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Asymptotical Mean-Square Stability of Split-Step θ Methods for Stochastic Pantograph Differential Equations Under Fully-Geometric Mesh

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Abstract. The paper deals with the numerical asymptotical mean-square stability of split-step θ methods for stochastic pantograph differential equations, which is the generalization of deterministic pantograph equations. Instead of the quasi-geometric mesh, a fully-geometric mesh, widely used for deterministic problems, is employed. A useful technique, the limiting equation, for deterministic problems is also extended to stochastic problems based on Kronecker product. Under the exact stability condition, the stability region of the split-step θ methods is discussed, which is an improvement of some existing results. Moreover, such technique is also available to stochastic pantograph differential equations with Poisson jumps. Meanwhile, compared with the destabilization of Wiener process, the stabilization of Poisson jumps is replicated by numerical processes. Finally, numerical examples are given to illustrate that our numerical stability condition is nearly necessary for stochastic problems.

1. Introduction

Stochastic pantograph differential equations (SPDEs) are a special kind of Itô-type stochastic delay differential equations with unbounded memory, which plays an important role in economics, biology, medicine and many of other fields. The fundamental theory about existence and uniqueness for the solution of SPDEs has been studied in [1, 4], and the stability proprieties can be found in [5, 7, 22, 24].

Exact solutions for SPDEs can rarely be obtained. Thus, it is necessary to develop numerical methods and to study the properties of these methods. Based on the propriety of convergence, the stability analysis is important. For example, Fan et al. investigated the mean-square stability of the semi-implicit Euler methods in [5]. Guo and Li obtained the global mean-square asymptotical stability of the Euler-Maruyama method in [7]. In recently years, a lot work have been done about split-step θ methods. Xiao et al. investigated the convergence and the mean-square stability of split-step θ methods in [21]. Guo and Li considered the sufficient conditions of the almost sure stability with general decay rate for the split-step θ methods in [8].

For the asymptotical mean-square stability of the linear θ-methods, we proposed the limiting equation by Kronecker product, and then we obtained an optimal stability condition in [25]. The split-step θ methods

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can formulate to the form in linear θ-methods sense. In order to overcome the influence of noise, some stronger results are needed. Hence, a developed approach give rise to stronger conditions than linear θ-methods, but weaker than the existing conditions for split-step θ methods.

Another key contribution in our development method is the fully-geometric mesh. When we investigate the numerical methods on uniform mesh, the difficulty is the limited computer memory. For deterministic pantograph equations, many authors apply geometric mesh to solve this problem, including fully-geometric mesh and quasi-geometric mesh. Wang [19] investigate fully-geometric mesh one-leg methods for the generalized pantograph equations. Bellen et. al [2] gave asymptotic stability properties of θ-methods for the pantograph equations on quasi-geometric mesh. Fully-geometric mesh and quasi-geometric mesh are the same in essence but slightly different in detail, and many results are the same with two kinds of geometric mesh (see [18]). As for SPDEs, authors use the variable stepsizes to deal with the storage problem [22, 27]. Regardless, up to the best of our knowledge, there is no such work for SPDEs under the fully-geometric mesh. Instead of the quasi-geometric mesh, the fully-geometric mesh is considered since its same features both globally and locally (with respect to intervals). Here, we study the behavior of split-step θ methods implemented on a fully-geometric mesh.

On the other hand Poisson jumps have received increasing attention in area of stochastic finance and model population dynamics. Instead of the Poisson process, Many authors consider the compensated Poisson process, which has a better stability proprieties (see [9, 10, 23]). So, we discuss the compensated Poisson process and generalize our study to SPDEs. Some similar results were also derived. Under some conditions, the Poisson process can stabilize the system by enlarging the intensity of Poisson process, which is different from the Wiener process.

The structure of this paper is organized as follows. We will review the fully-geometric mesh in Section 2. Applying the Kronecker product and limiting equation, we derive the asymptotical mean-square stability in the section 3. We extend our study to SPDEs with Poisson jumps in the section 4. Some numerical results are presented in the Section 5.

2. Fully-geometric mesh

In this section, we will review some knowledge about fully-geometric mesh, which was introduced by [2] and [14].

Here, the mesh $H = \{m; t_0, t_1 \cdots, t_n \cdots\}$ is defined as follows. Let $T_0 > 0$ be given, $t_0 = T_0$ and $t_m = q^{-1}T_0$. Firstly, we are interested in the initial interval $[0, T_0]$. We define $t_{-m} = qT_0$ and choose $m-1$ grid points $t_{-m+1} < t_{-m+2} < \cdots < t_{-1}$ in (t_{-m}, t_0), the grid points in (t_{-m}, t_0) is defined as

$$t_{n+1} = q^{-\frac{1}{m}} t_n \text{ for } n = -m+1, -m+2, \cdots, 0.$$

The stepsize in (t_{-m}, t_0) can be defined as $h_n = t_{n+1} - t_n$, for $n = -m+1, -m+2, \cdots, 0$. We choose the minimum stepsize as the initial stepsize and give an initial value x_0, then all the values on the mesh can be obtained over the initial interval $[0, T_0]$. For the items with pantograph $x(qt_n)$, we choose its left limit to approximate $x(qt_n)$ proposed in [17]. Unlike the quasi-geometric mesh, we don't need to define the global mesh by partitioning every primary interval into a fixed number m of subintervals of the same size. This way, after setting the first initial interval, we would have

$$t_{n+1} = q^{-\frac{1}{m}} t_n \text{ for } n \geq 0.$$

The stepsize $h_n = t_{n+1} - t_n$, which satisfy

$$h_{n+1} = q^{-\frac{1}{m}} h_n, \text{ for all } n \geq 1, \text{ and } \lim_{n\to\infty} h_n = \infty.$$

The fully-geometric mesh has been widely studied in the literature concerning numerical method for delay differential equation, neutral delay differential equation, and also for Volterra integral equation with proportional delay (see [3, 19]).

