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Slant Helices in Minkowski 3-Space $\mathbb{E}_1^3$ with Sasai's Modified Frame Fields

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Abstract. In this paper, we study slant helix using modified orthogonal frame in Minkowski space $\mathbb{E}_1^3$ with timelike, lightlike and spacelike axes. We also study a general slant helix with the Killing vector field axis. Furthermore, we give a non-trivial example and find the relations for curvature and torsion of f-biharmonic slant helix.

1. Introduction

In differential geometry curves and surfaces play an important role. Many important curves such as general helix, slant helix, Mannheim curve, Bertrand curve etc were investigated. The characterizations of such curves in different ambient spaces have been studied. In classical differential geometry general helices are defined as a regular space curves whose tangent **T** makes a constant angle with a fixed direction **U**, called the axis of the helix [10]. These are geodesics of cylinders shaped over a plane curve. Recently, general helices with different slope axis were investigated in [5, 18, 19]. Slant helices are the successor curves of the general helices [15]. Slant helices were introduced by Izumiya and Takeuchi [12]. A regular curve $\varphi(s)$ with non-zero curvature k and torsion τ is said to be slant helix if the principal normal vector makes a constant angle with a fixed axis. Some characterizations of slant helices in Euclidean space were investigated in [13].

In 2011, Ali and Lopez [2] studied slant helices and gave a useful characterization of slant helices in Minkowski space $\mathbb{E}_1^3$. The slant helices in $\mathbb{E}_1^3$ is defined as follows.

Definition 1.1. *A unit speed curve φ is called a slant helix if there exists a non-zero constant vector field* **U** *in $\mathbb{E}_1^3$ such that $g(\mathbf{N}, \mathbf{U}) = c$* (a constant)*.*

Ergut et al. [6] obtained some results on k-slant helices in Minkowski 3-space. Next, Barros [10, 11] discussed slant helices in the three-dimensional sphere S^3 and anti-de Sitter space $\mathbb{H}_1^3$ with the Killing vector field along the curve with constant length.

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In [17], T. Sasai defined an orthogonal frame in $\mathbb{E}^3$ called a *modified orthogonal frame* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, corresponding to the classical Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ which satisfies

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ k^2 & \frac{k'}{k} & \tau \\ 0 & -\tau & \frac{k'}{k} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

where $g(\mathbf{T}, \mathbf{T}) = 1$, $g(\mathbf{N}, \mathbf{N}) = k^2$, $g(\mathbf{B}, \mathbf{B}) = k^2$. It is noted that the modified orthogonal frame coincides with Frenet-Serret frame for $k = 1$. Thus modified orthogonal frame generalizes Frenet-Serret frame.

The modified orthogonal frame in Minkowski space $\mathbb{E}^3_1$ was investigated by Bukcu and Karacan in [4]. In [9], M. S. Lone and E. S. Hassan studied helices and Bertrand curves with modified frames in Euclidean 3-space. M. Najdanovic and Ljubica Velimirovic [16] discussed the second order infinitesimal bending of a circle and helix in Euclidean 3-space.

In this paper, we characterize the timelike, pseudo null and null slant helices by using the modified orthogonal frame and generalizing some results of [2]. Further, we find some useful relations for the curvature of the slant helices with lightlike, timelike and spacelike slope axes and provide a non-trivial example. Finally, we also give some useful relations for curvature and torsion relation of a f-biharmonic slant helix.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the Euclidean 3-space endowed with the standard indefinite flat metric given by

$$g = -u_1 v_1 + u_2 v_2 + u_3 v_3,$$

for any two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}^3_1$. Any vector $u \in \mathbb{E}^3_1$ can have three causal character namely: spacelike, timelike and lightlike if $g(u, u) = 1$, $g(u, u) = -1$ and $g(u, u) = 0$ respectively. In particular vector $u = 0$ is spacelike.

The following theorem represents a modified orthogonal frame in Minkowski 3-space $\mathbb{E}^3_1$.

Theorem 2.1. *[4] Suppose φ be a unit speed curve in Minkowski 3-space. The relation between the modified orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and the classical Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ at non-zero points of k are*

$$\mathbf{T} = \mathbf{t}, \ \mathbf{N} = k\mathbf{n}, \ \mathbf{B} = k\mathbf{b}.$$

If φ is timelike curve, then the modified orthogonal frame is

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ k^2 & \frac{k'}{k} & \tau \\ 0 & -\tau & \frac{k'}{k} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{1}$$

where $g(\mathbf{T}, \mathbf{T}) = -1$, $g(\mathbf{N}, \mathbf{N}) = k^2$ and $\mathrm{g}(\mathbf{B}, \mathbf{B}) = \mathrm{k}^2$.

If φ is a spacelike curve with a spacelike principal normal vector $\mathbf{N}$, then the modified orthogonal frame is

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -k^2 & \frac{k'}{k} & \tau \\ 0 & \tau & \frac{k'}{k} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2}$$

where $g(\mathbf{T}, \mathbf{T}) = 1$, $g(\mathbf{N}, \mathbf{N}) = k^2$ and $\mathrm{g}(\mathbf{B}, \mathbf{B}) = -\mathrm{k}^2$.

If φ is a spacelike curve with a spacelike binormal vector $\mathbf{B}$, then the modified orthogonal frame is

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ k^2 & \frac{k'}{k} & \tau \\ 0 & \tau & \frac{k'}{k} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{3}$$

where $g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=-k^2$ and $g(\mathbf{B},\mathbf{B})=k^2$.

If φ is pseudo null curve, then the orthogonal modified frame is given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{k'}{k}+\tau & 0 \\ -k^2 & 0 & \frac{k'}{k}-\tau \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{4}$$

where $g(\mathbf{T},\mathbf{T})=1,\ g(\mathbf{N},\mathbf{N})=0,\ g(\mathbf{B},\mathbf{B})=0$ and $g(\mathbf{N},\mathbf{B})=k^2$.

