Filomat 35:15 (2021), 5237–5252
https://doi.org/10.2298/FIL2115237S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On $\mathbb{A}$-Numerical Radius Inequalities for 2×2 Operator Matrices-II

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Abstract. Rout et al. [Linear Multilinear Algebra 2020, DOI: 10.1080/03081087.2020.1810201] presented certain $\mathbb{A}$-numerical radius inequalities for 2×2 operator matrices and further results on $\mathbb{A}$-numerical radius of certain 2×2 operator matrices are obtained by Feki [Hacet. J. Math. Stat., 2020, DOI:10.15672/hujms.730574], very recently. The main goal of this article is to establish certain $\mathbb{A}$-numerical radius equalities for operator matrices. Several new upper and lower bounds for the $\mathbb{A}$-numerical radius of 2×2 operator matrices has been proved, where $\mathbb{A}$ be the 2×2 diagonal operator matrix whose diagonal entries are positive bounded operator A. Further, we prove some refinements of earlier A-numerical radius inequalities for operators.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. The *numerical range* of $T \in \mathcal{L}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The *numerical radius* of T is defined as $w(T) = \sup\{|z| : z \in W(T)\}$. It is well-known that $w(\cdot)$ defines a norm on $\mathcal{H}$, and is equivalent to the usual operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $T \in \mathcal{L}(\mathcal{H})$, the operator norm and the numerical radius is always comparable by the following inequality

$$\frac{1}{2}\|T\| \le w(T) \le \|T\|. \quad (1)$$

An interested reader is referred to the recent articles [4, 17, 25, 26] for different generalizations, refinements and applications of numerical radius inequalities.

Let $\|\cdot\|$ be the norm induced from $\langle\cdot,\cdot\rangle$. Let the symbol I and O stand for the identity operator and the null operator on $\mathcal{H}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is called *selfadjoint* if $A = A^*$, where A^* denotes the adjoint of A. An operator $A \in \mathcal{L}(\mathcal{H})$ is called *positive* if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$, and is called *strictly positive* if $\langle Ax, x\rangle > 0$ for all non-zero $x \in \mathcal{H}$. We denote a positive (strictly positive) operator A by $A \ge O$ ($A > O$). We denote $\mathcal{R}(A)$ as the range space of A and $\overline{\mathcal{R}(A)}$ as the norm closure of $\mathcal{R}(A)$ in $\mathcal{H}$. Let $\mathbb{A}$ be a 2×2 diagonal operator matrix whose diagonal entries are positive operator A. Then $\mathbb{A} \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ and $\mathbb{A} \ge 0$. Then any such A induces a positive semidefinite sesquilinear form, $\langle\cdot,\cdot\rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ defined by $\langle x, y\rangle_A = \langle Ax, y\rangle$, $x, y \in \mathcal{H}$. Naturally,

2020 *Mathematics Subject Classification*. Primary 47A12; 47A30; 47A63; 47A05.
Keywords. A-numerical radius; Positive operator; Semi-inner product; Inequality; Operator matrix.
Received: 09 January 2021; Revised: 07 May 2021; Accepted: 11 September 2021
Communicated by Fuad Kittaneh
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this semi-inner product $\langle\cdot,\cdot\rangle_A$, induces a seminorm $\|\cdot\|_A$ defined by $\|x\|_A = \sqrt{\langle x,x\rangle_A}$ for all $x \in \mathcal{H}$. Then $\|x\|_A$ is a norm if and only if $A > O$. Also, $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. Throughout this paper, we fix A and $\mathbb{A}$ for positive operators on $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$, respectively.

Given $T \in \mathcal{L}(\mathcal{H})$ if there exists $c > 0$ satisfying $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \overline{\mathcal{R}(A)}$, then A-operator seminorm of T is defined as follows:

$$\|T\|_A = \sup_{x\in\overline{\mathcal{R}(A)},\ x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

Let

$$\mathcal{L}^A(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \|T\|_A < \infty\}.$$

Then $\mathcal{L}^A(\mathcal{H})$ is not a sub-algebra of $\mathcal{L}(\mathcal{H})$, and $\|T\|_A = 0$ if and only if $ATA = O$. Moreover, for $T \in \mathcal{L}^A(\mathcal{H})$, we have

$$\|T\|_A = \sup\{|\langle Tx, y\rangle_A| : x, y \in \overline{\mathcal{R}(A)}, \|x\|_A = \|y\|_A = 1\}.$$

If $AT \geq 0$, then the operator T is called *A-positive*. Note that if T is A-positive, then

$$\|T\|_A = \sup\{\langle Tx, x\rangle_A : x \in \overline{\mathcal{R}(A)}, \|x\|_A = 1\}.$$

For $T \in \mathcal{L}(\mathcal{H})$, an operator $X \in \mathcal{L}(\mathcal{H})$ is called an *A-adjoint operator* of T if $\langle Tx, y\rangle_A = \langle x, Xy\rangle_A$ for every $x, y \in \mathcal{H}$, i.e., $AX = T^*A$. By Douglas theorem [9, 18], the existence of an A-adjoint operator is not guaranteed. An operator $T \in \mathcal{L}(\mathcal{H})$ may admit none, one or many A-adjoints. A-adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$ exists if and only if $\mathcal{R}(T^*A) \subseteq \mathcal{R}(A)$. Let us now denote

$$\mathcal{L}_A(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

Note that $\mathcal{L}_A(\mathcal{H})$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{L}(\mathcal{H})$. Moreover, the following inclusions

$$\mathcal{L}_A(\mathcal{H}) \subseteq \mathcal{L}^A(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$$

hold with equality if A is injective and has a closed range.