3. Asymptotical mean-square stability

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_{t\geq 0}\}$ satisfying the usual conditions. We consider the linear SPDEs of the form

$$\begin{cases} \mathrm{d}x(t) &= [ax(t) + bx(qt)]\mathrm{d}t + [cx(t) + dx(qt)]\mathrm{d}W(t), \quad t > 0, \\ x(0) &= x_0, \end{cases} \tag{1}$$

where $0 < q < 1$, a, b, c, d are constants, $x_0 \in \mathbb{R}$ and $W(t)$ is a standard Wiener process. For the equation (1), the asymptotical mean-square stability has been obtained as follows.

Lemma 3.1. *[5] If the coefficient of equation (1) are satisfied*

$$a < -|b| - \frac{1}{2}(|c| + |d|)^2, \tag{2}$$

then, the trivial solution is asymptotically mean-square stable, that is

$$\lim_{t\to\infty} E|x(t)|^2 = 0.$$

We follow the numerical form in [21]

$$\begin{cases} x_n^* = x_n + \theta h_n(ax_n^* + bx_{n-m}^*), \\ x_{n+1} = x_n + h_n(ax_n^* + bx_{n-m}^*) + (cx_n^* + dx_{n-m}^*)\Delta W_n, \end{cases} \tag{3}$$

where x_n is an approximation to $x(t_n)$, θ is a parameter with $\theta \in [0, 1]$. The increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0, h_n)$-distributed Gaussian random variables. Assuming that x_n is $\mathcal{F}_{t_n}$-measurable at the mesh-point t_n, we can easily know that x_n^* and x_{n-m}^* are $\mathcal{F}_{t_n}$-measurable at related mesh points.

Since the convergence of split-step θ methods for SPDEs is investigated in [21], we are interested in the asymptotical mean-square stability of split-step θ methods.

Definition 3.2. *The split-step θ methods on the fully-geometric mesh are said to be asymptotically mean-square stable, if for any given mesh H,*

$$\lim_{n\to\infty} E|x_n|^2 = 0.$$

Remark 3.3. *Xiao et al. [21] proved that if $\theta > \frac{1}{2}$, then $\lim_{n\to\infty} E|\theta x_{n+1} + (1-\theta)x_n|^2 = 0$ implies $\lim_{n\to\infty} E|x_n|^2 = 0$. In this paper, we use the same technique to prove the asymptotical mean-square stability.*

From the equation (3), we have

$$\theta x_{n+1} + (1-\theta)x_n = x_n^* + \theta(cx_n^* + dx_{n-m}^*)\Delta W_n.$$

Note that $E \mid \Delta W_n \mid^2 = h_n$ and x_n^*, x_{n-m}^* are $\mathcal{F}_{t_n}$-measurable. Using the inequality $|a+b+c|^2 \leq 3|a|^2 + 3|b|^2 + 3|c|^2$ and taking the expectation on both sides, we derive that

$$E|\theta x_{n+1} + (1-\theta)x_n|^2 \leq 3E|x_n^*|^2 + 3E|\theta c x_n^* h_n^{\frac{1}{2}}|^2 + 3E|\theta d x_{n-m}^* h_n^{\frac{1}{2}}|^2. \tag{4}$$

By the inequality (4), we reformulate the conditions in [21], which is a key point in the following analysis.

Lemma 3.4. *If $\theta > \frac{1}{2}$ and $\lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_n^*|^2 = 0$, then the numerical solutions are asymptotically mean-square stable, i.e., $\lim_{n\to\infty} E|x_n|^2 = 0$.*

Proof. Using the condition $\lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_n^*|^2 = 0$ and the properties of the fully-geometric mesh, we have

$$\lim_{n\to\infty} E|x_n^*|^2 = \lim_{n\to\infty} \frac{1}{h_n} E|h_n^{\frac{1}{2}} x_n^*|^2 = \lim_{n\to\infty} \frac{1}{h_n} \lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_n^*|^2 = 0.$$

We also get

$$\lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_{n-m}^*|^2 = q^{-1} \lim_{n\to\infty} E|h_{n-m}^{\frac{1}{2}} x_{n-m}^*|^2 = 0.$$

In view of the inequality (4) and the Remark 3.3, we can immediately obtain

$$\lim_{n\to\infty} E|x_n|^2 = 0.$$

□

It follows from Lemma 3.4 that the numerical stability is resulted from the asymptotical behaviors of the split solutions x_n^*, which satisfies the following scheme

$$\begin{aligned} x_{n+1}^* =& x_n^* + (1-\theta)h_n(ax_n^* + bx_{n-m}^*) + \theta h_{n+1}(ax_{n+1}^* + bx_{n-m+1}^*) \\ &+ (cx_n^* + dx_{n-m}^*)\Delta W_n. \end{aligned} \tag{5}$$

