Filomat 35:15 (2021), 5279–5291
https://doi.org/10.2298/FIL2115279F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Tauberian Theorems for Cesàro Summability of Double Integrals over $\mathbb{R}_+^2$

Gökşen Fındık[a], İbrahim Çanak[a]

[a] *Ege University, Department of Mathematics, İzmir, Turkey*

Abstract. In this paper, we obtain one-sided and two-sided Tauberian conditions of Landau and Hardy types for $(C,1,0)$ and $(C,0,1)$ summability methods for improper double integrals under which convergence of improper double integrals follows from $(C,1,0)$ and $(C,0,1)$ summability of improper double integrals. We give similar results for $(C,1,1)$ summability method of improper double integrals. In general, we obtain Tauberian conditions in terms of the concepts of slowly decreasing (resp. oscillating) and strong slowly decreasing (resp. oscillating) functions in different senses for Cesàro summability methods of real or complex-valued locally integrable functions on $[0,\infty)\times[0,\infty)$ in different senses.

1. Introduction

Cesàro summability for double sequences has been investigated by Móricz [3] and Totur [6]. Móricz [3] obtained one-sided and two-sided Tauberian conditions for double sequences. Later, Totur [6] obtained one-sided Tauberian conditions based on the difference between the double sequences and its means in different senses under which convergence of double sequences follow from Cesàro summability of double sequences. Moreover, Totur [6] proved the generalized Littlewood Tauberian theorem for double sequences.

Cesàro summability for double improper integrals was studied by Móricz [4] and he obtained one-sided Tauberian conditions for real-valued functions and two-sided Tauberian conditions for complex-valued functions under which convergence of improper double integral follows from Cesàro summability of improper double integrals. Totur and Çanak [7] introduced (C,α,β) summability method where $\alpha > -1$ and $\beta > -1$ and they proved that (C,α,β) summability implies $(C,\alpha+h,\beta+k)$ summability for all $h > 0$ and $k > 0$. In addition to that, they obtained that (C,α,β) summability method is regular. Using the proving techniques of Laforgia [2], they proved a Tauberian theorem for $(C,1,1)$ summability method. Recently, Fındık and Çanak [1] has introduced the weighted mean method of type (α,β) determined by two weighted functions $p(x)$ and $q(x)$ and obtained analogous results as in [7] for this summability method.

In this paper, we obtain one-sided and two-sided Tauberian conditions of Landau and Hardy types for $(C,1,0)$ and $(C,0,1)$ summability methods for improper double integrals under which convergence of improper double integrals follows from $(C,1,0)$ and $(C,0,1)$ summability of improper double integrals.

2020 *Mathematics Subject Classification*. Primary 40B05; Secondary 40E05
Keywords. one-sided and two-sided Tauberian conditions, improper double integrals, Cesàro summability $(C,1,1)$, $(C,1,0)$ and $(C,0,1)$, convergence in Pringsheim's sense, slow decrease and strong slow decrease in different senses, slow oscillation and strong slow oscillation in different senses
Received: 01 November 2020; Revised: 01 March 2021; Accepted: 01 March 2021
Communicated by Eberhard Malkowsky
Email addresses: findikgoksen@gmail.com (Gökşen Fındık), ibrahim.canak@ege.edu.tr (İbrahim Çanak)

We give similar results for $(C,1,1)$ summability method of improper double integrals. In general, we obtain Tauberian conditions in terms of the concepts of slowly decreasing (resp. oscillating) and strong slowly decreasing (resp. oscillating) functions in different senses for Cesàro summability methods of real or complex-valued locally integrable functions on $[0,\infty)\times[0,\infty)$ in different senses.

2. Preliminaries

Suppose that f is a real or complex-valued locally integrable function on $\mathbb{R}^2_+ := [0,\infty)\times[0,\infty)$ and $s(u,v) = \int_0^u\int_0^v f(x,y)dxdy$ for $0 < u, v < \infty$. The mean $(C,1,1)$ (or Cesàro mean in sense $(1,1)$) of $s(u,v)$ is defined by

$$\begin{aligned}\sigma(s(u,v)) &= \sigma_{11}(s(u,v)) = \frac{1}{uv}\int_0^u\int_0^v s(x,y)dxdy \\ &= \int_0^u\int_0^v\Big(1-\frac{x}{u}\Big)\Big(1-\frac{y}{v}\Big)f(x,y)dxdy \qquad (1)\end{aligned}$$

for $u, v > 0$. The integral

$$\int_0^\infty\int_0^\infty f(x,y)dxdy \qquad (2)$$

is said to be $(C,1,1)$ summable (or Cesàro summable in sense $(1,1)$) to a finite number L if

$$\lim_{u,v\to\infty}\sigma(s(u,v)) = L. \qquad (3)$$

Throughout this work, convergence is always used in Pringsheim's sense for convergence of improper double integral [5]. Namely, both u and v tend to ∞ independently of each other in (3).