The *Moore-Penrose inverse* of $A \in \mathcal{L}(\mathcal{H})$ [21] is the operator $X : \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \longrightarrow \mathcal{H}$ which satisfies the following four equations:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ XA = P_{\mathcal{N}(A)^\perp},\ (4)\ AX = P_{\overline{\mathcal{R}(A)}}|_{\mathcal{R}(A)\oplus\mathcal{R}(A)^\perp}.$$

Here $\mathcal{N}(A)$ and P_L denote the null space of A and the orthogonal projection onto L, respectively. The Moore-Penrose inverse is unique, and is denoted by $A^\dagger$. In general, $A^\dagger \notin \mathcal{L}(\mathcal{H})$. It is bounded if and only if $\mathcal{R}(A)$ is closed. If $A \in \mathcal{L}(\mathcal{H})$ is invertible, then $A^\dagger = A^{-1}$. If $T \in \mathcal{L}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\#_A}$ (see [2, 19]). Note that $T^{\#_A} = A^\dagger T^*A$. If $T \in \mathcal{L}_A(\mathcal{H})$, then $AT^{\#_A} = T^*A$, $\mathcal{R}(T^{\#_A}) \subseteq \overline{\mathcal{R}(A)}$ and $\mathcal{N}(T^{\#_A}) = \mathcal{N}(T^*A)$ (see [9]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *A-selfadjoint* if AT is selfadjoint, i.e., $AT = T^*A$. Observe that if T is A-selfadjoint, then $T \in \mathcal{L}_A(\mathcal{H})$. However, in general, $T \neq T^{\#_A}$. But, $T = T^{\#_A}$ if and only if T is A-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. If $T \in \mathcal{L}_A(\mathcal{H})$, then $T^{\#_A} \in \mathcal{L}_A(\mathcal{H})$, $(T^{\#_A})^{\#_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$, and $\left((T^{\#_A})^{\#_A}\right)^{\#_A} = T^{\#_A}$. Also, $T^{\#_A}T$ and $TT^{\#_A}$ are A-positive operators, and

$$\|T^{\#_A}T\|_A = \|TT^{\#_A}\|_A = \|T\|_A^2 = \|T^{\#_A}\|_A^2 = w_A(TT^{\#_A}) = w_A(T^{\#_A}T). \tag{2}$$

An operator T is called A-bounded if there exists $\alpha > 0$ such that $\|Tx\|_A \leq \alpha\|x\|_A,\ \forall\, x \in \mathcal{H}$. By applying Douglas theorem [9], one can easily see that the subspace of all operators admitting $A^{1/2}$-adjoints, denoted by $\mathcal{L}_{A^{1/2}}(\mathcal{H})$, is equal the collection of all A-bounded operators, i.e.,

$$\mathcal{L}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H})\,;\ \exists\, \alpha > 0\,;\ \|Tx\|_A \leq \alpha\|x\|_A,\ \forall\, x \in \mathcal{H}\}.$$

Notice that $\mathcal{L}_A(\mathcal{H})$ and $\mathcal{L}_{A^{1/2}}(\mathcal{H})$ are two sub-algebras of $\mathcal{L}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathcal{L}(\mathcal{H})$. Moreover, we have $\mathcal{L}_A(\mathcal{H}) \subset \mathcal{L}_{A^{1/2}}(\mathcal{H})$ (see [2, 3]).

An operator $U \in \mathcal{L}_A(\mathcal{H})$ is said to be *A-unitary* if $\|Ux\|_A = \|U^{\#_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. For $T, S \in \mathcal{L}_A(\mathcal{H})$, we have $(TS)^{\#_A} = S^{\#_A}T^{\#_A}$, $(T+S)^{\#_A} = T^{\#_A} + S^{\#_A}$, $\|TS\|_A \le \|T\|_A\|S\|_A$ and $\|Tx\|_A \le \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. In 2012, Saddi [24] introduced *A-numerical radius* of T for $T \in \mathcal{L}(\mathcal{H})$, which is denoted as $w_A(T)$, and is defined as follows:

$$w_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}. \tag{3}$$

The A-numerical radius of an operator is one of the extensions of the numerical radius. When $A = I$, we will get the usual numerical radius.

From (3), it follows that

$$w_A(T) = w_A(T^{\#_A}) \text{ for any } T \in \mathcal{L}_A(\mathcal{H}).$$

A fundamental inequality for the A-numerical radius is the power inequality (see [20]) which says that for $T \in \mathcal{L}_A(\mathcal{H})$,

$$w_A(T^n) \le w_A^n(T), \ \ n \in \mathbb{N}. \tag{4}$$

Notice that the A-numerical radius of semi-Hilbertian space operators satisfies the weak A-unitary invariance property which asserts that

$$w_A(U^{\#_A}TU) = w_A(T), \tag{5}$$

for every $T \in \mathcal{L}_A(\mathcal{H})$ and every A-unitary operator $U \in \mathcal{L}_A(\mathcal{H})$ (see [7, Lemma 3.8]).

An interested reader may refer [1, 2] for further properties of operators on Semi-Hilbertian space.

Let

$$\Re_A(T) := \frac{T + T^{\#_A}}{2} \quad \text{and} \quad \Im_A(T) := \frac{T - T^{\#_A}}{2i},$$

for any arbitrary operator $T \in \mathcal{L}_A(\mathcal{H})$. Recently, in 2019 Zamani [28, Theorem 2.5] showed that if $T \in \mathcal{L}_A(\mathcal{H})$, then

$$w_A(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}T)\right\|_A = \sup_{\theta\in\mathbb{R}} \left\|\Im_A(e^{i\theta}T)\right\|_A. \tag{6}$$

In 2019, Zamani [28] showed that if $T \in \mathcal{L}_A(\mathcal{H})$, then

$$w_A(T) = \sup_{\theta\in\mathbb{R}} \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\#_A}}{2} \right\|_A. \tag{7}$$

The author then extended the inequality (1) using A-numerical radius of T, and the same is produced below:

$$\frac{1}{2}\|T\|_A \le w_A(T) \le \|T\|_A. \tag{8}$$

Furthermore, if T is A-selfadjoint, then $w_A(T) = \|T\|_A$. In 2019, Moslehian *et al.* [20] again continued the study of A-numerical radius and established some inequalities for A-numerical radius. Further generalizations and refinements of A-numerical radius are discussed in [5, 6, 22, 29]. In 2020, Bhunia *et al.* [8] obtained several $\mathbb{A}$-numerical radius inequalities. For more results on $\mathbb{A}$-numerical radius inequalities we refer the reader to visit [10–15, 23, 27].

In 2020, the concept of the A-spectral radius of A-bounded operators was introduced by Feki in [16] as follows:

$$r_A(T) := \inf_{n\ge1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|_A^{\frac{1}{n}}. \tag{9}$$

Here we want to mention that the proof of the second equality in (9) can also be found in [16, Theorem 1]. Like the classical spectral radius of Hilbert space operators, it was shown in [16] that $r_A(\cdot)$ satisfies the commutativity property, i.e.

$$r_A(TS) = r_A(ST), \tag{10}$$

for all $T, S \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. For the sequel, if $A = I$, then $\|T\|$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator T.