It is clear that the numerical scheme for split solutions x_n^* is almost same as the linear θ-methods. Hence the asymptotical mean-square stability of x_n^* is directly yielded from our previous analysis in [25], which also implies the asymptotical mean-square stability of numerical solutions x_n from (4) for $c = d = 0$. However, a stronger stability result for split solutions x_n^* is needed by (4), i.e., $\lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_n^*|^2 = 0$, whenever there is a stochastic noise indeed. Multiplying both sides of equation (5) by $h_{n+1}^{\frac{1}{2}}$, we have

$$\begin{aligned} (1 - a\theta h_{n+1})h_{n+1}^{\frac{1}{2}} x_{n+1}^* =& h_{n+1}^{\frac{1}{2}}((1 + (1-\theta)h_n a)x_n^* + (1-\theta)h_n b x_{n-m}^* \\ &+ \theta h_{n+1} b x_{n-m+1}^*) + (cx_n + dx_{n-m})\Delta W_n. \end{aligned} \tag{6}$$

Here, we denote $h_n^{\frac{1}{2}} x_n^* = \mathrm{x}_n$ for simplicity. Then the equation (6) can be rewritten by the fully-geometric meshes as

$$\begin{aligned} (1 - a\theta h_{n+1})\mathrm{x}_{n+1} =& (1 + (1-\theta)h_n a)q^{-\frac{1}{2m}} \mathrm{x}_n + (1-\theta)h_n b q^{-\frac{m+1}{2m}} \mathrm{x}_{n-m} \\ &+ \theta h_{n+1} b q^{-\frac{1}{2}} \mathrm{x}_{n-m+1} + (cq^{-\frac{1}{m}} \mathrm{x}_n + dq^{-\frac{m+1}{2m}} \mathrm{x}_{n-m})\Delta W_n, \end{aligned}$$

which has an equivalent form

$$M_{0n} X_{n+1} = M_{1n} X_n + M_{2n} X_n \Delta W_n,$$

where $X_n = (\mathrm{x}_n, \mathrm{x}_{n-1}, \cdots, \mathrm{x}_{n-m+1}, \mathrm{x}_{n-m})^T$,

$$M_{0n} = \begin{pmatrix} 1 - a\theta h_{n+1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_{1n} = \begin{pmatrix} (1 + (1-\theta)ah_n)q^{-\frac{1}{2m}} & 0 & \cdots & 0 & b\theta h_{n+1} q^{-\frac{1}{2}} & b(1-\theta)h_n q^{-\frac{m+1}{2m}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$M_{2n} = \begin{pmatrix} cq^{-\frac{1}{2m}} & 0 & \cdots & 0 & dq^{-\frac{m+1}{2m}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It follows from (2) that $1 - a\theta h_{n+1} \neq 0$, which yields that

$$X_{n+1} = M_1^n X_n + \widetilde{M}_2^n X_n \Delta W_n,$$

where

$$M_1^n = M_{0n}^{-1} M_{1n} = \begin{pmatrix} \frac{(1+(1-\theta)ah_n)q^{-\frac{1}{2m}}}{1-a\theta h_{n+1}} & 0 & \cdots & 0 & \frac{b\theta h_{n+1} q^{-\frac{1}{2}}}{1-a\theta h_{n+1}} & \frac{b(1-\theta)h_n q^{-\frac{m+1}{2m}}}{1-a\theta h_{n+1}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{7}$$

$$\widetilde{M}_2^n = M_{0n}^{-1} M_{2n} = \begin{pmatrix} \frac{cq^{-\frac{1}{2m}}}{1-a\theta h_{n+1}} & 0 & \cdots & 0 & \frac{dq^{-\frac{m+1}{2m}}}{1-a\theta h_{n+1}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We obtain an optimal stability region of the linear θ-methods for linear SPDEs by Kronecker product $E|X_n X_n^T|$ in [25], which was introduced in [20] for the asymptotical mean-square stability and for the numerical stability in [28]. In this paper, we will apply the same technique in our analysis. It is easy to know that

$$E(X_{n+1} X_{n+1}^T) = M_1^n E(X_n X_n^T)(M_1^n)^T + h_n^{\frac{1}{2}} \widetilde{M}_2^n E(X_n X_n^T)(\widetilde{M}_2^n)^T (h_n^{\frac{1}{2}})^T. \tag{8}$$

Thus, the equation (8) can be rewritten as a new matrix equation

$$\mathcal{X}_{n+1} = M_1^n \mathcal{X}_n (M_1^n)^T + M_2^n \mathcal{X}_n (M_2^n)^T,$$

with the notations $\mathcal{X}_{n+1} = E(X_{n+1} X_{n+1}^T)$, $h_n^{\frac{1}{2}} \widetilde{M}_2^n = M_2^n$.