If φ is a null curve, then the orthogonal modified frame is given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ k\tau & \frac{k'}{k} & -k \\ 0 & -\tau & \frac{k'}{k} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{5}$$

where $g(\mathbf{T},\mathbf{T})=0,\ g(\mathbf{N},\mathbf{N})=k^2,\ g(\mathbf{T},\mathbf{B})=k$ and $g(\mathbf{B},\mathbf{B})=0$.

Biharmonic map is defined as the critical point of a bienergy functional [7]

$$E_2(\varphi)=\frac{1}{2}\int_M |\tau(\varphi)|^2 dv_g. \tag{6}$$

As a generalization of biharmonic map, *f-biharmonic* map is defined as the critical point of f-bienergy functional.

$$E_{2,f}(\varphi)=\frac{1}{2}\int_M f|\tau(\varphi)|^2 dv_g.$$

A curve φ parametrized by the arc length $'s'$ is a *f-biharmonic* unit speed curve in $\mathbb{E}^3_1$ if and only if [8]

$$\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T}+\frac{2f'}{f}\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T}+\frac{f''}{f}\nabla_{\mathbf{T}}\mathbf{T}=0. \tag{7}$$

3. Slant helices with the modified orthogonal frame

In this section, by using the modified orthogonal frame, we obtain some relations for the curvature and torsion of a slant helix. First, we prove the following.

Theorem 3.1. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed lightlike curve with modified orthogonal frame. Then φ is a slant helix if and only if*

$$\left(\frac{k'}{k^3}\right)'+\left(\frac{\tau}{k}\right)'\left(s+\frac{c_1}{c}\right)+\frac{2\tau}{k}=0, \tag{8}$$

where c and c_1 are constants.

Proof. Suppose φ is a lightlike slant helix, then the constant slope axis $\mathbf{U}$ can be written as

$$\mathbf{U}=a_1(s)\mathbf{T}(s)+\frac{c}{k^2}\mathbf{N}(s)+a_3(s)\mathbf{B}(s). \tag{9}$$

Differentiating $\mathbf{U}$ and applying modified orthogonal frame (5) for lightlike curve, we obtain

$$a_1'+c\frac{\tau}{k}=0,\ \ a_1-c\frac{k'}{k^3}-a_3\tau(s)=0,\ \ a_3'+a_3\frac{k'}{k}-\frac{c}{k}=0. \tag{10}$$

From the last relation of (10), we obtain

$$a_3 = \frac{cs + c_1}{k}, \tag{11}$$

Next, from the second relation of (10), we find

$$a_1 = \frac{ck'}{k^3} + \frac{\tau}{k}(cs + c_1). \tag{12}$$

Then, from first relation of (10) together with (12), we get the required result (8)

Conversely, let φ be a unit speed timelike slant helix and (8) holds identically. Define

$$\mathbf{U} = \Big[\frac{k'}{k^3} + \frac{\tau}{k}(s + \frac{c_1}{c})\Big]\mathbf{T} + \frac{c}{k^2}\mathbf{N} + \frac{(s + \frac{c_1}{c})}{k}\mathbf{B}. \tag{13}$$

By differentiating $\mathbf{U}$ with respect to $'s'$ and using (5) we get

$$\mathbf{U}' = \Big[\Big(\frac{k'}{k^3} + \frac{\tau}{k}(s + \frac{c_1}{c})\Big)' + \frac{\tau}{k}\Big]\mathbf{T} + \Big[\frac{k'}{k^2}(cs + c_1) - \frac{1}{k} + (\frac{cs + c_1}{k})'\Big]\mathbf{B} \tag{14}$$

Now, making use of (8), we get $\mathbf{U}' = 0$ i.e $\mathbf{U}$ is a constant vector. Furthermore $g(\mathbf{N}, \mathbf{U}) =$ constant. This concludes the proof of the theorem. □

In particular, if $k = 1$ in above theorem, then we have following special case of Theorem 3.1.

Corollary 3.2. *[2] Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed lightlike curve in $\mathbb{E}^3_1$. Then φ is a slant helix if and only if the torsion is*

$$\tau = \frac{a}{(bs + c)^2}, \tag{15}$$

where a, b and c are constants with $bs + c \neq 0$.

Next, we have the following result.

Theorem 3.3. *Any pseudo null curve φ with a modified orthogonal frame is slant helix if $ke^{c_2 \int \tau(s)ds} =$ constant.*

Proof. Let the slope axis $\mathbf{U}$ is given by

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{16}$$

Differentiating $\mathbf{U}$ and using the modified orthogonal frame (4) for pseudo null curve, we obtain

$$a_1' - a_3k^2 = 0, \;\; a_2' + a_2\frac{k'}{k} + a_2\tau + a_1 = 0, \;\; a_3' + a_3(\frac{k'}{k} - \tau) = 0. \tag{17}$$

From third relation of (17), we have

$$a_3 = \frac{e^{c_2 \int \tau(s)ds}}{k}. \tag{18}$$

Therefore, from (18) and (16) together with (4), we get

$$g(\mathbf{U}, \mathbf{N}) = ke^{c_2 \int \tau(s)ds}, \tag{19}$$

where $c_2 \in \mathbb{R}$. Hence, if $ke^{c_2 \int \tau(s)ds} =$ constant, we conclude that φ is a slant helix. □

Now, for the timelike slant helix, we have the following result.