The mean $(C,1,0)$ (or Cesàro mean in sense $(1,0)$) of $s(u,v)$ is defined by

$$\sigma_{10}(s(u,v)) = \frac{1}{u}\int_0^u s(x,v)dx = \int_0^u\int_0^v\Big(1-\frac{x}{u}\Big)f(x,y)dxdy \qquad (4)$$

for $u, v > 0$. The integral (2) is said to be $(C,1,0)$ summable (or Cesàro summable in sense $(1,0)$) to a finite number L if

$$\lim_{u,v\to\infty}\sigma_{10}(s(u,v)) = L.$$

Similarly, the mean $(C,0,1)$ (or Cesàro mean in sense $(0,1)$) of $s(u,v)$ is defined by

$$\sigma_{01}(s(u,v)) = \frac{1}{v}\int_0^v s(u,y)dy = \int_0^u\int_0^v\Big(1-\frac{y}{v}\Big)f(x,y)dxdy \qquad (5)$$

for $u, v > 0$. The integral (2) is said to be $(C,0,1)$ summable (or Cesàro summable in sense $(0,1)$) to a finite number L if

$$\lim_{u,v\to\infty}\sigma_{01}(s(u,v)) = L.$$

A function $s(u,v)$ is bounded if there exists a real number $H > 0$ such that $|s(u,v)| \le H$ for all $u, v > 0$. In this case, we write $s(u,v) = O(1)$. Moreover, a real-valued function $s(u,v)$ is said to be one-sided bounded if there exists a real number $H > 0$ such that $s(u,v) \ge -H$ for all $u, v > 0$.

It is clear from the definition of $s(u,v)$ that

$$s_u(u,v) = \frac{\partial s(u,v)}{\partial u} = \int_0^v f(u,y)dy$$

$$s_v(u,v) = \frac{\partial s(u,v)}{\partial v} = \int_0^u f(x,v)dx$$

$$s_{uv}(u,v) = \frac{\partial^2 s(u,v)}{\partial u\partial v} = f(u,v)$$

for $u, v > 0$.

The Kronecker identities for double integrals take the following forms. For $u, v > 0$, we have

$$s(u,v) - \sigma_{10}(s(u,v)) - \sigma_{01}(s(u,v)) + \sigma_{11}(s(u,v)) = V_{11}(s_{uv}(u,v)) \tag{6}$$

where $V_{11}(s_{uv}(u,v)) = \frac{1}{uv}\int_0^u\int_0^v xyf(x,y)dxdy$. We note that $V_{11}(s_{uv}(u,v))$ is the $(C,1,1)$ mean of $uvs_{uv}(u,v)$. Moreover, in analogy to the Kronecker identity for double sequences, we have

$$s(u,v) - \sigma_{10}(s(u,v)) = V_{10}(s_u(u,v)) \tag{7}$$

where $V_{10}(s_u(u,v)) = \frac{1}{u}\int_0^u\int_0^v xf(x,y)dxdy$ and

$$s(u,v) - \sigma_{01}(s(u,v)) = V_{01}(s_v(u,v))$$

where $V_{01}(s_v(u,v)) = \frac{1}{v}\int_0^u\int_0^v yf(x,y)dxdy$. We note that $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are the $(C,1,0)$ mean of $us_u(u,v)$ and the $(C,0,1)$ mean of $vs_v(u,v)$, respectively.

The generalized de la Vallée Poussin mean of a real or complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ are given as follows:

$$\tau^{>}(s(u,v),\lambda) = \frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v} s(x,y)dxdy$$

for $\lambda > 1$ and

$$\tau^{<}(s(u,v),\lambda) = \frac{1}{(u - \lambda u)(v - \lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v} s(x,y)dxdy$$

for $0 < \lambda < 1$.

Assume that $s(u,v)$ is bounded on $\mathbb{R}^2_+$. If the limit

$$\lim_{u,v\to\infty} s(u,v) = L \tag{8}$$

exists, then the limit (3) also exists. The converse of this implication is not true in general, even if $s(u,v)$ is bounded on $\mathbb{R}^2_+$. We may get the converse implication if we add some suitable condition(s) imposed on $s(u,v)$, which is called a Tauberian condition. Any theorem which states that convergence of the improper double integral follows from its Cesàro summability in sense $(1,1)$ and some Tauberian condition is said to be a Tauberian theorem for Cesàro summability in sense $(1,1)$. Similar situations are valid for the $(C,1,0)$ and $(C,0,1)$ summability methods.

3. Auxiliary Results

We need the following auxiliary results for the proofs of our main results.

