we deduce that $a \in (ab)$. Hence, $a(1 - bc) = 0$ for some $c \in R$. But $1 - bc$ is a unit, and so $a = 0$, a contradiction with the assumption $ab \neq 0$. Consequently, $\sqrt{0}\big(M \setminus \sqrt{0}\big) = \{0\}$.
(⇐) The ring $k_1 \times k_2$ admits three ideals $\{(0, 0)\}$, $0 \times k_2$ and $k_1 \times 0$, and all these ideals are weakly primary. Moreover, all ideals of an $UN$-ring are primary and so weakly primary.
Suppose that (3) holds. Let $I$ be an ideal of $R$. If $\sqrt{I} = M$, then $I$ is primary, and then weakly primary. Hence, assume that $\sqrt{I} = \sqrt{0}$ and let $0 \neq ab \in I$ with $a \notin I$. Suppose that $b \notin \sqrt{I}$. Then, $a \in \sqrt{0}$ since $ab \in I \subseteq \sqrt{I} = \sqrt{0}$. Thus, $ab = 0$, a contradiction. Then, $b \in \sqrt{I}$. □

The following result follows easily from Theorem 2.5 and Corollary 2.8.

**Corollary 2.9.** *Let R be a reduced ring. Then, the following are equivalent:*

1. *Every ideal of R is weakly 1-absorbing primary*
2. *Every ideal of R is weakly primary*
3. *R satisfies one of the following statement:*
   (a) $R \cong k_1 \times k_2$ *where* $k_1$ *and* $k_2$ *are fields, or*
   (b) *R is a field, or*
   (c) *R is a domain with a unique nonzero prime ideal.*

## 3. Local Noetherian rings over which every (weakly) 1-absorbing primary ideal is (weakly) primary

We begin this section with a brief discussion on the following question posed by Badawi and Yetkin in [6].

**Question.** Does, over a non-local ring, every weakly 1-absorbing primary ideal is weakly primary?

A partial answer is given in [6, Theorem 5] as follows: Let $R$ be a non-local ring and $I$ be an ideal of $R$ such that $\operatorname{ann}(i)$ is not a maximal ideal of $R$ for every element $i \in I$. Then $I$ is a weakly 1-absorbing primary ideal of $R$ if and only if $I$ is a weakly primary ideal of $R$.
In this paper, we are not able to give an affirmative or a negative answer to this Question in the general case. However, we prove that this is true over non-local reduced rings. To do so, we need the next result which gives some details about weakly 1-absorbing primary ideals that are not weakly primary (if any) over non-local rings.

**Theorem 3.1.** *Let $R$ be a non-local ring, and let $I$ be a weakly 1-absorbing primary ideal that is not weakly primary. Then*

1. *either $I^3 = (0)$, or*
2. *$I^2 = (e)$ with $e$ is an idempotent such that $(1 - e)$ is a maximal ideal of $R$.*