Theorem 3.4. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed timelike curve such that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the modified orthogonal frame along the curve. Then φ is a slant helix if and only if*

$$\left(\frac{\left(\frac{c\tau k'}{k^3}\right) \pm \sqrt{\frac{(ck')^2}{k^4} - (k^2-\tau^2)(\frac{c^2}{k^2} - c_1)}}{(k^2-\tau^2)}\right)' + \frac{k'}{k}\left(\frac{\left(\frac{c\tau k'}{k^3}\right) \pm \sqrt{\frac{(ck')^2}{k^4} - (k^2-\tau^2)(\frac{c^2}{k^2} - c_1)}}{(k^2-\tau^2)}\right) + \frac{c\tau}{k^2} = 0, \tag{20}$$

where c and c_1 are constants.

Proof. Let φ be a slant helix and the slope axis $\mathbf{U}$ be a constant vector. Then we have

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{21}$$

Since φ is a slant helix, then from (1) $g(U, \mathbf{N}) = k^2 a_2(s) = c$ (constant). Hence, we find

$$a_2(s) = \frac{c}{k^2}. \tag{22}$$

Therefore, $\mathbf{U}$ can be expressed as

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{23}$$

Differentiating (23) and using the modified orthogonal frame (1) for timelike curve, we derive

$$a_1' + c = 0, \quad a_3' + a_3\frac{k'}{k} + c\frac{\tau}{k^2} = 0, \quad a_1 - c\frac{k'}{k^3} - a_3\tau(s) = 0. \tag{24}$$

Since $g(\mathbf{U}, \mathbf{U}) = c_1$ (constant), then from (21), we have

$$-(a_1)^2 + (a_2)^2 k^2 + (a_3)^2 k^2 = c_1. \tag{25}$$

Using (22) and third relation of (24), we obtain

$$a_3 = \frac{\left(\frac{c\tau k'}{k^3}\right) \pm \sqrt{\frac{(ck')^2}{k^4} - (k^2-\tau^2)(\frac{c^2}{k^2} - c_1)}}{(k^2-\tau^2)}. \tag{26}$$

Consequently, second relation of (24) together with (26) provides (20). Converse directly follows similar as in Theorem 3.1. □

Particularly, if $k = 1$ in above theorem, then we have the following result as a special case of Theorem 3.4.

Corollary 3.5. *[2] Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed timelike curve . Then φ is a slant helix if and only if either of the next two functions*

$$\frac{k^2}{(k^2-\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{k}\right)' \quad \text{or} \quad \frac{k^2}{(\tau^2-k^2)^{\frac{3}{2}}}\left(\frac{\tau}{k}\right)' \tag{27}$$

is constant everywhere provided $\tau^2 - k^2$ does not vanish.

Next, taking the curvature constant, we have the following result from Theorem 3.4.

Corollary 3.6. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed timelike curve such that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the modified orthogonal frame along the curve with constant curvature. Then, φ is slant helix if and only if*

$$\frac{\tau'}{(k^2-\tau^2)^{\frac{3}{2}}} = c \text{ (constant)}. \tag{28}$$

4. Slant helices with lightlike, timelike and spacelike slope axes

In this section we obtain the necessary and sufficient condition for a slant helix to have a lightlike, a timelike and a spacelike slope axis by using the modified orthogonal frame in $\mathbb{E}^3_1$.

First, we prove the following useful results.

Theorem 4.1. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed spacelike slant helix in $\mathbb{E}^3_1$ with a modified orthogonal frame and a spacelike principal normal. Then, the slope axis of $\varphi(s)$ is a lightlike, a timelike or a spacelike axis for $\epsilon = 0, -1$ or 1, respectively if and only if*

$$\left(\int \frac{\tau(s)}{k(s)}ds - l_2\right)^2 = (s + l_1)^2 - \frac{\epsilon}{c^2} + \frac{1}{k^2}, \tag{29}$$

where c, l_1 and l_2 are constants.

Proof. Let $\varphi(s)$ be a slant helix and the constant slope axis $\mathbf{U}$ is given by

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_3(s)\mathbf{B}(s).$$

Since $\varphi(s)$ is a slant helix, thus we have $g(U, N(s)) = k^2 a_2(s) = c$ (constant). Then, we find

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{30}$$

Differentiating (30), w.r.t. $'s'$ and using the modified orthogonal frame (2) for the spacelike curve, we obtain

$$a_1' - c = 0, \quad a_1 - c\frac{k'}{k^3} + a_3\tau = 0, \quad a_3' + a_3\frac{k'}{k} + c\frac{\tau}{k^2} = 0. \tag{31}$$

Solving first and third relations of (31) and using the obtained values of a_1 and a_3 in (30), $\mathbf{U}$ can be written as

$$\mathbf{U} = (cs + c_1)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + \left[-\frac{c}{k}\left(\int \frac{\tau(s)}{k(s)}ds - \frac{c_2}{c}\right)\right]\mathbf{B}. \tag{32}$$

Since $g(\mathbf{U}, \mathbf{U}) = \epsilon$, then we have

$$\left(\int \frac{\tau(s)}{k(s)}ds - \frac{c_2}{c}\right)^2 = (s + \frac{c_1}{c})^2 - \frac{\epsilon}{c^2} + \frac{1}{k^2}, \tag{33}$$

which provides (29) with $l_1 = \frac{c_1}{c}$ and $l_2 = \frac{c_2}{c}$.