Lemma 3.1. *([4]) If (2) is $(C,1,1)$ summable to a finite number L, then*

$$\lim_{u,v\to\infty} \tau^{>}(s(u,v),\lambda) = L$$

and

$$\lim_{u,v\to\infty} \tau^{<}(s(u,v),\lambda) = L.$$

Lemma 3.2. *Let $s(u,v)$ be a double integral over the rectangle $[0,u]\times[0,v]$. For sufficiently large u and v,*
(i) *If $\lambda > 1$,*

$$s(u,v)-\sigma_{11}(s(u,v)) = \left(\frac{\lambda}{\lambda-1}\right)^2(\sigma_{11}(s(\lambda u,\lambda v))-\sigma_{11}(s(u,v))) + \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(\lambda u,v))) + \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(u,\lambda v))) - \frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}(s(x,y)-s(u,v))\,dxdy.$$

(ii) *If $0<\lambda<1$,*

$$s(u,v)-\sigma_{11}(s(u,v)) = \left(\frac{\lambda}{1-\lambda}\right)^2(\sigma_{11}(s(\lambda u,\lambda v))-\sigma_{11}(s(u,v))) + \frac{\lambda}{(1-\lambda)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(\lambda u,v))) + \frac{\lambda}{(1-\lambda)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(u,\lambda v))) + \frac{1}{(u-\lambda u)(v-\lambda v)}\int_{\lambda u}^{u}\int_{\lambda v}^{v}(s(u,v)-s(x,y))\,dxdy.$$

Proof. For $\lambda>1$, we have

$$s(u,v)-\sigma_{11}(s(u,v)) = \tau^{>}(s(u,v),\lambda)-\sigma_{11}(s(u,v))-(\tau^{>}(s(u,v),\lambda)-s(u,v)). \tag{9}$$

From the expression of the generalized de la Vallée Poussin mean of $s(u,v)$, we have

$$\begin{aligned}
\tau^{>}(s(u,v),\lambda) &= \frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}s(x,y)dxdy\\
&= \frac{1}{(\lambda-1)^2uv}\left(\int_0^{\lambda u}-\int_0^{u}\right)\left(\int_0^{\lambda v}-\int_0^{v}\right)s(x,y)dxdy\\
&= \left(\frac{\lambda}{\lambda-1}\right)^2\sigma_{11}(s(\lambda u,\lambda v))-\frac{\lambda}{(\lambda-1)^2}\sigma_{11}(s(\lambda u,v))\\
&- \frac{\lambda}{(\lambda-1)^2}\sigma_{11}(s(u,\lambda v))+\frac{1}{(\lambda-1)^2}\sigma_{11}(s(u,v))\\
&= \left(\frac{\lambda}{\lambda-1}\right)^2(\sigma_{11}(s(\lambda u,\lambda v))-\sigma_{11}(s(u,v)))\\
&+ \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(\lambda u,v)))\\
&+ \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(u,\lambda v)))\\
&+ \sigma_{11}(s(u,v)).
\end{aligned}$$

From the identity above, we get

$$\begin{aligned}
\tau^{>}(s(u,v),\lambda)-\sigma_{11}(s(u,v)) &= \left(\frac{\lambda}{\lambda-1}\right)^2(\sigma_{11}(s(\lambda u,\lambda v))-\sigma_{11}(s(u,v)))\\
&+ \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(\lambda u,v)))\\
&+ \frac{\lambda}{(\lambda-1)^2}(\sigma_{11}(s(u,v))-\sigma_{11}(s(u,\lambda v))).
\end{aligned} \tag{10}$$

We obtain the identity (i) from (10).

The identity (ii) can be shown in a similar way. □

The following Lemma represents some relations between $(C,1,0)$, $(C,0,1)$ and $(C,1,1)$ means of $s(u,v)$.

Lemma 3.3. *Let $s(u,v)$ be a double integral over the rectangle $[0,u]\times[0,v]$. Then we have*

$$\sigma_{10}(\sigma_{01}(s(u,v))) = \sigma_{01}(\sigma_{10}(s(u,v))) = \sigma_{11}(s(u,v)) \tag{11}$$

$$\sigma_{10}(\sigma_{11}(s(u,v))) = \sigma_{11}(\sigma_{10}(s(u,v))) \tag{12}$$

$$\sigma_{01}(\sigma_{11}(s(u,v))) = \sigma_{11}(\sigma_{01}(s(u,v))) \tag{13}$$

for $u,v>0$.

Proof. First, we prove the identity (11). By (4) and (5), we have

$$\begin{aligned}\sigma_{10}(\sigma_{01}(s(u,v))) &= \int_0^u\int_0^v \left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)f(x,y)dxdy\\ &= \int_0^u\int_0^v \left(1-\frac{y}{v}\right)\left(1-\frac{x}{u}\right)f(x,y)dxdy\\ &= \sigma_{01}(\sigma_{10}(s(u,v))).\end{aligned}$$

By (1), we get

$$\begin{aligned}\sigma_{10}(\sigma_{01}(s(u,v))) &= \int_0^u\int_0^v \left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)f(x,y)dxdy\\ &= \sigma_{11}(s(u,v)).\end{aligned}$$