The first objective of this paper is to present a few new $\mathbb{A}$-numerical radius equalities for 2×2 operator matrices. Further, we provide some upper and lower bounds for the $\mathbb{A}$-numerical radius of 2×2 operator matrices. Finally, we aim to obtain some refinements of the 1st inequality in (8). In this aspect, the rest of the paper is broken down as follows. In Section 2, we collect a few results about $\mathbb{A}$-numerical radius inequalities which are required to state and prove the results in the subsequent section. Section 3 contains our main results, and is of three parts. In the first part, we establish $\mathbb{A}$-numerical radius equalities for 2×2 operator matrices. Motivated by the work of Hirzallah *et al.* [17], the second part presents several $\mathbb{A}$-numerical radius inequalities of 2×2 operator matrices while the next part focuses on some A-numerical radius inequalities. We provide several examples to demonstrate our results.

2. Preliminaries

We need the following lemmas to prove our results.

Lemma 2.1. [16, Theorem 7 and corollary 2] *If* $T \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$.*Then*

$$w_A(T) \leq \frac{1}{2}(\|T\|_A + \|T^2\|_A^{1/2}). \tag{11}$$

Further, if $AT^2 = 0$, *then*

$$w_A(T) = \frac{\|T\|_A}{2}. \tag{12}$$

Lemma 2.2. [16, Corollary 3] *Let* $T \in \mathcal{L}(\mathcal{H})$ *is an A-self-adjoint operator. Then,*

$$\|T\|_A = w_A(T) = r_A(T).$$

Lemma 2.3. [7, Lemma 6] *Let* $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ *be such that* $T_1, T_2, T_3, T_4 \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. *Then,* $T \in \mathcal{L}_{\mathbb{A}^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ *and*

$$r_{\mathbb{A}}(T) \leq r\left(\begin{bmatrix} \|T_1\|_A & \|T_2\|_A \\ \|T_3\|_A & \|T_4\|_A \end{bmatrix}\right).$$

The following lemma is already proved by Bhunia *et al.* [8] for the case strictly positive operator A. Very recentely the same result proved by Rout *et al.* [23] by dropping the assumption A is strictly positive is stated next for our purpose.

Lemma 2.4. [23, Lemma 2.4] *Let* $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. *Then*

(i) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}\right) = \max\{w_A(T_1), w_A(T_2)\}$.

(ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_1 & O \end{bmatrix}\right)$.

(iii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ e^{i\theta}T_2 & O \end{bmatrix}\right) = w_{\mathbb{A}}\left(\begin{bmatrix} O & T_1 \\ T_2 & O \end{bmatrix}\right)$ *for any* $\theta \in \mathbb{R}$.

(iv) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix}\right) = \max\{w_A(T_1+T_2), w_A(T_1-T_2)\}$. *In particular,* $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = w_A(T_2)$.

The following Lemma is proved by Rout *et al.* [23].

Lemma 2.5. [23, Lemma 2.2] *Let* $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$. *Then*

(i) $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right) \le w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

(ii) $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \le w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right)$.

Lemma 2.6. [7, 15, Lemma 2.4 and Lemma 3.1] *Let* $T_1, T_4 \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. *Then, the following assertions hold*

(i) $\left\|\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right\|_{\mathbb{A}} = \left\|\begin{bmatrix} O & T_1 \\ T_4 & O \end{bmatrix}\right\|_{\mathbb{A}} = \max\left\{\|T_1\|_A, \|T_4\|_A\right\}$.

(ii) *If* $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$, *then* $\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}^{\#_A} = \begin{bmatrix} T_1^{\#_A} & T_3^{\#_A} \\ T_2^{\#_A} & T_4^{\#_A} \end{bmatrix}$.

In order to prove our result the following identity is essential for our purpose. If $T \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$ and $\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}^2 = \begin{bmatrix} O & O \\ O & O \end{bmatrix}$, so by (12)

$$w_{\mathbb{A}}\left(\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right) = \frac{1}{2}\left\|\begin{bmatrix} T & T \\ -T & -T \end{bmatrix}\right\|_A = \|T\|_A. \tag{13}$$

3. Results

We will split our results into three subsections. The first and second part deals with $\mathbb{A}$-numerical radius of 2×2 operator matrices. The third part concerns some upper bounds for A numerical radius inequalities.

3.1. *$\mathbb{A}$-numerical radius equalities of operator matrices*

Here, we provide some $\mathbb{A}$-numerical radius equalities of 2×2 block operator matrices. The first result deals with $\mathbb{A}$-numerical radius estimate of a special 2×2 operator matrix.

Theorem 3.1. *Let* $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ iT_2 & T_1 \end{bmatrix}\right) = \max\left\{w_A\left(T_1 + \frac{1+i}{\sqrt{2}}T_2\right), w_A\left(T_1 - \frac{1+i}{\sqrt{2}}T_2\right)\right\}.$$

Proof. Let $T = \begin{bmatrix} T_1 & T_2 \\ iT_2 & T_1 \end{bmatrix}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ \frac{1+i}{\sqrt{2}}I & -\frac{1+i}{\sqrt{2}}I \end{bmatrix}$. It is not very difficult to show that U is $\mathbb{A}$-unitary. So, by using Lemma 2.6 (ii) we have

$$U^{\#_{\mathbb{A}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & \frac{1-i}{\sqrt{2}}P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & -\frac{1-i}{\sqrt{2}}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}.$$

Therefore, using Lemma 2.6, we have

$$U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U = \frac{1}{2}\begin{bmatrix} 2T_1^{\#_A}P_{\overline{\mathcal{R}(A)}} + \frac{2(1-i)}{\sqrt{2}}T_2^{\#_A}P_{\overline{\mathcal{R}(A)}} & O \\ O & 2T_1^{\#_A}P_{\overline{\mathcal{R}(A)}} - \frac{2(1-i)}{\sqrt{2}}T_2^{\#_A}P_{\overline{\mathcal{R}(A)}} \end{bmatrix}$$
$$= \begin{bmatrix} T_1^{\#_A} + \frac{1-i}{\sqrt{2}}T_2^{\#_A} & O \\ O & T_1^{\#_A} - \frac{1-i}{\sqrt{2}}T_2^{\#_A} \end{bmatrix} \quad \because \mathcal{R}(T_i^{\#_A}) \subseteq \overline{\mathcal{R}(A)}$$
$$= \begin{bmatrix} T_1 + \frac{1+i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1+i}{\sqrt{2}}T_2 \end{bmatrix}^{\#_{\mathbb{A}}}.$$