Conversely, let $\varphi(s)$ be a unit speed spacelike slant helix in $\mathbb{E}^3_1$ with a modified frame and a spacelike principal normal $\mathbf{N}$. Suppose that (29) holds then from (32), we find

$$\mathbf{U} = (cs + c_1)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) - \left(\frac{c}{k}\sqrt{\left(s + \frac{c_1}{c}\right)^2 - \frac{\epsilon}{c^2} + \frac{1}{k^2}}\right)\mathbf{B}$$

which gives $g(\mathbf{U}, \mathbf{U}) = \epsilon$ and therefore $\mathbf{U}$ is a constant vector. □

Proposition 4.2. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed spacelike slant helix in $\mathbb{E}^3_1$ with a modified frame field and a timelike principal normal. Then, the slope axis of $\varphi(s)$ is a lightlike, a timelike or a spacelike axis for $\epsilon = 0, -1$ or 1, respectively if and only if*

$$\left(\int \frac{\tau(s)}{k(s)}ds + l_2\right)^2 = \frac{\epsilon}{c^2} - (s + l_1)^2 + \frac{1}{k^2}, \tag{34}$$

where $l_1 = \frac{c_1}{c}$ and $l_2 = \frac{c_2}{c}$ are constants.

Proof. The constant slope axis of spacelike slant helix with timelike principal normal is given by

$$\mathbf{U} = a_1(s)\mathbf{T}(s) - \frac{c}{k^2}\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{35}$$

Differentiating (35) w.r.t. s and using the modified orthogonal frame (2), we obtain

$$a_1' - c = 0, \quad a_1 - 3c\frac{k'}{k^3} + a_3\tau = 0, \quad a_3' + a_3\frac{k'}{k} - c\frac{\tau}{k^2} = 0. \tag{36}$$

Solving the relations of (36) for a_1 and a_3 and use them in (35), thus $\mathbf{U}$ can be written as

$$\mathbf{U} = (cs + c_1)\mathbf{T}(s) - \frac{c}{k^2}\mathbf{N}(s) + \left[\frac{c}{k}\left(\int \frac{\tau(s)}{k(s)}ds + \frac{c_2}{c}\right)\right]\mathbf{B}. \tag{37}$$

Since $g(\mathbf{U}, \mathbf{U}) = \epsilon,$, then we derive

$$\left(\int \frac{\tau(s)}{k(s)}ds + \frac{c_2}{c}\right)^2 = \frac{\epsilon}{c^2} - \left(s + \frac{c_1}{c}\right)^2 + \frac{1}{k^2}, \tag{38}$$

which gives (34) with $l_1 = \frac{c_1}{c}$ and $l_2 = \frac{c_2}{c}$. Converse follows directly like Theorem 4.1. □

Theorem 4.3. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed spacelike curve in $\mathbb{E}^3_1$ with a modified frame field and a lightlike normal vector.*
(i) If $c = 0$, then

- *(a) The slope axis is a constant lightlike vector if and only if the normal component of the slope axis is given by $c_2 e^{\int \tau ds}$.*
- *(b) The slope axis is constant spacelike vector if and only if*

$$a_2 = \frac{1}{k}\left(c_2 e^{-\int \tau ds} - e^{-\int \tau ds}\int k e^{\int \tau ds} ds\right).$$

- *(c) There does not exist any pseudo null slant helix with the constant timelike slope axis.*

(ii) If $c \neq 0$, then the slope axis is given as

$$\mathbf{U} = (cs + c_1)\mathbf{T} - \left(\frac{cs^2}{2} + c_1 s + \frac{c_1^2 - \epsilon}{2c}\right)\mathbf{N} + \frac{c}{k^2}\mathbf{B}. \tag{39}$$

Proof. Let $\varphi(s)$ be a slant helix and the constant slope axis $\mathbf{U}$ is given by

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{40}$$

Since $\varphi(s)$ is a slant helix and $\mathbf{N}$ is a lightlike normal, then we have $g(\mathbf{U}, \mathbf{N}(s)) = a_3(s)k^2 = c$ (*constant*). Using this fact, (40) can be written as

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2(s)\mathbf{N}(s) + \frac{c}{k^2}\mathbf{B}(s). \tag{41}$$

Differentiating (41) w.r.t. 's' and using the modified orthogonal frame (4) for pseudo null curve, we find

$$a_1' - c = 0, \quad a_1 + a_2' + a_2(\frac{k'}{k} + \tau) = 0, \quad \frac{c}{k^2}(\frac{k'}{k} + \tau) = 0. \tag{42}$$

From the third relation of (42), two cases arise:

Case (i). If $c = 0$, then from first relation of 42, we obtain $a_1 = c_1$. Hence, we write

$$\mathbf{U} = c_1\mathbf{T} + a_2\mathbf{N}.$$

Since, $g(\mathbf{U},\mathbf{U}) = \epsilon$, then we have

(a) If $\mathbf{U}$ is a constant lightlike vector, then we have $g(\mathbf{U},\mathbf{U}) = 0$, which implies $c_1 = 0$ and hence $a_1 = 0$. Now, from second relation of (42), we derive

$$a_2 = c_2\frac{1}{k}e^{\int \tau ds}.$$

Therefore,

$$\mathbf{U} = c_2\frac{1}{k}e^{\int \tau ds}\mathbf{N}.$$

Hence, the normal component of lightlike slope axis is $c_2e^{\int \tau ds}$.

(b) If $\mathbf{U}$ is a constant spacelike vector, then $g(\mathbf{U},\mathbf{U}) = 1$, which implies $c_1 = \pm 1$ and hence $a_1 = \pm 1$. Now, from second relation of (42), we get

$$a_2 = \frac{1}{k}\left(c_2e^{-\int \tau ds} - e^{-\int \tau ds}\int ke^{\int \tau ds}ds\right),$$

which gives

$$\mathbf{U} = \mathbf{T} + \frac{1}{k}\left(c_2e^{-\int \tau ds} - e^{-\int \tau ds}\int ke^{\int \tau ds}ds\right)\mathbf{N}.$$

(c) If $\mathbf{U}$ is a constant timelike vector, then $g(\mathbf{U},\mathbf{U}) = -1$, which implies $c_1^2 = -1$ and which is a contradiction.

Case(ii) If $c \neq 0$, then $(\frac{k'}{k} + \tau) = 0$.