Now, we prove the identity (12). By (1) and (4), we have

$$\begin{aligned}\sigma_{10}(\sigma_{11}(s(u,v))) &= \int_0^u\int_0^v \left(1-\frac{x}{u}\right)\left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)f(x,y)dxdy\\ &= \int_0^u\int_0^v \left(1-\frac{x}{u}\right)\left(1-\frac{y}{v}\right)\left(1-\frac{x}{u}\right)f(x,y)dxdy\\ &= \sigma_{11}(\sigma_{10}(s(u,v))).\end{aligned}$$

The identity (13) can be shown similarly. □

4. Tauberian theorems for Cesàro summability methods of real-valued continuous functions on $\mathbb{R}_+^2$

We need the following definitions for the real-valued functions defined on $\mathbb{R}_+^2$:

A real-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be slowly decreasing in sense $(1,0)$ [4] if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{u\le x\le \lambda u}[s(x,v)-s(u,v)]\ge 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\lambda u\le x\le u}[s(u,v)-s(x,v)]\ge 0.$$

Analogously, a real-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be slowly decreasing in sense $(0,1)$ [4] if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{v\le y\le \lambda v}[s(u,y)-s(u,v)]\ge 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\lambda v\le y\le v}[s(u,v)-s(u,y)]\ge 0.$$

A real-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be strong slowly decreasing in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{\substack{u\le x\le \lambda u\\ v\le y\le \lambda v}}[s(x,y)-s(u,y)]\ge 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}[s(u,y)-s(x,y)]\ge 0.$$

Analogously, a real-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be strong slowly decreasing in sense $(0,1)$ if

$$\lim_{\lambda\to 1^+}\liminf_{u,v\to\infty}\min_{\substack{u\le x\le \lambda u\\ v\le y\le \lambda v}}[s(x,y)-s(x,v)]\ge 0$$

or equivalently

$$\lim_{\lambda\to 1^-}\liminf_{u,v\to\infty}\min_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}[s(x,v)-s(x,y)]\ge 0.$$

In the following theorem, we give one-sided Tauberian conditions of Landau type for improper double integrals under which convergence follows from $(C,1,0)$ and $(C,0,1)$ summability of (2).

Theorem 4.1. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,0)$ and $(C,0,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$uV_{11_u}(s_{uv}(u,v))\ge -H \tag{14}$$

and

$$vV_{11_v}(s_{uv}(u,v))\ge -H \tag{15}$$

are satisfied for all $(u,v)\in\mathbb{R}_+^2$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

The following two theorems are Tauberian theorems of Landau type for $(C,1,1)$ summability of improper double integrals.

Theorem 4.2. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$us_u(u,v)\ge -H \tag{16}$$

and

$$vs_v(u,v)\ge -H \tag{17}$$

are satisfied for all $(u,v)\in\mathbb{R}_+^2$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

Theorem 4.3. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$uV_{11_u}(s_{uv}(u,v))\ge -H,\quad vV_{11_v}(s_{uv}(u,v))\ge -H, \tag{18}$$

$$uV_{10_u}(s_u(u,v))\ge -H,\quad vV_{10_v}(s_u(u,v))\ge -H, \tag{19}$$

$$uV_{01_u}(s_v(u,v))\ge -H,\quad vV_{01_v}(s_v(u,v))\ge -H \tag{20}$$

are satisfied for all $(u,v)\in\mathbb{R}_+^2$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

In the next three theorems, Tauberian conditions are given in terms of slow decreasing and strong slow decreasing in different senses for Cesàro summability methods in different senses.

Theorem 4.4. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,0)$ and $(C,0,1)$ summable to a finite number L and $V_{11}(s_{uv}(u,v))$ slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$ or slowly decreasing in sense $(1,0)$ and strong slowly decreasing in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

Theorem 4.5. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and $s(u,v)$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$ or slowly decreasing in sense $(1,0)$ and strong slowly decreasing in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

Theorem 4.6. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$ or slowly decreasing in sense $(1,0)$ and strong slowly decreasing in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

5. Proofs

Proof of Theorem 4.1 Suppose that a bounded double integral $s(u,v)$ is $(C,1,0)$ and $(C,0,1)$ summable to L and conditions (14) and (15) hold. It can be easily verified that $(C,1,0)$ and $(C,0,1)$ summability of (2) implies $(C,1,1)$ summability of (2) by Lemma 3.3. Since $(C,1,1)$ summable method is regular and $\lim_{u,v\to\infty}\sigma_{11}(s(u,v)) = L$, $\sigma_{11}(s(u,v))$ is $(C,1,1)$ summable to L. Analogously, $\sigma_{10}(s(u,v))$ and $\sigma_{01}(s(u,v))$ are $(C,1,1)$ summable to L. Hence it follows from Kronecker identity (6) that $V_{11}(s_{uv}(u,v))$ is $(C,1,1)$ summable to 0. For $\lambda > 1$, if we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$ in (9), we have

$$V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right) = \left(\tau^{>}\left(V_{11}(s_{uv}(u,v)),\lambda\right) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) - \frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy. \quad (21)$$