Using the fact that $w_{\mathbb{A}}(S) = w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}SU)$ for any $S \in \mathcal{L}_A(\mathcal{H})$, we get

$$w_{\mathbb{A}}(T) = w_{\mathbb{A}}(T^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U) = w_{\mathbb{A}}\left(\begin{bmatrix} T_1 + \frac{1+i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1+i}{\sqrt{2}}T_2 \end{bmatrix}^{\#_{\mathbb{A}}}\right)$$
$$= w_{\mathbb{A}}\left(\begin{bmatrix} T_1 + \frac{1+i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1+i}{\sqrt{2}}T_2 \end{bmatrix}\right)$$
$$= \max\left\{w_A\left(T_1 + \frac{1+i}{\sqrt{2}}T_2\right), w_A\left(T_1 - \frac{1+i}{\sqrt{2}}T_2\right)\right\}.$$

□

Example 3.2. *Let* $T_1 = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{2+2i}{\sqrt{2}} \end{bmatrix}$, $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, *and* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then* $w_A(T_1) = w_A(T_2) = 1$. *By Theorem 3.1, we have* $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ iT_2 & T_1 \end{bmatrix}\right) = 2$.

Theorem 3.3. *Let* $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -iT_2 & T_1 \end{bmatrix}\right) = \max\left\{w_A\left(T_1 + \frac{1-i}{\sqrt{2}}T_2\right), w_A\left(T_1 - \frac{1-i}{\sqrt{2}}T_2\right)\right\}.$$

Proof. Let $T = \begin{bmatrix} T_1 & T_2 \\ -iT_2 & T_1 \end{bmatrix}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ \frac{1-i}{\sqrt{2}}I & -\frac{1-i}{\sqrt{2}}I \end{bmatrix}$. Therefore, using Lemma 2.6, we have

$$U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U = \begin{bmatrix} T_1^{\#_A} + \frac{1+i}{\sqrt{2}}T_2^{\#_A} & O \\ O & T_1^{\#_A} - \frac{1+i}{\sqrt{2}}T_2^{\#_A} \end{bmatrix}$$
$$= \begin{bmatrix} T_1 + \frac{1-i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1-i}{\sqrt{2}}T_2 \end{bmatrix}^{\#_{\mathbb{A}}}.$$

Using the fact that $w_{\mathbb{A}}(S) = w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}SU)$ for any $S \in \mathcal{L}_A(\mathcal{H})$, we get

$$w_{\mathbb{A}}(T) = w_{\mathbb{A}}(T^{\#_{\mathbb{A}}}) = w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U) = w_{\mathbb{A}}\left(\begin{bmatrix} T_1 + \frac{1-i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1-i}{\sqrt{2}}T_2 \end{bmatrix}^{\#_{\mathbb{A}}}\right)$$
$$= w_{\mathbb{A}}\left(\begin{bmatrix} T_1 + \frac{1-i}{\sqrt{2}}T_2 & O \\ O & T_1 - \frac{1-i}{\sqrt{2}}T_2 \end{bmatrix}\right)$$
$$= \max\left\{w_A\left(T_1 + \frac{1-i}{\sqrt{2}}T_2\right), w_A\left(T_1 - \frac{1-i}{\sqrt{2}}T_2\right)\right\}.$$

□

Example 3.4. *Let* $T_1 = \begin{bmatrix} \frac{2-2i}{\sqrt{2}} & 0 \\ 0 & \frac{4-4i}{\sqrt{2}} \end{bmatrix}$, $T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, *and* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then* $w_A(T_1) = 2$ *and* $w_A(T_2) = 1$. *By Theorem 3.3, we have* $w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -iT_2 & T_1 \end{bmatrix}\right) = \max\{3, 1\} = 3.$

3.2. *Certain* $\mathbb{A}$*-numerical radius inequalities of operator matrices*

Here, we establish our results dealing with different upper and lower bounds for $\mathbb{A}$-numerical radius of 2×2 block operator matrices. The very first result is stated next.

Theorem 3.5. *Let* $T_2, T_3 \in \mathcal{L}_A(\mathcal{H})$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \min\{w_A(T_2), w_A(T_3)\} + \min\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. So, by using Lemma 2.6 (ii) we have

$$U^{\#_{\mathbb{A}}} = \frac{1}{\sqrt{2}}\begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & P_{\overline{\mathcal{R}(A)}} \\ -P_{\overline{\mathcal{R}(A)}} & P_{\overline{\mathcal{R}(A)}} \end{bmatrix}.$$

This in turn implies $UU^{\#_{\mathbb{A}}} = \begin{bmatrix} P_{\overline{\mathcal{R}(A)}} & O \\ O & P_{\overline{\mathcal{R}(A)}} \end{bmatrix} = U^{\#_{\mathbb{A}}}U$. Thus, U is an $\mathbb{A}$-unitary operator. Using the identity $w_{\mathbb{A}}(T) = w_{\mathbb{A}}(U^{\#_{\mathbb{A}}}TU)$, we have

$$\begin{aligned}
w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) &= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}\right) \\
&= w_{\mathbb{A}}\left(U^{\#_{\mathbb{A}}}\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} U\right) \\
&= \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} T_3^{\#_A} + T_2^{\#_A} & T_3^{\#_A} - T_2^{\#_A} \\ -T_3^{\#_A} + T_2^{\#_A} & -T_3^{\#_A} - T_2^{\#_A} \end{bmatrix}\right) \\
&= \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ -(T_2 - T_3) & -(T_2 + T_3) \end{bmatrix}^{\#_{\mathbb{A}}}\right) \\
&= \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ -(T_2 - T_3) & -(T_2 + T_3) \end{bmatrix}\right) \quad (as\ w_A(T) = w_A(T^{\#_{\mathbb{A}}})) \\
&= \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix} + \begin{bmatrix} O & -2T_3 \\ 2T_3 & O \end{bmatrix}\right) \\
&\leq \frac{1}{2}\left\{w_{\mathbb{A}}\left(\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right) + w_{\mathbb{A}}\left(\begin{bmatrix} O & -2T_3 \\ 2T_3 & O \end{bmatrix}\right)\right\}.
\end{aligned}$$

