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The Spectral Norm and Spread of g-Circulant Matrices Involving Generalized Tribonacci Numbers

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Abstract. In this paper, we consider a g-circulant matrix $\mathcal{A}_g(T)$, whose the first row entries are generalized Tribonacci numbers $T_i^{(a)}$. We give an explicit formula of the spectral norm of this matrix. When $g = 1$, we also present upper and lower bounds for the spread of the *1*-circulant matrix $\mathcal{A}_1(T)$.

1. Introduction and preliminaries

Let g be a nonnegative integer. A matrix $\mathcal{A}_g \in M_n$ is called a g-circulant matrix if it is of the form

$$\mathcal{A}_g = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-g} & a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g-1} \\ a_{n-2g} & a_{n-2g+1} & a_{n-2g+2} & \cdots & a_{n-2g-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_g & a_{g+1} & a_{g+2} & \cdots & a_{g-1} \end{pmatrix} \tag{1}$$

where each of the subscripts is understood to be reduced modulo n. Obviously, when $g = 1$ or $g = n + 1$, the g-circulant matrix $\mathcal{A}_g$ reduces to the standard circulant matrix.

Circulant type matrices not only have many connections to problems in physics, statistics and numerical analysis, but also have important applications in signal and image processing [1], networks engineering [2, 3], solving ordinary and partial differential equations [4, 5]. In recent years, there are several papers focus on the norms and spread of some special matrices [6–21]. For example, Solak [6] gave upper and lower bounds for the spectral norms of circulant matrices whose entries are Fibonacci and Lucas numbers. İpek [7] improved the estimation for the spectral norms of these matrices. Kizilateş and Tuglu [14] established upper and lower bounds for the spectral norms of geometric circulant matrices involving generalized Fibonacci and hyperharmonic Fibonacci numbers. Zhou and Jiang [16] derived some explicit formulas for the spectral norms of g-circulant matrices whose the first row entries are Fibonacci number, Lucas number and their powers. In addition, Johnson et al. [19] derived some lower bounds for the spread of a normal matrix. Li

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et al. [21] investigated the norms and spread of circulant matrices with Tribonacci and generalized Lucas numbers.

Let a, u, v and w be arbitrary positive integers. The generalized Tribonacci sequence $\{T_n^{(a)}\}$ is defined by the following recurrence relations:

$$T_n^{(a)} = uT_{n-1}^{(a)} + vT_{n-2}^{(a)} + wT_{n-3}^{(a)}, \tag{2}$$

where $T_0^{(a)} = 0, T_1^{(a)} = a, T_2^{(a)} = au$. Some results concerning this sequence were given in [22–24]. When $a = u = v = w = 1$, the generalized Tribonacci sequence reduces to the Tribonacci sequence $\{T_n\}$ in [25]. Let γ_1, γ_2 and γ_3 be the roots of the characteristic equation $x^3 - ux^2 - vx - w = 0$. Then we have

$$\begin{cases} \gamma_1 + \gamma_2 + \gamma_3 = u, \\ \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3 = -v, \\ \gamma_1\gamma_2\gamma_3 = w. \end{cases}$$

Throughout this paper, we assume that γ_1, γ_2 and γ_3 are distinct. The sequence $\{T_n^{(a)}\}$ can be defined for negative values of n by using the recurrence (2) to extend the sequence backwards, that is

$$T_{-n}^{(a)} = (-vT_{-n+1}^{(a)} - uT_{-n+2}^{(a)} + T_{-n+3}^{(a)})/w.$$

In this paper, let $\mathcal{A}_g(T)$ be a g-circulant matrix, whose the first row entries are $(T_0^{(a)}, T_1^{(a)}, \cdots, T_{n-1}^{(a)})$. We give an explicit formula of the spectral norm of this matrix, which is only related to the generalized Tribonacci numbers. Afterwards, we also present upper and lower bounds for the spread of the circulant matrix $\mathcal{A}_1(T)$.