Taking the lim sup of both sides of the previous equation as $u,v\to\infty$, we get

$$\limsup_{u,v\to\infty}\left(V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) \le \limsup_{u,v\to\infty}\left(\tau^{>}\left(V_{11}(s_{uv}(u,v))\right),\lambda) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) + \limsup_{u,v\to\infty}\left(-\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy\right). \quad (22)$$

The first term on the right-hand side of the previous inequality is vanished by Lemma 3.1 and we have

$$\limsup_{u,v\to\infty}\left(V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) \le \limsup_{u,v\to\infty}\left(-\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy\right). \quad (23)$$

In addition, we obtain by (14) and (15) that

$$\begin{aligned} V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v)) &= \int_u^x V_{11_r}(s_{ry}(r,y))dr + \int_v^y V_{11_t}(s_{ut}(u,t))dt && (24)\\ &\ge -H\left(\int_u^x \frac{dr}{r} + \int_v^y \frac{dt}{t}\right) && (25)\\ &= -H\left(\ln\left(\frac{x}{u}\right) + \ln\left(\frac{y}{v}\right)\right) && (26)\end{aligned}$$

for some $H > 0$. From (23) and (24), we have

$$\limsup_{u,v\to\infty} (V_{11}(s_{uv}(u,v)) - \sigma_{11}(V_{11}(s_{uv}(u,v)))) \leq H \limsup_{u,v\to\infty} \left(\ln\left(\frac{\lambda u}{u}\right) + \ln\left(\frac{\lambda v}{v}\right)\right).$$

Taking the limit of both sides of the previous inequality as $\lambda \to 1^+$, we get

$$\limsup_{u,v\to\infty} (V_{11}(s_{uv}(u,v)) - \sigma_{11}(V_{11}(s_{uv}(u,v)))) \leq \lim_{\lambda\to 1^+} 2H\ln\lambda$$

and we obtain

$$\limsup_{u,v\to\infty} (V_{11}(s_{uv}(u,v)) - \sigma_{11}(V_{11}(s_{uv}(u,v)))) \leq 0. \tag{27}$$

For $0 < \lambda < 1$, in a similar way from Lemma 3.2 (ii) we have

$$\liminf_{u,v\to\infty} (V_{11}(s_{uv}(u,v)) - \sigma_{11}(V_{11}(s_{uv}(u,v)))) \geq 0. \tag{28}$$

By (27) and (28), we obtain $\lim_{u,v\to\infty} V_{11}(s_{uv}(u,v)) = 0$. Thus, $s(u,v)$ is convergent to L by Kronecker identity (6). □

Proof of Theorem 4.2 Assume that bounded double integral $s(u,v)$ is $(C,1,1)$ summable to L and conditions (16) and (17) hold. By $(C,1,1)$ summability of $s(u,v)$, we write $\lim\limits_{u,v\to\infty} \sigma(s(u,v)) = L$.

For $\lambda > 1$, using the identity (9) we have

$$s(u,v) - \sigma_{11}(s(u,v)) = \tau^{>}(s(u,v),\lambda) - \sigma_{11}(s(u,v)) - \frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} (s(x,y) - s(u,v))dxdy.$$

Taking the lim sup of both sides of the previous equation as $u,v \to \infty$, we get

$$\begin{aligned}\limsup_{u,v\to\infty} (s(u,v) - \sigma_{11}(s(u,v))) &\leq \limsup_{u,v\to\infty} (\tau^{>}(s(u,v),\lambda) - \sigma_{11}(s(u,v))) \\ &\quad + \limsup_{u,v\to\infty} \left(-\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} (s(x,y) - s(u,v))dxdy\right).\end{aligned}$$

The first term on the right-hand side of the previous inequality vanishes by Lemma 3.1, and we have

$$\limsup_{u,v\to\infty} (s(u,v) - \sigma_{11}(s(u,v))) \leq \limsup_{u,v\to\infty} \left(-\frac{1}{(\lambda u - u)(\lambda v - v)} \int_u^{\lambda u} \int_v^{\lambda v} (s(x,y) - s(u,v))dxdy\right). \tag{29}$$

In addition, we obtain by (16) and (17) that

$$\begin{aligned} s(x,y) - s(u,v) &= s(x,y) - s(u,y) + s(u,y) - s(u,v) \\ &= \int_u^x \int_0^y f(r,t)dtdr + \int_0^u \int_v^y f(r,t)dtdr \\ &\geq -H\left(\int_u^x \frac{dr}{r} + \int_v^y \frac{dt}{t}\right) \\ &\geq -2H\ln\lambda \end{aligned} \tag{30}$$

for some $H > 0$. Taking (30) into consideration, we obtain from (29)

$$\limsup_{u,v\to\infty} (s(u,v) - \sigma_{11}(s(u,v))) \leq 2H\ln\lambda.$$

Hence taking the limit of both sides of the last inequality as $\lambda \to 1^+$, we have