In the investigation of matrix equations, it is often convenient to consider members of M_1^n and M_2^n as vectors by ordering their entries in a conventional way. We adopt the common convention of stacking columns, left to right, and then apply the propriety of the Kronecker product

$$vec(\mathcal{X}_{n+1}) = A_n vec(\mathcal{X}_n), \tag{9}$$

where $A_n = M_1^n \otimes M_1^n + M_2^n \otimes M_2^n$, $\otimes$ is Kronecker product. Before applying the limiting equation in [25], we will investigate the limit of A_n firstly. The existence of limit can be ensured by the fully-geometric. In view of $\lim_{n\to\infty} h_n = \infty$, one concludes that

$$A = M_1 \otimes M_1,$$

where

$$M_1 = \lim_{n\to\infty} M_1^n = \begin{pmatrix} -\frac{(1-\theta)q^{\frac{1}{2m}}}{\theta} & 0 & \cdots & 0 & -\frac{b}{a\sqrt{q}} & -\frac{b(1-\theta)q^{\frac{1}{2m}}}{a\theta\sqrt{q}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{10}$$

$$M_2 = \lim_{n\to\infty} M_2^n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

The limiting equation of (9) is

$$Z_{n+1} = AZ_n.$$

Theorem 3.5. *Under the condition (2), if $\theta \in (\frac{1}{2}, 1]$ and $|b| < -\sqrt{q}a$, the split-step θ methods are asymptotically mean-square stable for any mesh H, i.e.*

$$\lim_{n\to\infty} E|x_n|^2 = 0.$$

Proof. It is easy to see that $\lim_{n\to\infty} E|x_n|^2 = 0$ is equivalent to $\lim_{n\to\infty} \mathcal{X}_n = 0$. Referring to [15], if two conditions are satisfied, i.e., $\Sigma_{n=0}^{\infty}\|A_n - A\|_\infty < \infty$ and the algebraic multiplicity of the eigenvalue for A is 1, we can know that $\lim_{n\to\infty} Z_n = 0$ implies $\lim_{n\to\infty} \mathcal{X}_n = 0$.

The problem turns out to verify the conditions in [15]. Applying the comparison test to (7) and (10), we get

$$\begin{aligned}
\Sigma_{n=0}^{\infty}\left|\frac{(1+(1-\theta)ah_n)q^{-\frac{1}{2m}}}{1-a\theta h_{n+1}} + \frac{(1-\theta)q^{\frac{1}{2m}}}{\theta}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{(1-\theta)q^{\frac{1}{2m}} + \theta q^{-\frac{1}{2m}}}{(1-a\theta h_{n+1})\theta}\right| < \infty, \\
\Sigma_{n=0}^{\infty}\left|\frac{b\theta h_{n+1}q^{-\frac{1}{2}}}{1-a\theta h_{n+1}} + \frac{b}{a\sqrt{q}}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{1}{(1-a\theta h_{n+1})a\sqrt{q}}\right| < \infty, \\
\Sigma_{n=0}^{\infty}\left|\frac{b(1-\theta)h_n q^{-\frac{m+1}{2m}}}{1-a\theta h_{n+1}} + \frac{b(1-\theta)q^{\frac{1}{2m}}}{a\theta\sqrt{q}}\right| &= \Sigma_{n=0}^{\infty}\left|\frac{b(1-\theta)q^{\frac{1-m}{2m}}}{(1-a\theta h_{n+1})a\theta}\right| < \infty,
\end{aligned}$$

which yields $\Sigma_{n=0}^{\infty}\|M_1^n - M_1\|_\infty < \infty$. By a similar calculation, $\Sigma_{n=0}^{\infty}\|M_2^n - M_2\|_\infty < \infty$. Hence from the properties of Kronecker product of matrix, we obtain $\Sigma_{n=0}^{\infty}\|A_n - A\|_\infty < \infty$.

We can now determine the eigenvalues of the Kronecker product A of two square matrices. That is, we only need to consider the eigenvalues of M_1, which is given by the roots of the equations

$$|\lambda I - M_1| = (\lambda + \frac{(1-\theta)q^{\frac{1}{2m}}}{\theta})(\lambda^m + \frac{b}{a\sqrt{q}}),$$

where I is the identity matrix of order $m+1$. Namely the eigenvalues are

$$\lambda_m = \frac{(\theta-1)q^{\frac{1}{2m}}}{\theta}, \lambda_k = \sqrt[m]{\left|\frac{b}{a\sqrt{q}}\right|}e^{i\frac{1}{m}(\arg(-\frac{b}{a\sqrt{q}})+2k\pi)}, k = 0, \cdots, m-1.$$

Every eigenvalue of A arises as such a product of eigenvalues of M_1 and M_1. Therefore, the algebraic multiplicity of the eigenvalue for A is 1.

It follows from $|b| < -\sqrt{q}a$ and $\theta \in (\frac{1}{2}, 1]$ that $\rho(M_1) < 1$, which implies that $\rho(A) < 1$. Hence the proof is complete. □

Remark 3.6. *In [21], the stability conditions for nonlinear system have been discussed, which can be rewritten for linear system as*

$$\theta > \tfrac{1}{2} \text{ and } q(2a+b) - |b| - (1+q)(c^2+d^2+cd) > 0 \text{ (with our notations).}$$

The stability conditions are improved in the Theorem 3.5.

Remark 3.7. *In [25], the stability condition for linear θ-methods have been established by $|b| < -a$. However, a stronger asymptotical behavior for the split solutions x_n^* is required $\lim_{n\to\infty} E|h_n^{\frac{1}{2}} x_n^*|^2 = 0$. Namely, not only the tendency of x_n^* to zero, but also its speed is considered, which yields the stronger stability condition $|b| < -\sqrt{q}a$ in Theorem 3.5.*

Remark 3.8. *In the case of $\theta = 1$, the split-step θ methods reduce to the split-step backward Euler methods introduced in [11], which has the scheme of the form*

$$\begin{aligned} x_n^* &= x_n + h_n(ax_n^* + bx_{n-m}^*), \\ x_{n+1} &= x_n^* + (cx_n^* + dx_{n-m}^*)\Delta W_n. \end{aligned}$$

Therefore, Theorem 3.5 illustrates the asymptotical mean-square stability of split-step backward Euler methods under fully-geometric mesh.