Now we give some preliminaries related to this paper. For any $\mathcal{A} = [a_{ij}] \in M_n$, the well-known Frobenius (or Euclidean) norm of the matrix $\mathcal{A}$ is

$$\|\mathcal{A}\|_F = \Big[\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\Big]^{\frac{1}{2}},$$

the spectral norm of $\mathcal{A}$ is

$$\|\mathcal{A}\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(\mathcal{A}^H\mathcal{A})},$$

and the spread of $\mathcal{A}$ is

$$s(\mathcal{A}) = \max_{1\le i,j\le n} |\lambda_i(\mathcal{A}) - \lambda_j(\mathcal{A})|,$$

where $\lambda_i(\mathcal{A})$ is eigenvalue of $\mathcal{A}$ and $\mathcal{A}^H$ is conjugate transpose of $\mathcal{A}$. Then the following inequality holds [26]:

$$s(\mathcal{A}) \le \Big[\Big(2\|\mathcal{A}\|_F^2 - \frac{2}{n}|\mathrm{tr}\mathcal{A}|^2\Big)^2 - 2\|\mathcal{A}\mathcal{A}^H - \mathcal{A}^H\mathcal{A}\|_F^2\Big]^{\frac{1}{4}}, \tag{3}$$

where $\mathrm{tr}\mathcal{A}$ is trace of $\mathcal{A}$.

Lemma 1.1. *[19] Let $\mathcal{A} = [a_{ij}]$ be an $n \times n$ matrix.*

(i) If $\mathcal{A}$ is real and normal, then $s(\mathcal{A}) \ge \frac{1}{n-1}|\sum_{i\neq j} a_{ij}|$;

(ii) If $\mathcal{A}$ is Hermitian, then $s(\mathcal{A}) \ge 2\max_{i\neq j} |a_{ij}|$.

Lemma 1.2. *[27] Let Q_g be an $n \times n$ g-circulant matrix with the first row $(1, 0, \cdots, 0)$. Then*

(i) Q_g is unitary if and only if $(n, g) = 1$;

(ii) $\mathcal{A}$ is a g-circulant matrix with the first row $(a_0, a_1, \cdots, a_{n-1})$ if and only if $\mathcal{A} = Q_g C$, where C is a circulant matrix with the first row $(a_0, a_1, \cdots, a_{n-1})$.

Lemma 1.3. *[28] Let $\mathcal{A} = [a_{ij}] \in M_n$ be a nonnegative matrix. Then its spectral radius $\rho(\mathcal{A})$ satisfies the following inequality*

$$\min_{1\le i\le n} \sum_{j=1}^{n} a_{ij} \le \rho(\mathcal{A}) \le \max_{1\le i\le n} \sum_{j=1}^{n} a_{ij}. \tag{4}$$

2. Main results

We consider three sequences $\{X_n\}, \{Y_n\}$ and $\{Z_n\}$, which satisfy the recurrence (2) and the following initial conditions:

$$X_0 = 0,\ \ X_1 = 0,\ \ X_2 = 1;$$
$$Y_0 = 0,\ \ Y_1 = 1,\ \ Y_2 = 0;$$
$$Z_0 = 1,\ \ Z_1 = 0,\ \ Z_2 = 0.$$

Then $T_i^{(a)} = aX_{i+1}$, and the Binet formulas of these sequences are given by [23]:

$$\begin{cases} X_n = A_1\gamma_1^n + A_2\gamma_2^n + A_3\gamma_3^n, \\ Y_n = B_1\gamma_1^n + B_2\gamma_2^n + B_3\gamma_3^n, \\ Z_n = C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n, \end{cases} \tag{5}$$

where

$$A_1 = \frac{1}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ A_2 = \frac{1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ A_3 = \frac{1}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)};$$

$$B_1 = \frac{-(\gamma_2+\gamma_3)}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ B_2 = \frac{-(\gamma_3+\gamma_1)}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ B_3 = \frac{-(\gamma_1+\gamma_2)}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)};$$

$$C_1 = \frac{\gamma_2\gamma_3}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ C_2 = \frac{\gamma_3\gamma_1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ C_3 = \frac{\gamma_1\gamma_2}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}.$$

By solving the equations in (5), we obtain

$$\begin{cases} \gamma_1^n = \gamma_1^2 X_n + \gamma_1 Y_n + Z_n, \\ \gamma_2^n = \gamma_2^2 X_n + \gamma_2 Y_n + Z_n, \\ \gamma_3^n = \gamma_3^2 X_n + \gamma_3 Y_n + Z_n. \end{cases} \tag{6}$$

Next we give an explicit formula of the spectral norm of the matrix $\mathcal{A}_g(T)$, which is only related to the generalized Tribonacci numbers $T_i^{(a)}$.