$$\limsup_{u,v\to\infty} (s(u,v) - \sigma_{11}(s(u,v))) \leq 0. \tag{31}$$

For $0 < \lambda < 1$, we have

$$\liminf_{u,v\to\infty} (s(u,v) - \sigma_{11}(s(u,v))) \geq 0. \tag{32}$$

By the inequalities (31) and (32), we obtain $s(u,v)$ is convergent to L. □

Proof of Theorem 4.3 Assume that bounded double integral $s(u,v)$ is $(C,1,1)$ summable to L and conditions (18)-(20) hold. By $(C,1,1)$ summability of $s(u,v)$, we have $\lim_{u,v\to\infty} \sigma(s(u,v)) = L$.

Taking (C,1,1) means of the Kronecker equality (7), we get

$$\sigma_{11}(s(u,v)) - \sigma_{10}(\sigma_{11}(s(u,v))) = \sigma_{11}(V_{10}(s_u(u,v))) \tag{33}$$

by Lemma 3.3. Since $(C,1,0)$ summability method is regular under the boundedness condition, we obtain that $V_{10}(s_u(u,v))$ is $(C,1,1)$ summable to 0 by taking (33) into consideration. Similarly, it can be easily seen that $V_{01}(s_v(u,v))$ is $(C,1,1)$ summable to 0.

From Kronecker equality (6), we get

$$(s(u,v) - \sigma_{10}(s(u,v))) + (s(u,v) - \sigma_{01}(s(u,v))) - (s(u,v) - \sigma_{11}(s(u,v))) = V_{11}(s_{uv}(u,v)).$$

From the previous identity and Kronecker identity, we have

$$s(u,v) - \sigma_{11}(s(u,v)) = V_{10}(s_u(u,v)) + V_{01}(s_v(u,v)) - V_{11}(s_{uv}(u,v)). \tag{34}$$

Since $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are $(C,1,1)$ summable to 0, taking $(C,1,1)$ means of the previous equality we obtain that $V_{11}(s_{uv}(u,v))$ is $(C,1,1)$ summable to 0. If we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ in Theorem 4.2 respectively, we obtain that $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are convergent to 0. Hence taking the limit of both sides of (34) as $u,v\to\infty$ we obtain that $s(u,v)$ is convergent to L.

Proof of Theorem 4.4 In the same way as in the proof of Theorem 4.1, we can show that $V_{11}(s_{uv}(u,v))$ is $(C,1,1)$ summable to 0.

For $\lambda > 1$, if we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$ in (9), we have . Taking the lim sup of both sides of (21) as $u,v\to\infty$, we get (22). The first term on the right-hand side of (22) vanishes by Lemma 3.1 and we have

$$\limsup_{u,v\to\infty} \left(V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) \leq -\liminf_{u,v\to\infty}\left(\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy\right). \tag{35}$$

Moreover, we have

$$\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy \geq \min_{\substack{u\leq x\leq\lambda u\\ v\leq y\leq\lambda v}}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big). \tag{36}$$

Taking the lim inf of both sides of (36) as $u,v\to\infty$, we get

$$\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)dxdy$$
$$\geq \liminf_{u,v\to\infty}\min_{\substack{u\leq x\leq\lambda u\\ v\leq y\leq\lambda v}}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uy}(u,y))\Big) + \liminf_{u,v\to\infty}\min_{v\leq y\leq\lambda v}\Big(V_{11}(s_{uy}(u,y)) - V_{11}(s_{uv}(u,v))\Big).$$

Since $V_{11}(s_{uv}(u,v))$ is slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$, we get

$$\liminf_{u,v\to\infty}\frac{1}{(\lambda u-u)(\lambda v-v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y))-V_{11}(s_{uv}(u,v))\Big)dxdy\geq 0 \tag{37}$$

by taking the limit of both sides of the last inequality as $\lambda\to 1^+$. Hence from (35) and (37), we obtain

$$\limsup_{u,v\to\infty}\left(V_{11}(s_{uv}(u,v))-\sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right)\leq 0. \tag{38}$$

For $0<\lambda<1$, in a similar way from Lemma 3.2 (ii) we have

$$\liminf_{u,v\to\infty}\left(V_{11}(s_{uv}(u,v))-\sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right)\geq 0. \tag{39}$$

By (38) and (39), we obtain $\lim_{u,v\to\infty}V_{11}(s_{uv}(u,v))=0$. Thus, $s(u,v)$ is convergent to L by the Kronecker identity (6). □

Proof of Theorem 4.5 Assume that bounded double integral $s(u,v)$ is $(C,1,1)$ summable to L and slowly decreasing in sense $(0,1)$ and strong slowly decreasing in sense $(1,0)$. If we apply a similar calculation for $V_{11}(s_{uv}(u,v))$ as in the proof of Theorem 4.4 to $s(u,v)$, we obtain that $s(u,v)$ is convergent to L. □