First two relations of (42) yield

$$a_1 = cs + c_1 \quad \textit{and} \quad a_2 = -(c\frac{s^2}{2} + c_1s + c_2).$$

Therefore, we find

$$\mathbf{U} = (cs + c_1)\mathbf{T} - \left(\frac{cs^2}{2} + c_1s + c_2\right)\mathbf{N} + \frac{c}{k^2}\mathbf{B}. \tag{43}$$

Furthermore, since $g(U,U) = \epsilon$, which gives $c_2 = \frac{c_1^2-\epsilon}{2c}$ and hence we derived the required Eq. (39) □

Theorem 4.4. *Let $\varphi : I \to \mathbb{E}_1^3$ be a unit speed lightlike slant helix in $\mathbb{E}_1^3$ with the modified frame field. Then, the slope axis of $\varphi(s)$ is a lightlike, a timelike or a spacelike axis for $\epsilon = 0, -1$ or 1 respectively if and only if*

$$\int \frac{\tau(s)}{k(s)}ds - l_2 = \frac{1}{2(s + l_1)}\left(\frac{1}{k^2} - \frac{\epsilon}{c^2}\right), \tag{44}$$

where, $l_2 = \frac{c_2}{c}$ and $l_1 = \frac{c_1}{c}$ are constants.

Proof. Let $\varphi(s)$ be a slant helix and the constant lightlike slope axis $\mathbf{U}$ is given by

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + a_2\mathbf{N}(s) + a_3\mathbf{B}(s). \tag{45}$$

Since $\varphi(s)$ is a slant helix, then we find

$$g(\mathbf{U}, \mathbf{N}(s)) = k^2 a_2(s) = c \text{ (constant)}.$$

Then, (45) will be

$$\mathbf{U} = a_1(s)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + a_3\mathbf{B}(s). \tag{46}$$

Differentiating (46) and applying the modified orthogonal frame (5) for a lightlike curve, we derive

$$a_1' + \frac{c\tau}{k} = 0, \quad a_3' + \frac{k' a_3}{k} - \frac{c}{k} = 0, \quad a_1 - \frac{ck'}{k^3} - a_3\tau = 0. \tag{47}$$

Solving (47) for a_1 and a_3 and using the obtained values in (45), we find

$$\mathbf{U} = \left(-c\int \frac{\tau}{k} ds + c_1\right)\mathbf{T} + \frac{c}{k^2}\mathbf{N} + \frac{(cs + c_2)}{k}\mathbf{B}. \tag{48}$$

Since $g(U, U) = \epsilon$, then we obtain

$$\int \frac{\tau(s)}{k(s)} ds - \frac{c_2}{c} = \frac{1}{2(s + \frac{c_1}{c})}\left(\frac{1}{k^2} - \frac{\epsilon}{c^2}\right), \tag{49}$$

which is (49) with $l_2 = \frac{c_2}{c}$ and $l_1 = \frac{c_1}{c}$. Converse follows similar as in theorem 4.1 □

Finally, we have the following result.

Proposition 4.5. *Let $\varphi : I \to \mathbb{E}^3_1$ be a unit speed timetlike slant helix in $\mathbb{E}^3_1$ with the modified frame field. Then, the slope axis of $\varphi(s)$ is a lightlike, a timelike or a spacelike axis for $\epsilon = 0, -1$ or 1, respectively if and only if*

$$\left(\int \frac{\tau(s)}{k(s)} ds + l_2\right)^2 = \frac{\epsilon}{c^2} + (s + l_1)^2 - \frac{1}{k^2}, \tag{50}$$

where $l_2 = \frac{c_2}{c}$ and $l_1 = \frac{c_1}{c}$ are constants.

Proof. Let $\varphi(s)$ be a unit speed timelike slant helix and $\mathbf{U}$ is given by (23). By solving first and third relation of (24) and using the obtained values of a_1 and a_3 in (23), $\mathbf{U}$ can be written as

$$\mathbf{U} = (cs + c_1)\mathbf{T}(s) - \frac{c}{k^2}\mathbf{N}(s) + [\frac{c}{k}(\int \frac{\tau(s)}{k(s)} ds + \frac{c_2}{c})]\mathbf{B}. \tag{51}$$

Since $g(\mathbf{U}, \mathbf{U}) = \epsilon$, then we derive

$$\left(\int \frac{\tau(s)}{k(s)} ds + \frac{c_2}{c}\right)^2 = \frac{\epsilon}{c^2} + (s + \frac{c_1}{c})^2 - \frac{1}{k^2}, \tag{52}$$

which is (50) with $l_1 = \frac{c_1}{c}$ and $l_2 = \frac{c_2}{c}$. The converse follows directly. □

Example 4.6. *From [1], the curve is given by*

$$\begin{aligned}\psi(t) = < &\frac{1}{8}[(3 - 2\sqrt{2})\sinh(1 + \sqrt{2})t - (3 + 2\sqrt{2})\sinh(1 - \sqrt{2})t + 2\sinh t], \\ &\frac{1}{8}[(3 - 2\sqrt{2})\cosh(1 + \sqrt{2})t - (3 + 2\sqrt{2})\cosh(1 - \sqrt{2})t + 2\cosh t], \quad \frac{1}{4\sqrt{2}\cosh(\sqrt{2}t)} >\end{aligned}$$

is a timelike slant helix having curvature $k = 1$ and torsion $\tau = \frac{s}{\sqrt{1+s^2}}$. From Proposition 4.5, we have

$$\int \frac{\tau(s)}{k(s)} ds = s + 1, \tag{53}$$

Hence, (50) is satisfied for the constant $l_1 = 1, \epsilon = 1$ and $c = 1$.