4. Stochastic pantograph differential equation with Poisson jumps

In this section, we extend the study of numerical stability to SPDEs with Poisson jumps by form

$$\begin{cases} \mathrm{d}y(t) = & [ay(t) + by(qt)]\mathrm{d}t + [cy(t) + dy(qt)]\mathrm{d}W(t) \\ & +[ey(t) + fy(qt)]\mathrm{d}N(t), \qquad t > 0, \\ y(0) = & y_0, \end{cases} \tag{11}$$

where $W(t)$ is a standard Wiener process and $N(t)$ is a scalar Poisson process with intensity λ. For the equation (11), we have the following stability result.

Theorem 4.1. *[26] If the coefficient of equation (11) are satisfied*

$$a + |b| + \lambda e + \lambda|f| + \frac{1}{2}(|c| + |d|)^2 + \frac{1}{2}\lambda(|e| + |f|)^2 < 0, \tag{12}$$

then the trivial solution is asymptotically mean-square stable, that is

$$\lim_{t\to\infty} E|y(t)|^2 = 0.$$

An adaption of the split-step θ methods on the fully-geometric mesh to (11) leads to

$$\begin{cases} y_n^* = y_n + \theta h_n(ay_n^* + by_{n-m}^*), \\ y_{n+1} = y_n + h_n(ay_n^* + by_{n-m}^*) + (cy_n^* + dy_{n-m}^*)\Delta W_n + (ey_n^* + fy_{n-m}^*)\Delta N_n, \end{cases}$$

where $\Delta N_n = N_{n+1} - N_n$ are independent Poisson distributed random variables with mean λh_n and variance λh_n.

For stochastic differential equation with Poisson jumps, the compensated split-step backward Euler methods are introduced in [10], which satisfies a better stability property than the split-step backward Euler methods. Lately, Huang [9] follow the idea in [10] to investigate the exponential mean square stability and find that split-step θ methods can be given a weaker assumption than compensated split-step θ methods. Therefore, we only consider the compensated split-step θ methods.

Noting that the compensated Poisson process $\tilde{N}(t) := N(t) - \lambda t$ is a martingale satisfying the properties

$$E(\tilde{N}(t+s) - \tilde{N}(t)) = 0,\ E(\tilde{N}(t+s) - \tilde{N}(t))^2 = \lambda s,\ t, s \geq 0,$$

we rewrite equation (11) in the equivalent form

$$\begin{cases} \mathrm{d}y(t) = & (\tilde{a}y(t) + \tilde{b}y(qt))\mathrm{d}t + (cy(t) + dy(qt))\mathrm{d}W(t) \\ & +(ey(t) + fy(qt))\mathrm{d}\tilde{N}(t), \qquad t > 0, \\ y(0) = & y_0, \end{cases} \tag{13}$$

with $\tilde{a} = a + \lambda e$ and $\tilde{b} = b + \lambda f$. An adaption of the split-step θ methods on the fully-geometric mesh to (13) leads to

$$\begin{cases} y_n^* = y_n + \theta h_n(\tilde{a}y_n^* + \tilde{b}y_{n-m}^*), \\ y_{n+1} = y_n + h_n(\tilde{a}y_n^* + \tilde{b}y_{n-m}^*) + (cy_n^* + dy_{n-m}^*)\Delta W_n + (ey_n^* + fy_{n-m}^*)\Delta\tilde{N}_n, \end{cases}$$

where $\Delta\tilde{N}_n = \tilde{N}_{n+1} - \tilde{N}_n$, which is called the compensated split-step θ methods. The convergence can be proved with the same method in [21]. Here, we only consider the numerical stability. In the following discussion, we will use the same technique in Sect. 3 and omit some simple parts. Denoting $h_n^{\frac{1}{2}} y_n^* = \mathrm{y}_n$ for simplicity, we have

$$\begin{aligned}(1-\tilde{a}\theta h_{n+1})\mathrm{y}_{n+1} =&(1+(1-\theta)h_n\tilde{a})q^{-\frac{1}{2m}}\mathrm{y}_n + (1-\theta)h_n\tilde{b}q^{-\frac{m+1}{2m}}\mathrm{y}_{n-m}\\ &+\theta h_{n+1}\tilde{b}q^{-\frac{1}{2}}\mathrm{y}_{n-m+1} + (cq^{-\frac{1}{m}}\mathrm{y}_n + dq^{-\frac{m+1}{2m}}\mathrm{y}_{n-m})\Delta W_n\\ &+(eq^{-\frac{1}{m}}\mathrm{y}_n + fq^{-\frac{m+1}{2m}}\mathrm{y}_{n-m})\Delta\tilde{N}_n.\end{aligned}$$