*Proof.* Suppose that (2) is not satisfied. Since $I$ is not weakly primary, there exists $x, y \in R$ such that $0 \neq xy \in I$, $x \notin I$, and $y \notin \sqrt{I}$. Clearly, $x$ and $y$ are nonunit.
Suppose that $wx \in I$ for all nonunit element $w \in R$. Let $u$ be a unit element of $R$. If $w + u$ is nonunit, then $(w + u)x \in I$, and so $ux \in I$, a contradiction since $x \notin I$. Hence, for each nonunit element $w \in R$ and each unit element $u \in R$, $w + u$ is a unit. Thus, by [4, Lemma 1], $R$ is local, a contradiction. Consequently, there exists a nonunit element $w \in R$ such that $wx \notin I$.
If $wxy \neq 0$, then $wx \in I$ since $y \notin \sqrt{I}$ and $I$ weakly 1-absorbing primary, a contradiction. Hence, $wxy = 0$.
Suppose that there exists $p \in I$ such that $wxp \neq 0$. Then, $0 \neq wxp = wx(y + p) \in I$. If $y + p$ is a unit, then $wx \in I$, a contradiction. Hence, since $wx \notin I$, we get $y + p \in \sqrt{I}$. Thus, $y \in \sqrt{I}$, a contradiction. Consequently, $wxI = (0)$.
Suppose that there exists $p \in I$ such that $wyp \neq 0$. Then, $0 \neq wyp = w(x + p)y \in I$. If $x + p = u$ is a unit, then $uy = xy + py \in I$, and so $y \in I$, a contradiction. Hence, $x + p$ is nonunit and $w(x + p) \in I$. So, $wx \in I$, a contradiction. Consequently, $wyI = (0)$.
Suppose that there exists $p, q \in I$ such that $wpq \neq 0$. Then, $0 \neq wpq = w(x + p)(y + q) \in I$. As above, $x + p$ and $y + q$ are nonunit. Hence, $w(x + p) \in I$. So, $wx \in I$, a contradiction. Consequently, $wI^2 = (0)$.
Suppose that there exists $p \in I$ such that $xyp \neq 0$. Then, $0 \neq xyp = (w + p)xy \in I$. Suppose that $u = w + p$ is a unit. Then, $up^2 = p^3$. Hence, $\left(pu^{-1}\right)^2 = \left(pu^{-1}\right)^3$. Thus, $e = \left(pu^{-1}\right)^2$ is an idempotent element. For each $q, r \in I$, we have $qru = qrp$ and $qpu = qp^2$. Thus, $qru^2 = r(qpu) = rqp^2$. Hence, $qr = qre$. Then, $I^2 \subseteq (e) \subseteq I^2$. Therefore, $I^2 = (e)$.
By assumption $(1 - e)$ is non-maximal. If $(e)$ is a maximal ideal, then $I = I^2 = (e)$, a contradiction since $I$ is not a weakly primary ideal. Thus, neither $(1 - e)$ nor $(e)$ is maximal. Hence, $R \cong R/(e) \times R/(1 - e)$ is a product of two rings that are not fields. By [6, Theorem 13], $I$ is primary, a contradiction. Accordingly, $w + p$ is nonunit, and so $(w + p)x \in I$. Then, $wx \in I$, a contradiction. Consequently, $xyI = (0)$.
Suppose that there exists $p, q \in I$ such that $xpq \neq 0$. Then, $0 \neq xpq = x(w + p)(y + q) \in I$. As above, $w + p$ and $y + q$ are nonunit. Hence, $x(w + p) \in I$. So, $wx \in I$, a contradiction. Consequently, $xI^2 = (0)$.
Suppose that there exists $p, q \in I$ such that $ypq \neq 0$. Then, $0 \neq ypq = (w + p)(x + q)y \in I$. As above, $w + p$ and $x + q$ are nonunit. Hence, $(w + p)(x + q) \in I$. So, $wx \in I$, a contradiction. Consequently, $yI^2 = (0)$.
Let $p, q, r \in I$ such that $pqr \neq 0$. Then, $(w + p)(x + q)(y + r) = pqr \neq 0$. As above, $w + p$, $x + q$ and $y + r$ are nonunit. Then, $(w + p)(x + q) \in I$ or $y + r \in \sqrt{I}$. That is $wx \in I$ or $y \in \sqrt{I}$, a contradiction. Hence, $I^3 = (0)$. □

**Theorem 3.2.** *Let $R$ be a non-local reduced ring. Then, every weakly 1-absorbing primary ideal is weakly primary.*

*Proof.* Let $I$ be a weakly 1-absorbing primary of $R$. Suppose that $I$ is not weakly primary. If $I^3 = (0)$, then $I = (0)$ which is weakly primary, a contradiction. Then, following Theorem 3.1, $I^2 = (e)$ with $e$ is an idempotent such that $(1-e)$ is a maximal ideal of $R$. We have that $R \cong R/(e) \times R/(1-e)$ following the isomorphism $f$ defined by $r \mapsto (\overline{r}, \overline{r})$. Set $R_1 = R/(e)$ and $k = R/(1-e)$ which is clearly a field. We have also $f(I^2) = (0) \times k$. Without loss of generality, set $R = R_1 \times k$ and $I = I_1 \times I_2$ such that $I_1$ and $I_2$ are ideals of $R_1$ and $k$, respectively. Hence, since $I^2 = (0_{R_1}) \times k$ and $R_1$ is reduced, we conclude that $I = (0_{R_1}) \times k$. Moreover, $\sqrt{I} = \sqrt{(0_{R_1})} \times k = (0_{R_1}) \times k = I$ since $R_1$ is reduced. Hence, $I$ is a radical ideal. Hence, by Lemma 2.4, $I$ is prime, a contradiction since $I$ is not weakly primary. Consequently, every weakly 1-absorbing primary ideal is weakly primary. □

The next result characterizes local Noetherian rings over which every weakly 1-absorbing primary ideal is weakly primary.