5. Killing vector field along spacelike and null general slant helix

A regular curve $\varphi(s)$ in $\mathbb{E}^3_1$ is said to be a *general slant helix* if there is a Killing vector field $\mathbf{U}$ along with constant length such that it make constant angle with principal normal vector $\mathbf{N}$ [14]. $\mathbf{U}$ is called an axis of the general slant helix.

The vector field $\mathbf{U}(s)$ is said to be a Killing vector field along φ if the following conditions are satisfied [11]

$$\frac{\partial v}{\partial z}|_{z=0} = \frac{\partial k}{\partial z}|_{z=0} = \frac{\partial \tau}{\partial z}|_{z=0} = 0, \tag{54}$$

where v is the velocity of the curve φ. From [14], it is easy to see that $\mathbf{U}(s)$ is a Killing vector field along a spacelike curve φ with spacelike principal normal if and only if it satisfies the following conditions:

$$\begin{cases} (a) & g(\mathbf{U}', \mathbf{T}) = 0, \\ (b) & g(\mathbf{U}'', \mathbf{N})k = 0, \\ (c) & g(\frac{1}{k}\mathbf{U}''' - \frac{k'}{k^2}\mathbf{U}'' + k\mathbf{U}', \mathbf{B})\tau = 0. \end{cases} \tag{55}$$

Finally, in case of null curves the vector field $\mathbf{U}(s)$ is a Killing vector field along $\varphi(s)$ if the following conditions are satisfied

$$\begin{cases} (a) & g(\mathbf{U}', \mathbf{T}) = 0, \\ (b) & g(\mathbf{U}'', \mathbf{N}) = 0, \\ (c) & g(\mathbf{U}''' - \tau\mathbf{U}', \mathbf{B}) = 0. \end{cases} \tag{56}$$

Now, we have the following result.

Theorem 5.1. *Let $\varphi(s)$ be a spacelike general slant helix with a Killing vector field $\mathbf{U}(s)$ in $\mathbb{E}^3_1$ with the modified orthogonal frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Then*

$$\mathbf{U}(s) = (cs + c_1)\mathbf{T} + \frac{c}{k^2}\mathbf{N} \pm \frac{\sqrt{k^2 a_1^2 + c^2 - k^2}}{k^2}\mathbf{B} \tag{57}$$

and the relation between curvature and torsion is given by

$$\frac{1}{c}(a_3\tau)' = \frac{3(k')^2}{k^4} - \frac{k''}{k^3} + 1, \tag{58}$$

where $a_3 = \pm\frac{\sqrt{k^2a_1^2+c^2-k^2}}{k^2}$.

Proof. Let $\varphi(s)$ be a spacelike general slant helix with the Killing vector field $\mathbf{U}(s)$ and the constant slope axis $\mathbf{U}(s)$ is given by

$$\mathbf{U}(s) = a_1(s)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + a_3(s)\mathbf{B}(s), \tag{59}$$

where $a_1(s), a_3(s)$ are differentiable functions and 'c' is a nonzero constant. Since $g(\mathbf{U}, \mathbf{U}) = 1$, then from (59) we obtain

$$k^2a_1^2 + c^2 - k^4a_3^2 = k^2,$$

which implies that

$$a_3 = \pm\frac{1}{k^2}\sqrt{k^2a_1^2 + c^2 - k^2}. \tag{60}$$

From (59) and the modified frame (2), we get

$$\mathbf{U}' = \left(a_1' - c)\mathbf{T} + (a_1 - c\frac{k'}{k^3} + \tau a_3\right)\mathbf{N} + \left(\frac{c\tau}{k^2} + a_3' + \frac{k' a_3}{k}\right)\mathbf{B}. \tag{61}$$

Making use of first Killing Eq. (55), it yields

$$a_1' = c. \tag{62}$$

Integrating (62) with respect to $'s'$, we obtain

$$a_1 = cs + c_1. \tag{63}$$

Therefore, we find

$$\mathbf{U} = (cs + c_1)\,\mathbf{T} + \frac{c}{k^2}\mathbf{N} \pm \frac{\sqrt{k^2 a_1^2 + c^2 - k^2}}{k^2}\mathbf{B},$$

which is (57), and also we find

$$\mathbf{U}' = \left(a_1 - \frac{k'}{k^3}c + \tau a_3\right)\mathbf{N} + \left(\frac{\tau c}{k^2} + a_3' + \frac{k' a_3}{k}\right)\mathbf{B}. \tag{64}$$

Furthermore from the condition $\mathbf{U}' = \lambda(\mathbf{T} \times \mathbf{U})$, we have

$$\mathbf{U}' = \lambda\left(\frac{c}{k^2}\mathbf{B} - a_3\mathbf{N}\right). \tag{65}$$

Using (64) and (65), we find

$$\lambda + \tau = -\frac{a_1}{a_3} + \frac{k'c}{a_3 k^3}, \quad a_3' + \frac{k'}{k}a_3 = \frac{c}{k^2}(\lambda - \tau). \tag{66}$$

By solving the above equation for λ, we find

$$2\lambda = -\frac{a_1}{a_3} + \frac{ck'}{a_3 k^3} + \frac{a_3' k^2}{c} + \frac{kk' a_3}{c}. \tag{67}$$

Again differentiating (65), we derive

$$\mathbf{U}'' = \lambda\left[a_3 k^2\mathbf{T} + \left(\frac{c\tau}{k^2} - a_3' - a_3\frac{k'}{k}\right)\mathbf{N} - \left(a_3\tau + \frac{ck'}{k^3}\right)\mathbf{B}\right]. \tag{68}$$

Making use of second killing Eq. (55) gives

$$a_3' + a_3\frac{k'}{k} - \frac{c\tau}{k^2} = 0. \tag{69}$$

Using (69) in (68) and differentiating again, we get

$$\mathbf{U}''' = \lambda\left[(a_3 k^2)'\mathbf{T} + \left(a_3 k^2 - a_3\tau^2 - \frac{ck'\tau}{k^3}\right)\mathbf{N} - \left((a_3\tau)' + a_3\tau\frac{k'}{k} + \frac{ck''}{k^3} - \frac{2c(k')^2}{k^4}\right)\mathbf{B}\right]. \tag{70}$$

Then, from (65), (68) and (70) together with the third killing Eq. (55), we derive

$$\frac{1}{c}(a_3\tau)' = \frac{3(k')^2}{k^4} - \frac{k''}{k^3} + 1.$$

which is (58). Hence, the proof is complete. □

Next, we prove the following result.