Proof of Theorem 4.6 In the same way as in the proof of Theorem 4.3, we can show that $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are $(C,1,1)$ summable to 0. If we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ in Theorem 4.5 respectively, we obtain that $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are convergent to 0. Hence taking the limit of both sides of (34) in Theorem 4.3 as $u,v\to\infty$, we obtain that $s(u,v)$ is convergent to L. □

6. Tauberian theorems for Cesàro summability methods for complex-valued functions on $\mathbb{R}_+^2$

We need the following definitions for the complex-valued functions defined on $\mathbb{R}_+^2$:

A complex-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be slowly oscillating in sense $(1,0)$ [4] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{u\leq x\leq\lambda u}|s(x,v)-s(u,v)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\lambda u\leq x\leq u}|s(u,v)-s(x,v)|=0.$$

Analogously, a complex-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be slowly oscillating in sense $(0,1)$ [4] if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{v\leq y\leq\lambda v}\left|s(u,y)-s(u,v)\right|=0$$

or equivalently

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\lambda v\leq y\leq v}\left|s(u,v)-s(u,y)\right|=0.$$

A complex-valued function $s(u,v)$ defined on $\mathbb{R}_+^2$ is said to be strong slowly oscillating in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\substack{u\leq x\leq\lambda u\\ v\leq y\leq\lambda v}}|s(x,y)-s(u,y)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\substack{\lambda u\leq x\leq u\\ \lambda v\leq y\leq v}}|s(u,y)-s(x,y)|=0.$$

Analogously, a complex-valued function $s(u,v)$ defined on $\mathbb{R}^2_+$ is said to be strong slowly oscillating in sense $(0,1)$ if

$$\lim_{\lambda\to 1^+}\limsup_{u,v\to\infty}\max_{\substack{u\le x\le\lambda u\\ v\le y\le \lambda v}}|s(x,y)-s(x,v)|=0$$

or equivalently

$$\lim_{\lambda\to 1^-}\limsup_{u,v\to\infty}\max_{\substack{\lambda u\le x\le u\\ \lambda v\le y\le v}}|s(x,v)-s(x,y)|=0.$$

As a corollary of Theorem 4.1, we give two-sided Tauberian conditions of Hardy type for improper double integrals under which convergence follows from $(C,1,0)$ and $(C,0,1)$ summability of double integrals.

Theorem 6.1. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,0)$ and $(C,0,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$uV_{11_u}(s_{uv}(u,v))=O(1) \tag{40}$$

and

$$vV_{11_v}(s_{uv}(u,v))=O(1) \tag{41}$$

are satisfied for all $(u,v)\in\mathbb{R}^2_+$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

As a corollary of Theorem 4.2, we give Hardy type Tauberian theorem for $(C,1,1)$ summability of improper double integrals.

Theorem 6.2. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$us_u(u,v)=O(1) \tag{42}$$

and

$$vs_v(u,v)=O(1) \tag{43}$$

are satisfied for all $(u,v)\in\mathbb{R}^2_+$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

As a corollary of Theorem 4.3, we give Hardy type Tauberian theorem for $(C,1,1)$ summability of (2).

Theorem 6.3. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and there exist constants $H>0$ and $x_0\ge 0$ such that conditions*

$$uV_{11_u}(s_{uv}(u,v))=O(1),\quad vV_{11_v}(s_{uv}(u,v))=O(1), \tag{44}$$

$$uV_{10_u}(s_u(u,v))=O(1),\quad vV_{10_v}(s_u(u,v))=O(1), \tag{45}$$

$$uV_{01_u}(s_v(u,v))=O(1),\quad vV_{01_v}(s_v(u,v))=O(1) \tag{46}$$

are satisfied for all $(u,v)\in\mathbb{R}^2_+$ with $u,v>x_0$, then $s(u,v)$ is convergent to L.

In the next three theorems, Tauberian conditions are given in terms of slow oscillating and strong slow oscillating in different senses for Cesàro summability methods in different senses.

Theorem 6.4. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,0)$ and $(C,0,1)$ summable to a finite number L and $V_{11}(s_{uv}(u,v))$ slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$ or slowly oscillating in sense $(1,0)$ and strong slowly oscillating in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

Theorem 6.5. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and $s(u,v)$ is slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$ or slowly oscillating in sense $(1,0)$ and strong slowly oscillating in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

Theorem 6.6. *Let the double integral $s(u,v)$ be bounded. If (2) is $(C,1,1)$ summable to a finite number L and $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$ or slowly oscillating in sense $(1,0)$ and strong slowly oscillating in sense $(0,1)$, then $s(u,v)$ is convergent to L.*

7. Proofs

Proof of Theorem 6.1 It is clear that conditions (40) and (41) imply (14) and (15) in Theorem 4.1, respectively.

Proof of Theorem 6.2 It is clear that conditions (42) and (43) imply (16) and (17) in Theorem 4.2, respectively.