It follows from (12) that $1-(a+\lambda e)\theta h_{n+1} \neq 0$, which yields that

$$Y_{n+1} = U_1^n Y_n + \widetilde{U_2^n} Y_n \Delta W_n + \widetilde{U_3^n} Y_n \Delta\tilde{N}_n,$$

where $Y_n = (\mathrm{y}_n, \mathrm{y}_{n-1}, \cdots, \mathrm{y}_{n-m+1}, \mathrm{y}_{n-m})^T$,

$$U_1^n = \begin{pmatrix} \frac{(1+(1-\theta)\tilde{a}h_n)q^{-\frac{1}{2m}}}{1-\tilde{a}\theta h_{n+1}} & 0 & \cdots & 0 & \frac{\tilde{b}\theta h_{n+1}q^{-\frac{1}{2}}}{1-\tilde{a}\theta h_{n+1}} & \frac{\tilde{b}(1-\theta)h_n q^{-\frac{m+1}{2m}}}{1-\tilde{a}\theta h_{n+1}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$\widetilde{U_2^n} = \begin{pmatrix} \frac{cq^{-\frac{1}{2m}}}{1-\tilde{a}\theta h_{n+1}} & 0 & \cdots & 0 & \frac{dq^{-\frac{m+1}{2m}}}{1-\tilde{a}\theta h_{n+1}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\widetilde{U_3^n} = \begin{pmatrix} \frac{eq^{-\frac{1}{2m}}}{1-\tilde{a}\theta h_{n+1}} & 0 & \cdots & 0 & \frac{fq^{-\frac{m+1}{2m}}}{1-\tilde{a}\theta h_{n+1}} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence,

$$\boldsymbol{\mathcal{Y}}_{n+1} = U_1^n \boldsymbol{\mathcal{Y}}_n (U_1^n)^T + U_2^n \boldsymbol{\mathcal{Y}}_n (U_2^n)^T + U_3^n \boldsymbol{\mathcal{Y}}_n (U_3^n)^T,$$

with the notations $\boldsymbol{\mathcal{Y}}_{n+1} = E(Y_{n+1}Y_{n+1}^T)$, $h_n^{\frac{1}{2}}\widetilde{U_2^n} = U_2^n$, $(\lambda h_n)^{\frac{1}{2}}\widetilde{U_3^n} = U_3^n$. It has an equivalent form

$$vec(\boldsymbol{\mathcal{Y}}_{n+1}) = B_n vec(\boldsymbol{\mathcal{Y}}_n), \tag{14}$$

where $B_n = U_1^n \otimes U_1^n + U_2^n \otimes U_2^n + U_3^n \otimes U_3^n$. In view of $\lim_{n\to\infty} h_n = \infty$ and $h_{n+1} = q^{-1}h_n$, we have

$$B = U_1 \otimes U_1,$$

where

$$U_1 = \lim_{n\to\infty} U_1^n = \begin{pmatrix} -\frac{(1-\theta)q^{\frac{1}{2m}}}{\theta} & 0 & \cdots & 0 & -\frac{(b+\lambda f)}{(a+\lambda e)\sqrt{q}} & -\frac{(b+\lambda f)(1-\theta)q^{\frac{1}{2m}}}{(a+\lambda e)\theta\sqrt{q}} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$U_2 = \lim_{n\to\infty} U_2^n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$$U_3 = \lim_{n\to\infty} U_3^n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

The limiting equation of (14) is

$$Z_{n+1} = BZ_n. \tag{15}$$

Now, we are in a position to state the following result.

Theorem 4.2. *Under the condition (12), if $\theta \in (\frac{1}{2}, 1]$ and $|b + \lambda f| < -\sqrt{q}(a + \lambda e)$, the split-step θ methods are asymptotically mean-square stable for any mesh H, i.e.*

$$\lim_{n\to\infty} E|y_n|^2 = 0.$$

Proof. For the eigenvalues for B, we only need to consider the eigenvalues of U_1, which is given by the roots of the equations

$$|\lambda I - U_1| = (\lambda + \frac{(1-\theta)q^{\frac{1}{2m}}}{\theta})(\lambda^m + \frac{(b+\lambda f)}{(a+\lambda e)\sqrt{q}}),$$

where I is the identity matrix of order $m + 1$. Namely the eigenvalues are

$$\lambda_m = \frac{(\theta-1)q^{\frac{1}{2m}}}{\theta}, \lambda_k = \sqrt[m]{|\frac{(b+\lambda f)}{(a+\lambda e)\sqrt{q}}|} e^{i\frac{1}{m}(\arg(-\frac{(b+\lambda f)}{(a+\lambda e)\sqrt{q}})+2k\pi)}, k = 0, \cdots m-1.$$

Therefore, by the proprieties of the Kronecker product, the algebraic multiplicity of the eigenvalues for B is 1. Hence the proof is complete. □

Remark 4.3. *It is easy to verify that if $|f| + \sqrt{q}e < 0$ and $e + |f| + \frac{1}{2}(|e| + |f|)^2 < 0$, there exists a λ such that the conditions in Theorem 4.2 are satisfied. Furthermore, when λ is big enough, then the Poisson process becomes the dominant term to stabilize the system, which is a difference from the Wiener process.*

5. Numerical experiments

In this section, some numerical experiments are given to validate our theoretical results. For convenience, we choose $m = 5$ and $T_0 = 1$, and we simulate the mean-square curves of numerical solutions by 4000 sample paths.