Now, using identity (13) and Lemma 2.4, we have

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \frac{\|T_2 + T_3\|_A}{2} + w_A(T_3). \tag{14}$$

Replacing T_3 by $-T_3$ in the inequality (14) and using Lemma 2.4, we get

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \frac{\|T_2 - T_3\|_A}{2} + w_A(T_3). \tag{15}$$

From the inequalities (14) and (15), we have

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_A(T_3) + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}. \tag{16}$$

Again, in the inequality (16), interchanging T_2 and T_3 and using Lemma 2.4(ii), we get

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq w_A(T_2) + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}. \tag{17}$$

From the inequalities (16) and (17), we get

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \min\{w_A(T_2), w_A(T_3)\} + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}.$$

This completes the proof. □

Remark 3.6. *We give an example to show the bound obtained in Theorem 3.5 is better than the upper bounds obtained in [23, Lemma 2.14] and [23, Theorem 3.2]. If we consider* $T_2 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $T_3 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, *and* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then Theorem 3.5 gives* $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq \min\{2,1\} + \min\{1.5, 0.5\} = 1.5$, *whereas the right hand inequality of [23, Lemma 2.14] and [23, Theorem 3.2] both gives* $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \leq 2$.

Remark 3.7. *In Remark 3.6 it is calculated that* $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) = 1.5$. *So the inequality in Theorem 3.5 is sharp.*

Theorem 3.8. *Let* $T_2, T_3 \in \mathcal{L}_A(\mathcal{H})$. *Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq \max\{w_A(T_2), w_A(T_3)\} - \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}.$$

and

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq \max\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\} - \min\{w_A(T_2), w_A(T_3)\}.$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. It can be shown that U is $\mathbb{A}$-unitary. Then

$$\frac{1}{2}\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}^{\#_{\mathbb{A}}} = U^{\#_{\mathbb{A}}}\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} U - \begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}. \tag{18}$$

So,

$$\begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} = U^{\#_{\mathbb{A}}}\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} U - \frac{1}{2}\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}^{\#_{\mathbb{A}}}. \tag{19}$$

This implies

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}\right) \leq w_{\mathbb{A}}\left(U^{\#_{\mathbb{A}}}\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} U\right) + \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}^{\#_{\mathbb{A}}}\right).$$

Which in turn implies that

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}\right) \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}\right) + \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}\right)$$
$$= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}\right).$$

Thus, using inequality (13) and Lemma 2.4

$$w_A(T_3) \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \frac{\|T_2+T_3\|_A}{2}. \tag{20}$$

Replacing T_3 by $-T_3$ in the inequality (20) we have

$$w_A(T_3) \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \frac{\|T_2-T_3\|_A}{2}. \tag{21}$$

Now from inequality (20) and (21) that

$$w_A(T_3) \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}. \tag{22}$$

Interchanging T_2 and T_3 in the inequality (22), we get

$$w_A(T_2) \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}. \tag{23}$$

From inequalities (22) and (23), we have

$$\max\{w_A(T_2), w_A(T_3)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + \min\left\{\frac{\|T_2+T_3\|_A}{2}, \frac{\|T_2-T_3\|_A}{2}\right\}. \tag{24}$$

Which proves the first inequality.
Again, by identity (18) and inequality (13) that

$$\frac{1}{2}\|T_2+T_3\|_A = \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}\right)$$
$$= \frac{1}{2} w_{\mathbb{A}}\left(\begin{bmatrix} T_2+T_3 & T_2+T_3 \\ -(T_2+T_3) & -(T_2+T_3) \end{bmatrix}^{\#_{\mathbb{A}}}\right)$$
$$\leq w_{\mathbb{A}}\left(U^{\#_{\mathbb{A}}}\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}} U\right) + w_{\mathbb{A}}\left(\begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}\right)$$
$$= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}^{\#_{\mathbb{A}}}\right) + w_{\mathbb{A}}\left(\begin{bmatrix} O & -T_3 \\ T_3 & O \end{bmatrix}\right)$$
$$= w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + w_A(T_3) \ \text{(by Lemma 2.4)}.$$

Thus,

$$\frac{1}{2}\|T_2+T_3\|_A \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + w_A(T_3). \tag{25}$$

Replacing T_3 by $-T_3$ in the inequality (25) and using Lemma 2.4, we get

$$\frac{1}{2}\|T_2 - T_3\|_A \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + w_A(T_3). \tag{26}$$

It follows from inequalities (25) and (26) that

$$\max\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + w_A(T_3). \tag{27}$$

Interchanging T_2 and T_3 in the inequality (27) and using Lemma 2.4, we get

$$\max\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) + w_A(T_2). \tag{28}$$

Now combining (27) and (28), we have

$$\max\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\} - \min\{w_A(T_2), w_A(T_3)\} \leq w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right). \tag{29}$$

This completes the proof. □

Remark 3.9. *Using Theorem 3.5 and Theorem 3.8 we have* $w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix}\right) = w_A(T_2)$ *(see [23, Lemma 2.4(iii)]).*

Theorem 3.10. *Let* $T_2, T_3 \in \mathcal{L}_A(\mathcal{H})$*. Then*

$$w_{\mathbb{A}}^2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq \frac{1}{2}\max\Big\{w_A(T_2T_3 + T_3T_2), w_A(T_2T_3 - T_3T_2)\Big\}.$$

Proof. Let us consider $\mathbb{A}$-unitary operator $U = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$; $U^{\#_{\mathbb{A}}} = \begin{bmatrix} O & P_{\overline{\mathcal{R}(A)}} \\ P_{\overline{\mathcal{R}(A)}} & O \end{bmatrix}$; $T = \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}$. Now,

$$(T^{\#_{\mathbb{A}}})^2 + (U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U)^2 = \begin{bmatrix} T_2T_3 + T_3T_2 & O \\ O & T_3T_2 + T_2T_3 \end{bmatrix}^{\#_{\mathbb{A}}}.$$