**Theorem 3.3.** *Let $(R, M)$ be a local Noetherian ring. The following are equivalent:*

1. *Every weakly 1-absorbing primary ideal is weakly primary.*
2. *Every 1-absorbing primary ideal is weakly primary.*
3. *$R$ is:*
   (a) *either UN-ring, or*
   (b) $\mathrm{Spec}(R) = \{\sqrt{0}, M\}$ *such that* $\sqrt{0}\left(M \setminus \sqrt{0}\right) = \{0\}$.
4. *Every ideal of $R$ is weakly primary.*

*Proof.* (1) ⇒ (2) and (4) ⇒ (1) are clear.
(2) ⇒ (3) Assume that $R$ is not an *UN*-ring.
Suppose that $R$ is a domain. Let $0 \neq P$ be a non-maximal prime ideal of $R$. Then, $PM$ is a 1-absorbing primary ideal of $R$ and $\sqrt{PM} = P$ (by [4, Theorem 7]). Then $PM$ is a weakly primary ideal of $R$. Let $0 \neq x \in P$ and $y \in M \setminus P$. We have $0 \neq xy \in PM$ and $y \notin P = \sqrt{PM}$. Thus, $x \in PM$. Hence, $P = PM$. Since $R$ is Noetherian, by the Nakayama's lemma, we get $P = (0)$, a contradiction. Thus, $M$ is the unique nonzero prime ideal of $R$, and so (b) holds.
Suppose that $R$ is not a domain. Let $P, Q$ be two non-maximal prime ideals of $R$.
Assume that $P(M \setminus P) \neq \{0\}$. As above, $PM$ is a weakly primary ideal of $R$ and $\sqrt{PM} = P$. Let $x \in P$ and $y \in M \setminus P$ such that $xy \neq 0$. Then $xy \in PM$ and $y \notin P = \sqrt{PM}$. Thus, $x \in PM$. Now, let $p \in P$ arbitrary. If $py \neq 0$, then as above $p \in PM$. If $py = 0$, then $(p + x)y = xy \neq 0$, and so we obtain that $p + x \in PM$. Thus, $p \in PM$. Hence, we conclude that $P \subseteq PM$. Thus, $P = PM$. As above, we get $P = (0)$, a contradiction. Consequently, $P(M \setminus P) = \{0\}$.
Similarly, $Q(M \setminus Q) = \{0\}$. If $M = P \cup Q$, then $M = P$ or $M = Q$, a contradiction. Thus, there exists $\alpha \in M \setminus (P \cup Q)$. We have $\alpha P = (0) = \alpha Q$. Thus, $\alpha P \subseteq Q$, and so $P \subseteq Q$ since $\alpha \notin Q$. Similarly, we get $Q \subseteq P$. Thus, $P = Q$. Hence, the only nonzero prime non-maximal ideal of $R$ is $\sqrt{0}$ and $\sqrt{0}(M \setminus \sqrt{0}) = \{0\}$.
(3) ⇒ (4) Follows from Corollary 2.8. □

**Corollary 3.4.** *Let $(R, M)$ be a local Noetherian ring. The following are equivalent:*

1. *Every nonzero weakly 1-absorbing primary ideal is primary.*
2. *Every nonzero 1-absorbing primary ideal is primary.*
3. *$R$ is:*
   (a) *either UN-ring, or*
   (b) $\mathrm{Spec}(R) = \{\sqrt{0}, M\}$ *such that* $\sqrt{0}$ *is a minimal ideal.*
4. *Every nonzero ideal of $R$ is primary.*

*Proof.* (1) ⇒ (2) and (4) ⇒ (1) are clear.
(2) ⇒ (3) We have that every nonzero 1-absorbing primary ideal is primary, and so weakly primary. Moreover, the zero ideal is always weakly primary. Thus, by Theorem 3.3, if $R$ is not an *UN*-ring, then

$\text{Spec}(R) = \{\sqrt{0}, M\}$ such that $\sqrt{0}(M \setminus \sqrt{0}) = \{0\}$. Let $0 \neq I \subseteq \sqrt{0}$. Let $a, b, c \in R$ nonunit such that $abc \in I \subseteq \sqrt{0}$ and $c \notin \sqrt{I} = \sqrt{0}$. Hence, $a \in \sqrt{0}$ or $b \in \sqrt{0}$. In the both cases, $abc = 0$ since $\sqrt{0}(M \setminus \sqrt{0}) = \{0\}$. Thus, $I$ is weakly 1-aborbing primary, and so primary. Let $x \in \sqrt{0}$ and $y \in M \setminus \sqrt{0}$. We have $xy = 0 \in I$ and $y \notin \sqrt{0} = \sqrt{I}$. Thus, $x \in I$. Hence, $\sqrt{0} \subseteq I$, and so $\sqrt{0} = I$. Consequently, $\sqrt{0}$ is a minimal ideal of $R$.
(3) ⇒ (4) If $R$ is an *UN*-ring or $R$ is a domain with unique nonzero prime ideal, then every nonzero ideal $I$ is primary since $\sqrt{I}$ is always the maximal ideal of $R$.
Now, suppose that $R$ is not a domain such that $\text{Spec}(R) = \{\sqrt{0}, M\}$ and $\sqrt{0}$ is a minimal ideal. Let $I$ be a nonzero prime ideal of $R$. If $\sqrt{I} = M$, then $M$ is primary. Now, if $\sqrt{I} = \sqrt{0}$, then $I = \sqrt{0}$ since $\sqrt{0}$ is minimal and $(0) \neq I \subseteq \sqrt{I} = \sqrt{0}$. Hence, $I$ is prime. □