Theorem 2.1. *Let $\mathcal{A}_g(T)$ be as the matrix in (1), with $a_i = T_i^{(a)} (i = 0, 1, \cdots, n-1)$ in the first row of $\mathcal{A}_g(T)$. If $(n, g) = 1$, then we have*

$$\|\mathcal{A}_g(T)\|_2 = \frac{T_{n+1}^{(a)} + (1-u)T_n^{(a)} + wT_{n-1}^{(a)} - a}{u+v+w-1}.$$

Proof. Applying the results from Lemma 1.2, we obtain

$$(\mathcal{A}_g(T))^H\mathcal{A}_g(T) = (Q_gC)^HQ_gC = C^H(Q_g)^HQ_gC = C^HC,$$

where $C = [c_{ij}]$ is an $n \times n$ circulant matrix with the first row $(T_0^{(a)}, T_1^{(a)}, \cdots, T_{n-1}^{(a)})$. Hence the spectral norm of the matrix $\mathcal{A}_g(T)$ is the same as that of C.

Since the circulant matrix C is normal, there exists a unitary matrix $U \in M_n$ such that $U^HCU = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where λ_i is eigenvalue of C, hence

$$U^HC^HCU = diag(|\lambda_1|^2, |\lambda_2|^2, \cdots, |\lambda_n|^2).$$

Therefore, the spectral norm of C is given by its spectral radius. Note that C is nonnegative, hence its spectral radius $\rho(C)$ satisfies the following inequality:

$$\min_{1\le i\le n} \sum_{j=1}^{n} c_{ij} \le \rho(C) \le \max_{1\le i\le n} \sum_{j=1}^{n} c_{ij}.$$

Moreover, for each $i \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
\sum_{j=1}^{n} c_{ij} &= \sum_{k=0}^{n-1} T_k^{(a)} = a\sum_{k=0}^{n-1} X_{k+1} = a\sum_{k=0}^{n-1}(A_1\gamma_1^{k+1} + A_2\gamma_2^{k+1} + A_3\gamma_3^{k+1}) \\
&= a\Big[\frac{A_1\gamma_1(1-\gamma_1^n)}{1-\gamma_1} + \frac{A_2\gamma_2(1-\gamma_2^n)}{1-\gamma_2} + \frac{A_3\gamma_3(1-\gamma_3^n)}{1-\gamma_3}\Big] \\
&= \frac{a[X_{n+2} + (1-u)X_{n+1} + wX_n - 1]}{u+v+w-1} \\
&= \frac{T_{n+1}^{(a)} + (1-u)T_n^{(a)} + wT_{n-1}^{(a)} - a}{u+v+w-1}.
\end{aligned}$$

It follows that

$$\|\mathcal{A}_g(T)\|_2 = \|C\|_2 = \frac{T_{n+1}^{(a)} + (1-u)T_n^{(a)} + wT_{n-1}^{(a)} - a}{u+v+w-1}.$$

Thus the proof is completed. □

If we take $a = u = v = w = 1$ in Theorem 2.1, then we obtain the spectral norm of a g-circulant matrix involving the Tribonacci numbers.

Corollary 2.2. *Let $\mathcal{B}_g(T)$ be a g-circulant matrix with the first row $(T_0, T_1, \cdots, T_{n-1})$. If $(n, g) = 1$, then we have*

$$\|\mathcal{B}_g(T)\|_2 = \frac{T_{n+1} + T_{n-1} - 1}{2}.$$

In the sequel of this paper, we will investigate the spread of the circulant matrix $\mathcal{A}_1(T)$. Before presenting our main theorem, we need the following several lemmas.