So,

$$\begin{aligned} w_{\mathbb{A}}\left(\begin{bmatrix} T_2T_3 + T_3T_2 & O \\ O & T_3T_2 + T_2T_3 \end{bmatrix}\right) &= w_{\mathbb{A}}\left(\begin{bmatrix} T_2T_3 + T_3T_2 & O \\ O & T_3T_2 + T_2T_3 \end{bmatrix}^{\#_{\mathbb{A}}}\right) \\ &= w_{\mathbb{A}}\left((T^{\#_{\mathbb{A}}})^2 + (U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U)^2\right) \\ &\leq w_{\mathbb{A}}\left((T^{\#_{\mathbb{A}}})^2\right) + w_{\mathbb{A}}\left((U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U)^2\right) \\ &\leq w_{\mathbb{A}}^2\left(T^{\#_{\mathbb{A}}}\right) + w_{\mathbb{A}}^2\left(U^{\#_{\mathbb{A}}}T^{\#_{\mathbb{A}}}U\right) \\ &= w_{\mathbb{A}}^2\left(T^{\#_{\mathbb{A}}}\right) + w_{\mathbb{A}}^2\left(T^{\#_{\mathbb{A}}}\right) \\ &= w_{\mathbb{A}}^2(T) + w_{\mathbb{A}}^2(T) \\ &= 2w_{\mathbb{A}}^2(T) \quad \left(as\ w_{\mathbb{A}}(T) = w_{\mathbb{A}}(T^{\#_{\mathbb{A}}})\right). \end{aligned}$$

Hence by using Lemma 2.4 we obtain

$$w_A(T_2T_3 + T_3T_2) \leq 2w_{\mathbb{A}}^2(T). \tag{30}$$

Using similar argument to $(T^{\#_A})^2 - (U^{\#_A}T^{\#_A}U)^2$, we have

$$w_A(T_2T_3 - T_3T_2) \leq 2w_{\mathbb{A}}^2(T). \tag{31}$$

Combining (30) and (31) we get

$$w_{\mathbb{A}}^2\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq \frac{1}{2}\max\left\{w_A(T_2T_3 + T_3T_2), w_A(T_2T_3 - T_3T_2)\right\}.$$

□

Corollary 3.11. *Let $T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) \geq \max\left\{w_A(T_1), w_A(T_4), \frac{1}{\sqrt{2}}\left(w_A(T_2T_3 + T_3T_2)\right)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\left(w_A(T_2T_3 - T_3T_2)\right)^{\frac{1}{2}}\right\}.$$

Proof. Based on Lemma 2.5, Lemma 2.4 and Theorem 3.10 we have

$$\begin{aligned} w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}\right) &\geq \max\left\{w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & O \\ O & T_4 \end{bmatrix}\right), w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right)\right\} \\ &\geq \max\left\{w_A(T_1), w_A(T_4), \frac{1}{\sqrt{2}}\left(w_A(T_2T_3 + T_3T_2)\right)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\left(w_A(T_2T_3 - T_3T_2)\right)^{\frac{1}{2}}\right\}. \end{aligned}$$

□

Theorem 3.12. *Let $T_2, T_3 \in \mathcal{L}_A(\mathcal{H})$. Then for $n \in \mathbb{N}$*

$$w_{\mathbb{A}}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right) \geq \left[\max\{w_A((T_2T_3)^n), w_A((T_3T_2)^n)\}\right]^{\frac{1}{2n}}. \tag{32}$$

Proof. Let $T = \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}$. Then for $n \in \mathbb{N}$, $T^{2n} = \begin{bmatrix} (T_2T_3)^n & O \\ O & (T_3T_2)^n \end{bmatrix}$ and using Lemma 2.4 we obtain

$$\begin{aligned} \max\{w_A((T_2T_3)^n), w_A((T_3T_2)^n)\} &= w_{\mathbb{A}}\left(\begin{bmatrix} (T_2T_3)^n & O \\ O & (T_3T_2)^n \end{bmatrix}\right) \\ &= w_{\mathbb{A}}(T^{2n}) \\ &\leq w_{\mathbb{A}}^{2n}(T) \quad \text{(by inequality (4))} \\ &= w_{\mathbb{A}}^{2n}\left(\begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix}\right). \end{aligned}$$

□

The following lemma is already proved by Hirzallah *et al.* [17] for the case of Hilbert space operators. Using similar technique we can prove this lemma for the case of semi-Hilbert space. Now we state here the result without proof for our purpose.

Lemma 3.13. *Let $T = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \in \mathcal{L}_A(\mathcal{H} \oplus \mathcal{H})$ and $n \in \mathbb{N}$. Then $T^n = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$ for some $P, Q \in \mathcal{L}_A(\mathcal{H})$ such that $P + Q = (T_1 + T_2)^n$ and $P - Q = (T_1 - T_2)^n$.*

The forthcoming result is analogous to Theorem 3.12

Theorem 3.14. *Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \geq \left[\max\{w_A\left(((T_1 - T_2)(T_1 + T_2))^n\right), w_A\left(((T_1 + T_2)(T_1 - T_2))^n\right)\}\right]^{\frac{1}{2n}} \tag{33}$$

for $n \in \mathbb{N}$ and

$$w_{\mathbb{A}}\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq \frac{\max\{\|T_1 + T_2\|_A, \|T_1 - T_2\|_A\}}{2} + \frac{[\max\{\|(T_1 + T_2)(T_1 - T_2)\|_A, \|(T_1 - T_2)(T_1 + T_2)\|_A\}]^{\frac{1}{2}}}{2}. \tag{34}$$