Recall that a ring $R$ is called divided if for every prime ideal $P$ of $R$ and for every $x \in R$ we have either $(x) \subseteq P$ or $P \subseteq (x)$.

**Proposition 3.5.** *Let $(R, M)$ be a local Noetherian ring. The following are equivalent:*

1. *Every weakly 1-absorbing primary ideal is primary.*
2. *Every 1-absorbing primary ideal is primary.*
3. *$R$ is an UN-ring or $R$ is a domain with unique nonzero prime ideal.*
4. *Every ideal of $R$ is primary.*
5. *$R$ is a divided ring.*

*Proof.* (1) ⇒ (2) and (4) ⇒ (1) are clear, and the equivalence between (2), (3), (4), and (5) is exactly [1, Theorem 3.4]. □

We need the following well-known lemma.

**Lemma 3.6.** *Let $R$ be a ring. Then, $(0)$ is a primary ideal of $R$ if and only if $Z(R) = \sqrt{0}$.*

Recall that a ring $R$ is called Von Neumann regular (or absolutely flat ring) if, for every $x \in R$, there exists $y \in R$ such that $x^2y = x$. The following characterizations of Von Neumann regular rings can be found in [8, 9]. Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is Von Neumann regular.
2. $R$ has Krull dimension 0 and is reduced.
3. Every localization of $R$ at a maximal ideal is a field

The following result characterizes rings over which every nonzero weakly 1-absorbing primary ideal is prime.

**Theorem 3.7.** *Let $R$ be a ring. The following are equivalent:*

1. *Every nonzero weakly 1-absorbing primary ideal is prime.*
2. *Every nonzero 1-absorbing primary ideal is prime.*
3. *Every nonzero primary ideal is prime.*
4. *$R$ is either a*
   (a) *Von Neumann regular ring, or*
   (b) *$(R, \sqrt{0})$ is local non-domain and the only nonzero proper ideal of $R$ is $\sqrt{0}$.*

*Proof.* (1) ⇒ (2) ⇒ (3) Clear.
(3) ⇒ (4) Let $P$ be a prime ideal of $R$. Then, $R/P$ is a domain. Let $P \subseteq J$ be an ideal of $R$ such that $J/P$ be a nonzero primary ideal of $R/P$. Then, $J$ is a nonzero primary ideal of $R$. Thus, $J$ is prime and so $J/P$ is prime. Thus, over $R/P$, every primary ideal is prime. Hence, by [1, Theorem 3.1], $R/P$ is Von Neumann regular and so a field since $R/P$ is a domain. Hence, $P$ is maximal. Thus, every prime ideal of $R$ is maximal, and so the Krull dimension of $R$ is 0. It is known that over such rings, every regular element is unit.

If (0) is non-primary, then (3) is equivalent to that every primary ideal is prime. Hence, by [1, Theorem 3.1], $R$ is a Von Neumann regular ring.
Now, assume that (0) is primary. Thus, $Z(R) = \sqrt{0}$, and so $Z(R)$ is an ideal of $R$. Since every regular element is unit, $Z(R)$ is the unique maximal ideal of $R$, and so $R$ is local with maximal ideal $\sqrt{0}$ (an $UN$-ring).
If $R$ is a domain, then it is a field, and so a Von Neumann regular ring. Suppose now that $R$ is not a domain and let $I \neq 0$ be a proper ideal of $R$. We have $\sqrt{0} \subseteq \sqrt{I}$. Thus, $\sqrt{I} = \sqrt{0}$ is maximal, and so $I$ is primary. Thus, $I$ is prime. So, $I = \sqrt{0}$. Thus, $\sqrt{0}$ is the unique nonzero proper ideal of $R$.
(4) $\Rightarrow$ (1) If $(R, \sqrt{0})$ is local non-domain and the only nonzero ideal of $R$ is $\sqrt{0}$, then the result follows trivially. Hence, suppose that $R$ is Von Neuman regular. By [1, Theorem 3.1], every 1-absorbing primary ideal is prime. Let $I$ be a nonzero weakly 1-absorbing primary ideal of $R$. By [6, Theorem 4], $I$ is 1-absorbing primary, and so prime. □