Lemma 2.3. *For arbitrary integer $k \ge 0$, we have*

$$\sum_{i=1}^{3} A_i^2\gamma_i^k = \frac{2(v^2-3uw)X_k - (uv+9w)Y_k + 2(u^2+3v)Z_k}{\Delta^2} \tag{7}$$

$$\sum_{1\le i<j\le 3} A_iA_j(\gamma_i\gamma_j)^k = \frac{uwH_{k-1} - 2w^2H_{k-2} - H_{k+1}}{\Delta^2} \tag{8}$$

where

$$\begin{aligned}\Delta^2 &= u^2v^2 - 27w^2 + 4v^3 - 4u^3w - 18uvw,\\ H_r &= (v^2 - 2uw)X_r^2 - (uv + 3w)X_rY_r + 2(u^2 + 2v)X_rZ_r + 2uY_rZ_r\\ &\quad -vY_r^2 + 3Z_r^2 \ \ (r = -2, -1, 0, 1, 2, \cdots).\end{aligned}$$

Proof. Since

$$\begin{aligned}A_1^2 + A_2^2 + A_3^2 &= \frac{1}{(\gamma_1-\gamma_2)^2(\gamma_1-\gamma_3)^2} + \frac{1}{(\gamma_2-\gamma_3)^2(\gamma_2-\gamma_1)^2} + \frac{1}{(\gamma_3-\gamma_1)^2(\gamma_3-\gamma_2)^2}\\ &= \frac{2[(\gamma_1^2+\gamma_2^2+\gamma_3^2) - (\gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_1\gamma_3)]}{(\gamma_1-\gamma_2)^2(\gamma_2-\gamma_3)^2(\gamma_3-\gamma_1)^2}\\ &= \frac{2(u^2+3v)}{\Delta^2},\end{aligned}$$

and

$$\gamma_1A_1^2 + \gamma_2A_2^2 + \gamma_3A_3^2 = -\frac{uv+9w}{\Delta^2}, \quad \gamma_1^2A_1^2 + \gamma_2^2A_2^2 + \gamma_3^2A_3^2 = \frac{2(v^2-3uw)}{\Delta^2},$$

the formula (7) is valid for $k = 0, 1, 2$. In the case $k \geq 3$, according to (6), we obtain

$$\begin{aligned}\sum_{i=1}^{3} A_i^2\gamma_i^k &= \sum_{i=1}^{3} A_i^2(\gamma_i^2X_k + \gamma_iY_k + Z_k)\\ &= \Big(\sum_{i=1}^{3}\gamma_i^2A_i^2\Big)X_k + \Big(\sum_{i=1}^{3}\gamma_iA_i^2\Big)Y_k + \Big(\sum_{i=1}^{3}A_i^2\Big)Z_k\\ &= \frac{2(v^2-3uw)X_k - (uv+9w)Y_k + 2(u^2+3v)Z_k}{\Delta^2}.\end{aligned}$$

Next we will prove the formula (8). Since (6) is valid for $r \geq -2$, we have

$$\begin{aligned}\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^r &= \sum_{1\le i<j\le 3}(\gamma_i^2X_r + \gamma_iY_r + Z_r)(\gamma_j^2X_r + \gamma_jY_r + Z_r)\\ &= \Big(\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^2\Big)X_r^2 + \Big(\sum_{1\le i<j\le 3}(\gamma_i^2\gamma_j + \gamma_i\gamma_j^2)\Big)X_rY_r + \Big(\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)\Big)Y_r^2\\ &\quad +2\Big(\sum_{i=1}^{3}\gamma_i^2\Big)X_rZ_r + 2\Big(\sum_{i=1}^{3}\gamma_i\Big)Y_rZ_r + 3Z_r^2\\ &= (v^2-2uw)X_r^2 - (uv+3w)X_rY_r + 2(u^2+2v)X_rZ_r\\ &\quad +2uY_rZ_r - vY_r^2 + 3Z_r^2\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{1\le i<j\le 3} A_iA_j(\gamma_i\gamma_j)^k &= \frac{\gamma_1\gamma_2\gamma_3\sum_{1\le i<j\le 3}(\gamma_i^{k-1}\gamma_j^k+\gamma_i^k\gamma_j^{k-1})-\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^{k+1}-(\gamma_1\gamma_2\gamma_3)^2\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^{k-2}}{\Delta^2}\\ &= \frac{uw\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^{k-1}-2w^2\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^{k-2}-\sum_{1\le i<j\le 3}(\gamma_i\gamma_j)^{k+1}}{\Delta^2}\\ &= \frac{uwH_{k-1}-2w^2H_{k-2}-H_{k+1}}{\Delta^2}\end{aligned}$$