Theorem 5.2. *Let $\varphi(s)$ be a lightlike general slant helix with killing vector field $\mathbf{U}(s)$ in $\mathbb{E}_1^3$ with the modified orthogonal frame field $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Then $\mathbf{U}(s)$ is given by*

$$\mathbf{U} = \frac{k^2 - c^2}{2k^2(cs + c_1)}\mathbf{T} + \frac{c}{k^2}\mathbf{N} + \frac{cs + c_1}{k}\mathbf{B}, \tag{71}$$

and the relation between curvature and torsion is given by

$$\left(a_1' + \frac{c\tau}{k}\right)'' = \tau\left(a_1' + \frac{c\tau}{k}\right) - \left(a_1 k\tau - \frac{ck'\tau}{k^2} - a_3 k\tau^2\right)' \tag{72}$$

where a_1, a_3 are given by

$$a_1 = \frac{k^2 - c^2}{2k^2(cs + c_1)}, \quad a_3 = \frac{cs + c_1}{k}.$$

Proof. Let the constant slope axis $\mathbf{U}$ for lightlike general slant helix be given by

$$\mathbf{U}(s) = a_1(s)\mathbf{T}(s) + \frac{c}{k^2}\mathbf{N}(s) + a_3(s)\mathbf{B}(s). \tag{73}$$

From (73) and the modified frame (5), we have

$$\mathbf{U}' = \left(a_1' + \frac{c\tau}{k}\right)\mathbf{T} + \left(a_1 + \frac{k'}{k}c - a_3\tau\right)\mathbf{N} + \left(-\frac{c}{k} + a_3' + \frac{k'}{k}a_3\right)\mathbf{B}. \tag{74}$$

From the Killing Eq. (56), we obtain

$$a_3 = \frac{cs + c_1}{k}. \tag{75}$$

Since $g(\mathbf{U}, \mathbf{U}) = 1$, then we find

$$a_1 a_3 = \frac{1}{2k}\left(1 - \frac{c^2}{k^2}\right). \tag{76}$$

Using (75) and (76), we derive

$$a_1 = \frac{k^2 - c^2}{2k^2(cs + c_1)}. \tag{77}$$

Therefore the axis of the null slant helix will be

$$\mathbf{U} = \left(\frac{k^2 - c^2}{2k^2(cs + c_1)}\right)\mathbf{T} + \frac{c}{k^2}\mathbf{N} + \left(\frac{cs + c_1}{k}\right)\mathbf{B}$$

which is (71). Now, differentiating (71) and using (75), we obtain

$$\mathbf{U}'' = \left(a_1' + \frac{c\tau}{k}\right)'\mathbf{T} + \left[a_1' + \frac{c\tau}{k} + \frac{k'}{k}\left(a_1 - \frac{ck'}{k^3} - a_3\tau\right) + \left(a_1 - \frac{ck'}{k^3} - a_3\tau\right)'\right]\mathbf{N} - k\left(a_1 - \frac{ck'}{k^3} - a_3\tau\right)\mathbf{B}.$$

Making use of second Killing Eq. (56), it follows

$$a_1' + \frac{c\tau}{k} + \frac{k'}{k}\left(a_1 - \frac{ck'}{k^3} - a_3\tau\right) + \left(a_1 - \frac{ck'}{k^3} - a_3\tau\right)' = 0.$$

Further differentiating (74) and using the third Killing Eq. (56), we obtain

$$\left(a_1' + \frac{c\tau}{k}\right)'' - \tau\left(a_1' + \frac{c\tau}{k}\right) - \left(a_1 k\tau + \frac{ck'\tau}{k^2} - a_3 k\tau^2\right)' = 0,$$

which provides (72). □

6. f-biharmonic slant helix

In this section we prove the following result.

Theorem 6.1. *Let $\varphi : I \to \mathbb{E}_1^3$ be a unit speed lightlike f-biharmonic slant helix. Then, the curvature and torsion satisfies the following relation*

$$\left(\frac{\sqrt{8k^4 + c_2c_3k^3}}{c_2k^3}\right)' + \left(\frac{\tau}{k}\right)'\left(s + \frac{c_1}{c}\right) + \frac{2\tau}{k} = 0,$$

where $k = c_1 f^{-\frac{2}{3}}$ and $\tau = kc_2$.