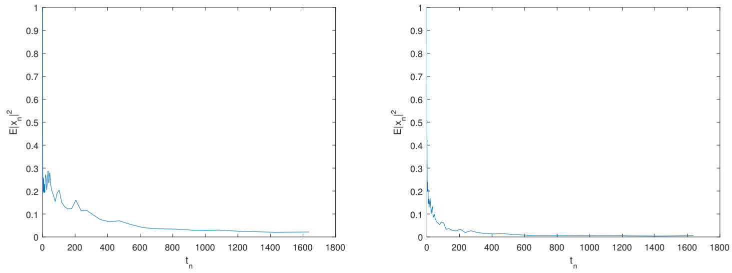
Firstly, we consider the following linear stochastic pantograph differential equation

$$\mathrm{d}x(t) = [ax(t) + bx(qt)]\mathrm{d}t + [cx(t) + dx(qt)]\mathrm{d}W(t), \tag{16}$$

with initial condition $x(0) = 1$.

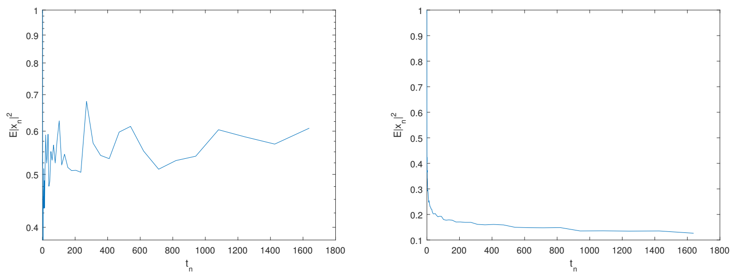
Case 1: we choose $a = -5, b = 1, c = 1, d = 1, q = 0.25$, which satisfies the conditions in Theorem 3.5. In Figure 1, the numerical curves are drawn for $\theta = 0.6, 0.8$. It is seen that the numerical solutions are asymptotically mean-square stable, which is in coincidence with Theorem 3.5. However, the stability condition in [21] is violated for the linear equation, and hence the asymptotical mean-square stability results for linear equations are improved in our paper.

Case 2: we choose $a = -6, b = 3, c = 1, d = 1, q = 0.25$, which satisfy the exact mean-square stability condition (2). In Figure 2, the mean-square curves of linear θ-methods and the split-step θ methods are drawn for $\theta = 0.8$. The numerical curves in Figure 2 illustrate that the asymptotical mean-square stability



(a) $\theta = 0.6$ (b) $\theta = 0.8$

Figure 1: The mean-square curves of numerical solutions to (16) against the parameter θ



(a) split-step θ methods (b) linear θ-methods

Figure 2: The mean-square curves of numerical solutions to (16) for linear θ-methods and split-step θ methods

holds for the linear θ methods, but not for the split-step θ methods. A possible reason is that the conditions in [25] are satisfied rather than the conditions in Theorem 3.5. Therefore, the condition in Theorem 3.5 for the split-step θ methods is nearly essential.

Case 3: to illustrate the perturbation of the white noise, we choose $a = -5, b = 3, q = 0.25, \theta = 0.8$. In Figure 3, the numerical solutions are drawn with and without white perturbations. It is seen that the deterministic stability is broken by white noise (not only the present term but also the delay history term). Therefore, for SPDEs, i.e., at least one of c and d is nontrivial, the mean-square stability condition ($|b| < -\sqrt{q}a$) in Theorem 3.5 is nearly essential under white noises.

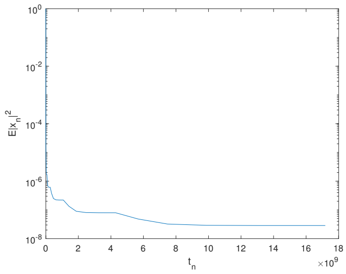
Secondly, we consider the following linear stochastic pantograph differential equation with Poisson jumps

$$\mathrm{d}x(t) = [ax(t) + bx(qt)]\mathrm{d}t + [cx(t) + dx(qt)]\mathrm{d}W(t) + [ex(t) + fx(qt)]\mathrm{d}N(t), \tag{17}$$

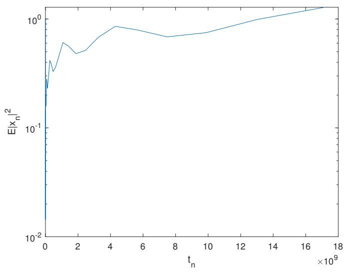
where the initial condition is given by $x(0) = 1$.

Case 1: we choose $a = -15, b = 1, c = 1, d = 1, e = -1, f = 1, q = 0.25, \theta = 0.8, \lambda = 4$. The mean-square curves of numerical solutions under split-step θ methods and compensated split-step θ methods are drawn in Figure 4. It shows that the mean-square stability of compensated split-step θ methods is better than that of split-step θ methods, which is in coincidence with the idea for stochastic differential equation in [10].

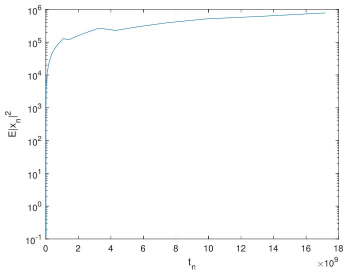
Case 2: compared with the destabilization of white noise, we choose $a = -3, b = 2$, $\lambda = 4$, $q = 0.25$ and $\theta = 0.8$. In Figure 5, the numerical curves are drawn for the deterministic pantograph equation and SPDEs with white noise and Poisson jumps. It is seen that the numerical solutions for deterministic equations are asymptotically stable (see (a) in Figure 5) and the stability is broken by the white noise (see (b) in Figure 5). Meanwhile, the mean-square stability is improved by Poisson jumps (see (c) and (d) in Figure 5). This implies us the the Poisson process $N(t)$ stabilizes possibly the white-noise perturbations in the mean-square sense, which coincides with the conclusion in [26].