Thus the proof is completed. □

For the convenience of the discussion, we denote

$$M_i := \frac{2(v^2-3uw)X_i-(uv+9w)Y_i+2(u^2+3v)Z_i}{\Delta^2}, \tag{9}$$

$$N_i := \frac{uwH_{i-1}-2w^2H_{i-2}-H_{i+1}}{\Delta^2}. \tag{10}$$

Lemma 2.4. *For the generalized Tribonacci sequence $\{T_n^{(a)}\}$, if $v\neq u+w+1$ and $v\neq w^2-uw-1$, then we have*

$$\begin{aligned}\sum_{j=0}^{n-1}(T_j^{(a)})^2 &= \frac{a^2[w^2(M_0-M_{2n})+(1-u^2-2v)(M_2-M_{2n+2})+(M_4-M_{2n+4})]}{(v-1)^2-(u+w)^2}\\ &\quad+\frac{2a^2[w^2(N_0-N_n)+(1+v)(N_1-N_{n+1})+(N_2-N_{n+2})]}{1+v+uw-w^2}.\end{aligned}$$

Proof. Since $v\neq u+w+1$ and $v\neq w^2-uw-1$, $\gamma_i\gamma_j\neq 1$ for any $i,j\in\{1,2,3\}$. Applying Binet formula of the sequence $\{X_n\}$, then we obtain

$$\begin{aligned}\sum_{j=0}^{n-1}(T_j^{(a)})^2 &= a^2\sum_{j=0}^{n-1}X_{j+1}^2 = a^2\sum_{j=0}^{n-1}\Big(\sum_{i=1}^{3}A_i\gamma_i^{j+1}\Big)^2\\ &= a^2\sum_{j=0}^{n-1}\Big(\sum_{i=1}^{3}A_i^2\gamma_i^{2(j+1)}+2\sum_{1\le i<k\le 3}A_iA_k(\gamma_i\gamma_k)^{j+1}\Big)\\ &= a^2\sum_{i=1}^{3}\frac{A_i^2\gamma_i^2(1-\gamma_i^{2n})}{1-\gamma_i^2}+2a^2\sum_{1\le i<k\le 3}\frac{A_iA_k(\gamma_i\gamma_k)[1-(\gamma_i\gamma_k)^n]}{1-\gamma_i\gamma_k}.\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}\sum_{i=1}^{3}\frac{A_i^2\gamma_i^2(1-\gamma_i^{2n})}{1-\gamma_i^2} &= \frac{w^2\sum_{i=1}^3A_i^2(1-\gamma_i^{2n})+(1-u^2-2v)\sum_{i=1}^3A_i^2\gamma_i^2(1-\gamma_i^{2n})+\sum_{i=1}^3A_i^2\gamma_i^4(1-\gamma_i^{2n})}{(1-\gamma_1^2)(1-\gamma_2^2)(1-\gamma_3^2)}\\ &= \frac{w^2(M_0-M_{2n})+(1-u^2-2v)(M_2-M_{2n+2})+(M_4-M_{2n+4})}{(v-1)^2-(u+w)^2}\end{aligned}$$

and

$$\sum_{1\le i<k\le 3}\frac{A_iA_k(\gamma_i\gamma_k)[1-(\gamma_i\gamma_k)^n]}{1-\gamma_i\gamma_k} = \frac{\begin{array}{l} w^2\sum\limits_{1\le i<k\le 3}A_iA_k[1-(\gamma_i\gamma_k)^n]+\sum\limits_{1\le i<k\le 3}A_iA_k(\gamma_i\gamma_k)^2[1-(\gamma_i\gamma_k)^n]+\\ (1+v)\sum\limits_{1\le i<k\le 3}A_iA_k(\gamma_i\gamma_k)[1-(\gamma_i\gamma_k)^n]\end{array}}{(1-\gamma_1\gamma_2)(1-\gamma_2\gamma_3)(1-\gamma_1\gamma_3)}$$