Proof of Theorem 6.3 It is clear that conditions (44), (45) and (46) imply (18), (19) and (20) in Theorem 4.3, respectively.

Proof of Theorem 6.4 In the same way as in the proof of Theorem 4.1, we can show that $V_{11}(s_{uv}(u,v))$ is $(C,1,1)$ summable to 0.

For $\lambda > 1$, if we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$ in (9), we have

$$V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right) = \left(\tau^{>}\left(V_{11}(s_{uv}(u,v)),\lambda\right) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right) - \frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy.$$

From above equality, we get

$$\left|V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right| \le \left|\tau^{>}\left(V_{11}(s_{uv}(u,v)),\lambda\right) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right| + \left|-\frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy\right|.$$

Taking the lim sup of both sides of the previous equation as $u,v \to \infty$, we get

$$\limsup_{u,v\to\infty}\left|V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right| \le \limsup_{u,v\to\infty}\left|\tau^{>}\left(V_{11}(s_{uv}(u,v)),\lambda\right) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right| + \limsup_{u,v\to\infty}\left|\frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy\right|. \quad (47)$$

The first term on the right-hand side of (47) is vanished by Lemma 3.1. Moreover,

$$\left|\frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy\right| \le \max_{\substack{u\le x\le \lambda u\\ v\le y\le \lambda v}}\left|V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\right| \quad (48)$$

Taking lim sup of both sides of (48) as $u,v \to \infty$, we have

$$\limsup_{u,v\to\infty}\left|\frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy\right| \le \limsup_{u,v\to\infty}\max_{\substack{u\le x\le \lambda u\\ v\le y\le \lambda v}}\left|V_{11}(s_{xy}(x,y)) - V_{11}(s_{uy}(u,y))\right| + \limsup_{u,v\to\infty}\max_{v\le y\le \lambda v}\left|V_{11}(s_{uy}(u,y)) - V_{11}(s_{uv}(u,v))\right|.$$

Since $V_{11}(s_{uv}(u,v))$ is slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$, we get

$$\limsup_{u,v\to\infty}\left|\frac{1}{(\lambda u - u)(\lambda v - v)}\int_u^{\lambda u}\int_v^{\lambda v}\Big(V_{11}(s_{xy}(x,y)) - V_{11}(s_{uv}(u,v))\Big)\,dxdy\right| \le 0 \quad (49)$$

by taking the limit of both sides of the last inequality as $\lambda \to 1^{+}$. From (47) and (49), we obtain

$$\limsup_{u,v\to\infty}\left|V_{11}(s_{uv}(u,v)) - \sigma_{11}\left(V_{11}(s_{uv}(u,v))\right)\right| \le 0.$$

Hence $\lim_{u,v\to\infty} V_{11}(s_{uv}(u,v)) = \sigma_{11}(V_{11}(s_{uv}(u,v))) = 0$ and the proof is completed by the Kronecker identity (6). □

Proof of Theorem 6.5 Assume that bounded double integral $s(u,v)$ is $(C,1,1)$ summable to L and slowly oscillating in sense $(0,1)$ and strong slowly oscillating in sense $(1,0)$. If we apply a similar calculation for $V_{11}(s_{uv}(u,v))$ as in the proof of Theorem 6.4 to $s(u,v)$, we obtain that $s(u,v)$ is convergent to L. □

Proof of Theorem 6.6 In the same way as in the proof of Theorem 4.3, we can show that the integral $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are $(C,1,1)$ summable to 0. If we replace $s(u,v)$ by $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ in Theorem 6.5 respectively, we obtain that $V_{11}(s_{uv}(u,v))$, $V_{10}(s_u(u,v))$ and $V_{01}(s_v(u,v))$ are convergent to 0. Hence taking the limit of both sides of (34) in Theorem 4.3 as $u,v\to\infty$ we obtain that $s(u,v)$ is convergent to L. □

Acknowledgment

The authors would like to thank the anonymous referee for his/her valuable remarks and suggestions to improve the quality and readability of the paper.

References

[1] G. Fındık, İ. Çanak, Some Tauberian theorems for weighted means of double integrals, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. 68 (2019), 1452–1461.
[2] A. Laforgia, A theory of divergent integrals, Appl. Math. Lett. 22 (2009), 834–840.
[3] F. Móricz, Tauberian theorems for Cesàro summable double sequences, Studia Math. 110 (1994), 83–96.
[4] F. Móricz, Tauberian theorems for Cesàro summable double integrals over $\mathbb{R}^2_+$, Stud. Math. 138 (2000), 41–52.
[5] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (1900), 289–321.
[6] Ü. Totur, Classical Tauberian theorems for the $(C,1,1)$ summability method, An. Ştiint. Univ. Al. I. Cuza Iaşi. Mat. (N.S.). 61 (2015), 401–414.
[7] Ü. Totur, İ.Çanak, On the Cesàro summability for functions of two variables, Miskolc Math. Notes. 19 (2018), 1203–1215.