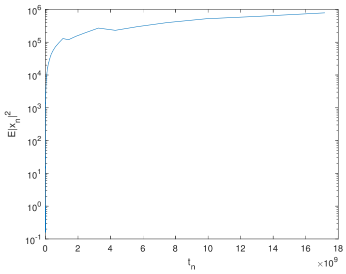
(a) $a = -5, b = 3, c = 0, d = 0$



(b) $a = -5, b = 3, c = 0.1, d = 0$

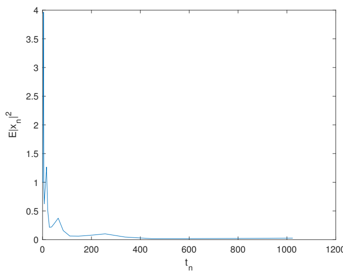


(c) $a = -5, b = 3, c = 0, d = 0.1$

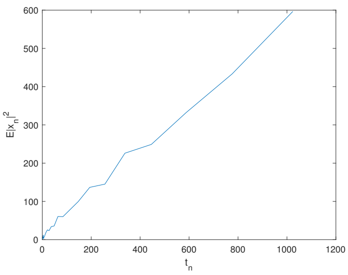


(d) $a = -5, b = 3, c = 0.1, d = 0.1$

Figure 3: The mean-square curves of numerical solutions to (16) with and without white noises

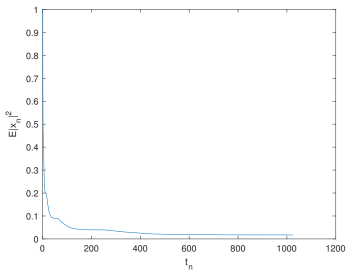


(a) compensated split-step θ method

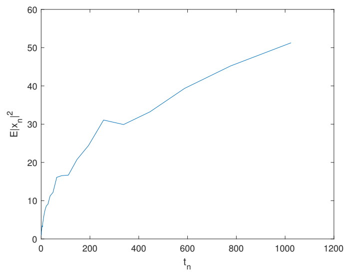


(b) split-step θ methods

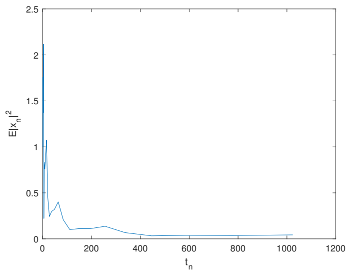
Figure 4: The mean-square curves of numerical solutions to (17) for two kinds of split-step θ methods



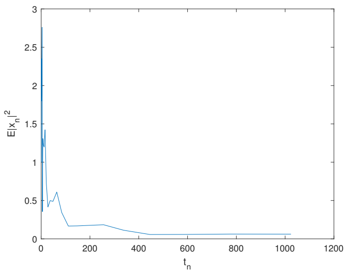
(a) $a = -3, b = 2, c, d = 0, e, f = 0$



(b) $a = -3, b = 2, c, d = 1, e, f = 0$

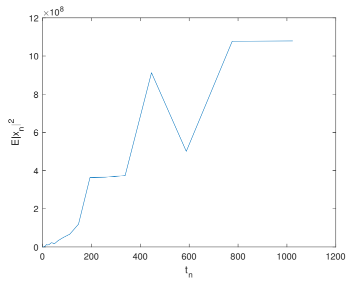


(c) $a = -3, b = 2, c, d = 0, e, f = -1$

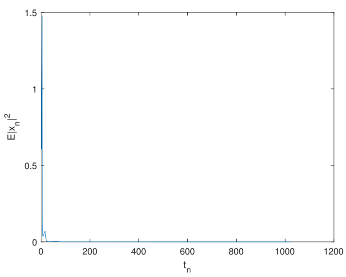


(d) $a = -3, b = 2, c, d = 1, e, f = -1$

Figure 5: The mean-square curves of numerical solutions to (17) with and without stochastic noises



(a) $\lambda = 1$



(b) $\lambda = 100$

Figure 6: The mean-square curves of numerical solutions to (17) against parameter λ

Case 3: for the stabilization, we choose $a = -2, b = 8, c = 1, d = 1, e = -1, f = 0.1, q = 0.25, \theta = 0.8$. It is easy to see that the coefficients do not satisfy the stability condition in Theorem 4.1 with $\lambda = 1$, which is shown in (a) Figure 6 that the numerical solutions are also unstable in the mean-square sense. However, from (b) Figure 6, the numerical solutions are asymptotically mean-square stable for $\lambda = 100$. Compared with the white noise, we are eager to further illustrate the way to stabilize the system by Poisson process. That is, under some conditions, the mean-square stability is obtained by enlarging the intensity of Poisson process.

6. Conclusion

The asymptotical mean-square stability of split-step θ methods under fully-geometric mesh for stochastic pantograph differential equations with Wiener processes and Poisson jumps has been discussed in this paper. From the approach in deterministic pantograph equations, a limiting equation is introduced by

Kronecker products and the stability region is improved compared with the existing results. Moreover, the stabilization of the Poisson process is indicated from the stability condition and the numerical experiments.

The sequel work will be focused on stochastic pantograph differential equations with variable coefficients and nonlinear stochastic pantograph equations.

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