$$= \frac{w^2(N_0-N_n)+(1+v)(N_1-N_{n+1})+(N_2-N_{n+2})}{1+v+uw-w^2}.$$

Thus the proof is completed. □

Theorem 2.5. *For the circulant matrix $\mathcal{A}_1(T)$, if $v\neq u+w+1$ and $v\neq w^2-uw-1$, then we have*

$$s(\mathcal{A}_1(T)) \ge \frac{n}{n-1}\left[\frac{T^{(a)}_{n+1}+(1-u)T^{(a)}_n+wT^{(a)}_{n-1}-a}{u+v+w-1}\right],$$

and

$$s(\mathcal{A}_1(T)) \le \sqrt{2na}\left[\frac{w^2(M_0-M_{2n})+(1-u^2-2v)(M_2-M_{2n+2})+(M_4-M_{2n+4})}{(v-1)^2-(u+w)^2}\right.$$
$$\left.+\frac{2(w^2(N_0-N_n)+(1+v)(N_1-N_{n+1})+(N_2-N_{n+2}))}{1+v+uw-w^2}\right]^{\frac{1}{2}},$$

where M_i, N_i are given by (9) and (10).

Proof. Since $\mathcal{A}_1(T)$ is a real and normal matrix, applying the results from Lemma 1.1 and Theorem 2.1, we have

$$s(\mathcal{A}_1(T)) \ge \frac{n}{n-1}\left[\sum_{k=0}^{n-1}T^{(a)}_k-T^{(a)}_0\right]=\frac{n}{n-1}\left[\frac{T^{(a)}_{n+1}+(1-u)T^{(a)}_n+wT^{(a)}_{n-1}-a}{u+v+w-1}\right].$$

On the other hand, since $(\mathcal{A}_1(T))^H\mathcal{A}_1(T)=\mathcal{A}_1(T)(\mathcal{A}_1(T))^H$, by using (3) and Lemma 2.4, we obtain

$$s(\mathcal{A}_1(T)) \le \sqrt{2\|\mathcal{A}_1(T)\|_F^2-\frac{2}{n}|\mathrm{tr}\mathcal{A}_1(T)|^2}=\sqrt{2}\|\mathcal{A}_1(T)\|_F=\sqrt{2}\left[n\sum_{k=0}^{n-1}(T^{(a)}_k)^2\right]^{\frac{1}{2}}$$
$$= \sqrt{2na}\left[\frac{w^2(M_0-M_{2n})+(1-u^2-2v)(M_2-M_{2n+2})+(M_4-M_{2n+4})}{(v-1)^2-(u+w)^2}\right.$$
$$\left.+\frac{2(w^2(N_0-N_n)+(1+v)(N_1-N_{n+1})+(N_2-N_{n+2}))}{1+v+uw-w^2}\right]^{\frac{1}{2}}.$$

Thus the proof is completed. □

If we take $a=u=v=w=1$ in Theorem 2.5, then we obtain upper and lower bounds for the spread of a circulant matrix involving the Tribonacci numbers.

Corollary 2.6. *Let $\mathcal{B}_1(T)$ be a circulant matrix with the first row $(T_0, T_1, \cdots, T_{n-1})$. Then we have*

$$s(\mathcal{B}_1(T)) \ge \frac{n}{n-1}\left(\frac{T_{n+1}+T_{n-1}-1}{2}\right),$$

and

$$s(\mathcal{B}_1(T)) \le \sqrt{2n}\left[\frac{11+7X_{2n+1}-2X_{2n}+X_{2n-1}-(H_{n+3}+2H_{n+2}+3H_{n-1}+2H_{n-2})}{44}\right]^{\frac{1}{2}},$$

where $H_r=-X_r^2-4X_rY_r+6X_rZ_r+2Y_rZ_r-Y_r^2+3Z_r^2$ $(r=0,1,2,\cdots)$.

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