Proof. Let $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}$ and $R = T^2 = \begin{bmatrix} T_1^2 - T_2^2 & T_1T_2 - T_2T_1 \\ T_1T_2 - T_2T_1 & T_1^2 - T_2^2 \end{bmatrix}$. Using Lemma 3.13 we have there exist $P, Q \in \mathcal{L}_A(\mathcal{H})$ such that $R^n = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$ with $P + Q = ((T_1^2 - T_2^2) + (T_1T_2 - T_2T_1))^n$ and $P - Q = ((T_1^2 - T_2^2) - (T_1T_2 - T_2T_1))^n$. So, $T^{2n} = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$ with $P + Q = ((T_1 - T_2)(T_1 + T_2))^n$ and $P - Q = ((T_1 + T_2)(T_1 - T_2))^n$. By using inequality (4), we have

$$\begin{aligned} w_{\mathbb{A}}^{2n}(T) &\geq w_{\mathbb{A}}(T^{2n}) \\ &= w_{\mathbb{A}}\left(\begin{bmatrix} P & Q \\ Q & P \end{bmatrix}\right) \\ &= \max\{w_A(P + Q), w_A(P - Q)\} \quad \text{(by Lemma 2.4)} \\ &= \max\{w_A\left(((T_1 - T_2)(T_1 + T_2))^n\right), w_A\left(((T_1 + T_2)(T_1 - T_2))^n\right)\}. \end{aligned} \tag{35}$$

This proves the inequality (33). In order to prove the inequality (34), let $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}$. Then, using Lemma 2.6 we have $T^{\#\mathbb{A}} = \begin{bmatrix} T_1^{\#A} & -T_2^{\#A} \\ T_2^{\#A} & -T_1^{\#A} \end{bmatrix}$, so
$TT^{\#\mathbb{A}} = \begin{bmatrix} T_1T_1^{\#A} + T_2T_2^{\#A} & -T_1T_2^{\#A} - T_2T_1^{\#A} \\ -T_2T_1^{\#A} - T_1T_2^{\#A} & T_2T_2^{\#A} + T_1T_1^{\#A} \end{bmatrix}$. Now it follows from (2) that

$$\begin{aligned} \|T\|_{\mathbb{A}}^2 &= \|TT^{\#\mathbb{A}}\|_{\mathbb{A}} \\ &= w_A(TT^{\#\mathbb{A}}) \\ &= \max\{w_A(T_1T_1^{\#A} + T_2T_2^{\#A} - T_1T_2^{\#A} - T_2T_1^{\#A}), w_A(T_1T_1^{\#A} + T_2T_2^{\#A} + T_1T_2^{\#A} + T_2T_1^{\#A})\} \\ &\qquad \text{(by Lemma 2.4)} \\ &= \max\{w_A((T_1 - T_2)(T_1 - T_2)^{\#A}), w_A((T_1 + T_2)(T_1 + T_2)^{\#A})\} \\ &= \max\{\|(T_1 - T_2)(T_1 - T_2)^{\#A}\|_A, \|(T_1 + T_2)(T_1 + T_2)^{\#A}\|_A\} \\ &= \max\{\|T_1 - T_2\|_A^2, \|T_1 + T_2\|_A^2\}. \end{aligned}$$

Thus

$$\|T\|_{\mathbb{A}} = \max\{\|T_1 - T_2\|_A, \|T_1 + T_2\|_A\}. \tag{36}$$

Similarly we can show that

$$\|T^2\|_{\mathbb{A}} = \max\{\|(T_1 - T_2)(T_1 + T_2)\|_A, \|(T_1 + T_2)(T_1 - T_2)\|_A\}. \tag{37}$$

From inequality (11), combining inequality (36) and (37), we obtain

$$
\begin{aligned}
w_{\mathbb{A}}(T) &\leq \frac{1}{2}(\|T\|_{\mathbb{A}} + \|T^2\|_{\mathbb{A}}^{1/2}) \\
&= \frac{\max\{\|T_1 + T_2\|_A, \|T_1 - T_2\|_A\}}{2} \\
&+ \frac{[\max\{\|(T_1 + T_2)(T_1 - T_2)\|_A, \|(T_1 - T_2)(T_1 + T_2)\|_A\}]^{\frac{1}{2}}}{2}.
\end{aligned}
$$

□

3.3. Some A-numerical radius inequalities for operators

In this subsection we establish some upper bounds for A-numerical radius of operators. In the next result, we derive an upper bound for A-numerical radius of product of operators on semi-Hilbertian space.

Theorem 3.15. *Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then*

$$w_A(T_1T_2) \leq \frac{1}{2}\left(\|T_2T_1\|_A + \|T_1\|_A\|T_2\|_A\right).$$

Proof. It is not difficult to see that $\Re_A(e^{i\theta}T_1T_2)$ is an A-selfadjoint operator. So, by Lemma 2.2 we have

$$\left\|\Re_A(e^{i\theta}T_1T_2)\right\|_A = w_A(\Re_A(e^{i\theta}T_1T_2)).$$

So,

$$
\begin{aligned}
\left\|\Re_A(e^{i\theta}T_1T_2)\right\|_A &= \frac{1}{2}w_A\left(e^{i\theta}T_1T_2 + e^{-i\theta}T_2^{\#_A}T_1^{\#_A}\right) \\
&= \frac{1}{2}w_{\mathbb{A}}\left(\begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^{\#_A}T_1^{\#_A} & O \\ O & O \end{bmatrix}\right).
\end{aligned}
$$

It can observed that

$$
\begin{aligned}
\begin{bmatrix} A & O \\ O & A \end{bmatrix}\begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^{\#_A}T_1^{\#_A} & O \\ O & O \end{bmatrix} &= \begin{bmatrix} e^{i\theta}AT_1T_2 + e^{-i\theta}AT_2^{\#_A}T_1^{\#_A} & O \\ O & O \end{bmatrix} \\
&= \begin{bmatrix} e^{i\theta}(T_2^{\#_A}T_1^{\#_A})^*A + e^{-i\theta}(T_1T_2)^*A & O \\ O & O \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\theta}T_2^{\#_A}T_1^{\#_A} + e^{i\theta}T_1T_2 & O \\ O & O \end{bmatrix}^* \begin{bmatrix} A & O \\ O & A \end{bmatrix}.
\end{aligned}
$$

Hence $\begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^{\#_A}T_1^{\#_A} & O \\ O & O \end{bmatrix}$ is $\mathbb{A}$-selfadjoint operator.
So by applying Lemma 2.2 we see that

$$
\begin{aligned}
\left\|\Re_A(e^{i\theta}T_1T_2)\right\|_A &= \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^{\#_A}T_1^{\#_A} & O \\ O & O \end{bmatrix}\right) \\
&= \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} e^{i\theta}T_1 & T_2^{\#_A} \\ O & O \end{bmatrix}\begin{bmatrix} T_2 & O \\ e^{-i\theta}T_1^{\#_A} & O \end{bmatrix}\right).
\end{aligned}
$$