Proof. Let $\varphi : I \to \mathbb{E}_1^3$ be a unit speed lightlike f-biharmonic slant helix curve with the modified orthogonal frame in $\mathbb{E}_1^3$. Then from (5), we have

$$\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T} = \nabla_{\mathbf{T}}\mathbf{N} = (k\tau)\mathbf{T} + \frac{k'}{k}\mathbf{N} - k\mathbf{B}, \tag{78}$$

$$\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T} = \nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{N} = (k\tau' + 2k'\tau)\mathbf{T} + \left[2k\tau + \left(\frac{k'}{k}\right)' + \left(\frac{k'}{k}\right)^2\right]\mathbf{N} - 3k'\mathbf{B}. \tag{79}$$

For the curve φ to be f−biharmonic then, we have

$$\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T} + \frac{2f'}{f}\nabla_{\mathbf{T}}\nabla_{\mathbf{T}}\mathbf{T} + \frac{f''}{f}\nabla_{\mathbf{T}}\mathbf{T} = 0. \tag{80}$$

Using Eqs. (78) and (79) in (80), we get

$$\left(2k'\tau + k\tau' + \frac{2f'}{f}k\tau\right)\mathbf{T} + \left[2k\tau + \left(\frac{k'}{k}\right)' + \left(\frac{k'}{k}\right)^2 + \frac{2f'k'}{fk} + \frac{f''}{f}\right]\mathbf{N}$$
$$- \left(3k' + \frac{2f'}{f}k\right)\mathbf{B} = 0. \tag{81}$$

Equating the coefficients of $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$, we obtain

$$2k'\tau + k\tau' + \frac{2f'}{f}k\tau = 0, \tag{82}$$

$$2k\tau + \left(\frac{k'}{k}\right)' + \left(\frac{k'}{k}\right)^2 + \frac{2f'k'}{fk} + \frac{f''}{f} = 0, \tag{83}$$

$$3k' + \frac{2f'}{f}k = 0. \tag{84}$$

By Eq. (84) it follows

$$\frac{k'}{k} = -\frac{2f'}{3f}, \tag{85}$$

which yields $k = c_1 f^{-2/3}$. By using (82) and (84), we have $\tau = kc_2$. Now from Eq. (83), we have

$$\left(\frac{k'}{k}\right)' - 2\left(\frac{k'}{k}\right)^2 + \frac{f''}{f} = -2c_2k^2. \tag{86}$$

From (85), it follows that

$$\frac{f''}{f} = \frac{9}{4}\left(\frac{k'}{k}\right)^2 - \frac{3}{2}\left(\frac{k'}{k}\right)'. \tag{87}$$

Then from (86) and (87), we obtain

$$2\left(\frac{k'}{k}\right)' - \left(\frac{k'}{k}\right)^2 = 8c_2k^2, \tag{88}$$

which on solving gives

$$k' = \frac{\sqrt{8k^4 + c_2c_3k^3}}{c_2}. \tag{89}$$

On the other hand, From Theorem 3.1, for a lightlike slant helix, we have

$$\left(\frac{k'}{k^3}\right)' + \left(\frac{\tau}{k}\right)'\left(s + \frac{c_1}{c}\right) + \frac{2\tau}{k} = 0 \tag{90}$$

Using (89) and (90), the result follows immediately. □

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References

[1] T. Ali, M. Turgut, *Position vector of a time-like slant helix in Minkowski 3-space*, J. Math. Anal. Appl. **365** (2010), 559–569.
[2] T. Ali and R. Lopez, *On Slant helices in Minkowski space* $\mathbb{E}_1^3$, J. Korean Math. Soc. **48** (2011), 159-167.
[3] T. Ali, S. R. Mahmoud, *Position vectors of spacelike slant helices in Minkowski 3-space*, Honam Math. J. **36** (2014), 233–251.
[4] B. Bukcu, M. K Karacan, *On the modified orthogonal frame with curvature and torsion in 3-space*, Math. Sci. Appl. E-Notes, **4** (2016), 184-188.
[5] C. Camci, K. Ilarslan, A. Ucum , *General helices with lightlike slope axis*, Filomat **32** (2) (2018), 355-367.
[6] M. Ergut, H.B. Oztekin, S. Aykurt, *Non-null k-slant helices and their spherical indicatrices in Minkowski 3-space*, J. Adv. Res. Dyn. Control. Syst. **2** (2010) 1-12.
[7] J. Eells Jr. and J.H.Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109-160.
[8] Li Du and Juan Zhang, *Classification of f-Biharmonic Curves in Lorentz Minkowski Space*, J. Math. (2020), Art. 7529284.
[9] M. S. Lone, E. S. Hasan, M. K. Karacan, B. Bukcu, *On Some Curves with Modified Orthogonal Frame in Euclidean 3-space.* Iranian J. Sci. Tech., Trans. A: Sci. **43** (2018), 1905–1916.
[10] M. Barros, *General helices and a theorem of Lancret*. Proc. Amer. Math. Soc. **125** (1997), 1503-1509.
[11] M. Barros, A. Ferrndez, P. Lucas, MA. Meroo, *General helices in the 3-dimensional Lorentzian space forms*, Rocky Mountain J. Math. **31** (2001), 373-388.
[12] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28** (2004), 153-163.
[13] L. Kula, N. Ekemkci, Y. Yayli, K. Ilarslan, *Characterizations of slant helices in Euclidean 3-space*, Turk. J. Math. **34** (2010), 261-273.
[14] P. Lucas, J. A. Ortega-Yagues, *Slant helices: a new approximation*, Turk. J. Math. **43** (2019), 473–485.
[15] A. Menninger, *Characterization of the slant helix as successor curve of the general helix*, Int. Electron. J. Geom. **7** (2) (2014), 84–91.
[16] M. Najdanovic, L. Velimirovic, *Second Order Infinitesimal Bending of Curves*, Filomat **31** (13) (2017), 4127–4137.
[17] T. Sasai, *The fundamental theorem of analytical space curves and apparent singularities of Fuchsian differential equations*, Tohoku Math. J. **36** (1984), 17-24.
[18] A. Ucum, C. Camci, K Ilarslan, *General helices with timelike slope axis in minkowski 3-space*, Adv. Appl. Clifford Algebras, **26** (2016), 793-807.
[19] A. Ucum, C. Camci, K Ilarslan, *General helices with spacelike slope axis in minkowski 3-space*, Asian-European J. Math. **12** (2019), Art. 1950076.
[20] J. Walrave, *Curves and surfaces in Minkowski space* Ph.D. dissertation, K. U. Leuven, Leuven,1995.