An ideal $I$ of $R$ is said to be semi-primary if $\sqrt{I}$ is prime. It is proved in [4, Theorem 2] that every 1-absorbing primary ideal of $R$ is semi-primary. However, this is not the case for weakly 1-absorbing primary ideals. The next results characterizes rings over which every weakly 1-absorbing primary ideal of $R$ is semi-primary.

**Proposition 3.8.** *Let $R$ be a ring. Then, every weakly 1-absorbing primary ideal of $R$ is semi-primary if and only if $\sqrt{0}$ is prime*

*Proof.* ($\Rightarrow$) Trivial since (0) is weakly 1-absorbing primary.
($\Leftarrow$) Let $I$ be a weakly 1-absorbing primary ideal of $R$. Let $a, b \in R$ with $ab \in \sqrt{I}$ and $a \notin \sqrt{I}$. We have to show that $b \in \sqrt{I}$. We may assume that $a$ is nonunit. There exists an integer $n \geq 1$ such that $a^n b^n \in I$. Hence, $a^{n+1}b^n \in I$ and $n + 1 \geq 2$. If $a^{n+1}b^n \neq 0$, then $a^{n+1} \in I$ or $b^n \in I$. Thus, $b \in \sqrt{I}$ since $a \notin \sqrt{I}$. So, we suppose that $a^{n+1}b^n = 0$. If $a^{n+1}I = (0) \subseteq \sqrt{0}$, then $I \subseteq \sqrt{0}$ since $\sqrt{0}$ is prime and $a \notin \sqrt{0}$. Thus, $\sqrt{I} = \sqrt{0}$ is prime. If $a^{n+1}I \neq 0$, then there exists $x \in I$ such that $a^{n+1}x \neq 0$, and so $0 \neq a^{n+1}(x + b^n) \in I$. If $x + b^n$ is a unit, then $a^{n+1} \in I$, a contradiction. Thus, $x + b^n$ is nonunit. Since $a^{n+1} \notin I$, we get $x + b^n \in \sqrt{I}$. Thus, $b \in \sqrt{I}$. Consequently, $\sqrt{I}$ is prime. □

**Proposition 3.9.** *Let $R$ be a ring. Then, every weakly 1-absorbing primary ideal of $R$ is 1-absorbing primary if and only if* (0) *is 1-absorbing primary.*
*In this case, we have either $Z(R) = \sqrt{0}$ or $R$ is local with maximal ideal $Z(R) = \mathrm{ann}(x)$ for some $x \in R$.*

*Proof.* ($\Rightarrow$) Trivial since (0) is weakly 1-absorbing primary.
($\Leftarrow$) Let $I$ be a weakly 1-absorbing primary ideal of $R$. Let $a, b, c \in R$ nonunit such that $abc \in I$ and $c \notin \sqrt{I}$. If $abc \neq 0$, then $ab \in I$. Now, suppose that $abc = 0$. Hence, $ab = 0$ or $c \in \sqrt{0}$ since (0) is 1-absorbing primary. But the second case is impossible since $c \notin \sqrt{I}$. Hence, $ab = 0 \in I$, as desired.
Now, suppose that (0) is 1-absorbing primary and $Z(R) \neq \sqrt{0}$. Let $a \in Z(R) \setminus \sqrt{0}$. There exists $0 \neq x \in R$ such that $ax = 0$. Assume that $R$ contains a nonunit regular element $r$. We have $rxa = 0$ and $a \notin \sqrt{0}$. Then, $rx = 0$, and so $x = 0$ which is impossible. Thus, nonunit elements of $R$ are zero divisors. Let $y \in Z(R)$. We have $yxa = 0$ and $a \notin \sqrt{0}$, and so $yx = 0$. Thus, $Z(R) = \mathrm{ann}(x)$. Consequently, $R$ is local with maximal ideal $Z(R)$. □

**Corollary 3.10.** *Let $R$ be a reduced ring. Then, every weakly 1-absorbing primary ideal of $R$ is 1-absorbing primary if and only if $R$ is a domain.*

*Proof.* ($\Rightarrow$) If every weakly 1-absorbing primary ideal of $R$ is 1-absorbing primary, then $(0) = \sqrt{0}$ is prime, as desired.
($\Leftarrow$) Clear. □

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