So, by using (10) we have

$$\begin{aligned}\left\|\Re_A(e^{i\theta}T_1T_2)\right\|_A &= \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} T_2 & O \\ e^{-i\theta}T_1^{\#_A} & O\end{bmatrix}\begin{bmatrix} e^{i\theta}T_1 & T_2^{\#_A} \\ O & O\end{bmatrix}\right)\\ &= \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} e^{i\theta}T_2T_1 & T_2T_2^{\#_A} \\ T_1^{\#_A}T_1 & T_1^{\#_A}T_2^{\#_A}\end{bmatrix}\right)\\ &\le \frac{1}{2}r\left(\begin{bmatrix} \|T_2T_1\|_A & \|T_2T_2^{\#_A}\|_A \\ \|T_1^{\#_A}T_1\|_A & \|T_1^{\#_A}T_2^{\#_A}\|_A\end{bmatrix}\right) \quad \text{(by Lemma 2.3)}\\ &= \frac{1}{2}\left(\|T_2T_1\|_A + \|T_1\|_A\|T_2\|_A\right).\end{aligned}$$

So, by taking supremum over $\theta \in \mathbb{R}$, then using (6) we get our desired result. □

We conclude the article with the following result which is a refinement of inequality (8). To do this we need the following lemma.

Lemma 3.16. *Let $z, y \in \mathcal{H}$ and $\lambda \in \mathbb{R}$, then*

$$\|z\|_A^2\|y\|_A^2 - |\langle z, y\rangle_A|^2 \le \|z\|_A^2\|y - \lambda z\|_A^2.$$

Proof. Since $|\Re_A\langle z, y\rangle_A| \le |\langle z, y\rangle_A|$, so the discriminant of the quadratic polynomial $p(\lambda) = \|z\|_A^4\lambda^2 - 2\Re_A\langle z, y\rangle_A\|z\|_A^2\lambda + |\langle z, y\rangle_A|^2$ is not positive, which implies that $p(\lambda) \ge 0$ for all $\lambda \in \mathbb{R}$. Hence

$$\|z\|_A^2\|y\|_A^2 - |\langle z, y\rangle_A|^2 \le \|z\|_A^4\lambda^2 - 2\Re_A\langle z, y\rangle_A\|z\|_A^2\lambda + \|z\|_A^2\|y\|_A^2 = \|z\|_A^2\|y - \lambda z\|_A^2.$$

□

The following result is a refinement of the inequality (8).

Theorem 3.17. *Let $T \in \mathcal{L}_A(\mathcal{H})$, $\alpha \in \mathbb{C} - \{0\}$ and $r \in \mathbb{R}$ are such that $\|T - \alpha I\|_A \le r$. Then for $r < |\alpha|$*

$$\frac{\|T\|_A}{2} \le \sqrt{1 - \frac{r^2}{|\alpha|^2}}\|T\|_A \le w_A(T). \tag{38}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$, put $z = Tx,\ y = \alpha x$ in Lemma 3.16, we have

$$\|Tx\|_A^2\|\alpha x\|_A^2 - |\langle Tx, \alpha x\rangle_A|^2 \le \|Tx\|_A^2\|\lambda Tx - \alpha x\|_A^2,$$

so

$$\|Tx\|_A^2 - |\langle Tx, x\rangle_A|^2 \le \|Tx\|_A^2\frac{\|\lambda Tx - \alpha x\|_A^2}{|\alpha|^2}.$$

Taking supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have

$$\|T\|_A^2 - w_A^2(T) \le \|T\|_A^2\frac{\|\lambda T - \alpha I\|_A^2}{|\alpha|^2}.$$

Since $\|T - \alpha I\|_A \le r$, taking $\lambda = 1$ gives

$$\left(1 - \frac{r^2}{|\alpha|^2}\right)\|T\|_A^2 \le w_A^2(T).$$

Which completes the proof. □

Remark 3.18. *Notice that the inequality* (38) *already proved by Saddi [24]. We remark here that the method we use to prove inequality* (38) *is different from the methods presented in [24].*

Very recentely, the following lemma is proved by Xu *et al.* [27]. We state here the result for our purpose to prove another inequality.

Lemma 3.19. *Let $x, y, z \in \mathcal{H} \oplus \mathcal{H}$ with $\|x\|_{\mathbb{A}} = 1$. Then*

$$2|\langle z, x\rangle_{\mathbb{A}}\langle x, y\rangle_{\mathbb{A}}| \leq (\|z\|_{\mathbb{A}}\|y\|_{\mathbb{A}} + |\langle z, y\rangle_{\mathbb{A}}|).$$

Using the inequality (38), we have the following result.

Theorem 3.20. *Let $T \in \mathcal{L}_A(\mathcal{H})$, $\alpha \in \mathbb{C} - \{0\}$ and $r \in \mathbb{R}$ are such that $\|T - \alpha I\|_A \leq r$. Then for $r < |\alpha|$*

$$\left(2 - \frac{|\alpha|^2}{|\alpha|^2 - r^2}\right) w_A^2(T) \leq w_A(T^2). \tag{39}$$

Proof. Putting $z = Tx,\ y = T^{\#_A}x$ with $\|x\|_A = 1$ in Lemma 3.19, we get

$$2|\langle Tx, x\rangle_A|^2 \leq \|Tx\|_A\|T^{\#_A}x\|_A + |\langle T^2x, x\rangle_A|.$$

Taking supremum over $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have

$$2w_A^2(T) \leq w_A(T^2) + \|T\|_A^2.$$

Using inequality (38), we have

$$2w_A^2(T) \leq w_A(T^2) + \frac{|\alpha|^2}{|\alpha|^2 - r^2} w_A^2(T).$$

Hence

$$\left(2 - \frac{|\alpha|^2}{|\alpha|^2 - r^2}\right) w_A^2(T) \leq w_A(T^2).$$

□

Acknowledgements

The author thank the referee for the valuable suggestions and comments on an earlier version. Incorporating appropriate responses to these in the article has led to a better presentation of the results. We also thank the **Government of India** for introducing the *work from home initiative* during the COVID-19 pandemic.

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