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Certain Properties of Lupaş–Kantorovich Operators

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Abstract. In the present paper we establish a link between the Lupaş operators and its Kantorovich type integral modification, by applying the methods of finite differences, also the difference between two operators is established in weighted spaces. Further, we consider a modification of the Kantorovich variant, which preserve the test functions $e^{\nu jx}, j = 1, 2$ and estimate a Voronovskaja type convergence estimate.

1. Introduction

In the recent years several problems concerning convergence of operators have been discussed by researchers, see for example [1], [2], [3], [4], [5], [10], [13] and [16] etc. In the recent book [12] a collection of moments of some operators is provided, based on different methods. A distinguished Romanian mathematician Alexandru Lupaş in [14] proposed an important discrete operator, which for $x \geq 0, n \in \mathbb{N}$ is defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} l_k(nx) F_{n,k}(f) = \sum_{k=0}^{\infty} l_k(nx) f\left(\frac{k}{n}\right), \tag{1}$$

where

$$l_k(nx) = \frac{(nx)_k}{2^k \cdot k!} \cdot 2^{-nx}$$

and the rising factorial is given by $(y)_m = \prod_{i=0}^{m-1}(y+i), m \geq 1; \quad (y)_0 = 1$. These operators are different from the other operators of exponential type (see [15]), as we can not find a function $p(x)$, such that $p(x)(L_n f)'(x) = n(L_n \psi_x^1 f), \psi_x^m = (e_1 - xe_0)^m, e_i(t) = t^i, i = 0, 1, 2, 3, ...$, which is the necessary condition for an operator to be of exponential type. If we denote $b^{F_{n,k}} = F_{n,k}(e_1) = \frac{k}{n}$, then $\mu_r^{F_{n,k}} = F_{n,k}\left(e_1 - e_0 F_{n,k}(e_1)\right)^r = 0$.

Remark 1.1. *Using the identity* $\frac{1}{(1-a)^\beta} = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} a^k, \ \ |a| < 1,$ *we have*

$$L_n(e^{\lambda t}, x) = (2 - e^{\lambda/n})^{-nx},$$

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which provides the moment generating function (abrv. m.g.f.) of the Lupaş operators, and if we denote the r-th order moment of the operators (1) are given by $(L_n e_r)(x), e_r(t) = t^r$*, then, few moments are given by*

$$\begin{aligned}
(L_n e_0)(x) &= 1\\
(L_n e_1)(x) &= x\\
(L_n e_2)(x) &= x^2 + \frac{2x}{n}\\
(L_n e_3)(x) &= x^3 + \frac{6x^2}{n} + \frac{6x}{n^2}\\
(L_n e_4)(x) &= x^4 + \frac{12x^3}{n} + \frac{36x^2}{n^2} + \frac{26x}{n^3}.
\end{aligned}$$

The Kantorovich variant of Lupaş operators defined by Agratini [6] is defined as follows:

$$(K_n f)(x) = \sum_{k=0}^{\infty} l_k(nx) G_{n,k}(f), x \geq 0 \tag{2}$$

where

$$G_{n,k}(f) = n\int_{k/n}^{(k+1)/n} f(t)dt.$$

Remark 1.2. *By simple computation, we have* $G_{n,k}(e_1) = \frac{2k+1}{2n}$ *and*

$$\begin{aligned}
\mu_2^{G_{n,k}} &= G_{n,k}\left(e_1 - e_0 G_{n,k}(e_1)\right)^2 = \frac{1}{12n^2},\\
\mu_6^{G_{n,k}} &= G_{n,k}\left(e_1 - e_0 G_{n,k}(e_1)\right)^6\\
&= G_{n,k}(e_6) - 6G_{n,k}(e_5)G_{n,k}(e_1) + 15G_{n,k}(e_4)\left(G_{n,k}(e_1)\right)^2 - 20G_{n,k}(e_3)\left(G_{n,k}(e_1)\right)^3\\
&\quad +15G_{n,k}(e_2)\left(G_{n,k}(e_1)\right)^4 - 6G_{n,k}(e_1)\left(G_{n,k}(e_1)\right)^5 + G_{n,k}(e_0)\left(G_{n,k}(e_1)\right)^6\\
&= \frac{\left((k+1)^7 - k^7\right)}{7n^6} - 6\frac{\left((k+1)^6 - k^6\right)}{6n^5}\left(\frac{2k+1}{2n}\right) + 15\frac{\left((k+1)^5 - k^5\right)}{5n^4}\left(\frac{2k+1}{2n}\right)^2\\
&\quad -20\frac{\left((k+1)^4 - k^4\right)}{4n^3}\left(\frac{2k+1}{2n}\right)^3 + 15\frac{\left((k+1)^3 - k^3\right)}{3n^2}\left(\frac{2k+1}{2n}\right)^4\\
&\quad -6\frac{\left((k+1)^2 - k^2\right)}{2n}\left(\frac{2k+1}{2n}\right)^5 + \left(\frac{2k+1}{2n}\right)^6\\
&= \frac{1}{448n^6}.
\end{aligned}$$

Remark 1.3. *By simple computation, the moment generating function of the Lupaş-Kantorovich operators (2) is given by*

$$(K_n e^{At})(x) = \frac{n(e^{A/n} - 1)}{A}\left(2 - e^{A/n}\right)^{-nx}.$$

The r-th order moment of (2) satisfy

$$(K_n e_r)(x) = \left[\frac{\partial^r}{\partial \lambda^r}\frac{n(e^{\lambda/n} - 1)}{\lambda}\left(2 - e^{\lambda/n}\right)^{-nx}\right]_{\lambda=0}.$$

Using this expression, some of the moments are given by

$$\begin{aligned}(K_ne_0)(x) &= 1\\(K_ne_1)(x) &= x+\frac{1}{2n}\\(K_ne_2)(x) &= x^2+\frac{3x}{n}+\frac{1}{3n^2}\\(K_ne_3)(x) &= x^3+\frac{15x^2}{2n}+\frac{10x}{n^2}+\frac{1}{4n^3}\\(K_ne_4)(x) &= x^4+\frac{14x^3}{n}+\frac{50x^2}{n^2}+\frac{43x}{n^3}+\frac{1}{5n^4}.\end{aligned}$$

We may point out that there was a minor misprint in the first moment given in [6, pp. 48]. Also, it is seen from Remark 1.3 that the operators (K_nf) preserve only constant function.

The link between the Lupaş operators (1) and its Kantorovich variant was not established earlier, as standard differential operator does not work for such operators. In the present article we establish a connection between the two operators, using the methods of finite differences. We also provide the difference between Lupaş operator and its Kantorovich variant. In the last section, we modify the Kantorovich variant in such a way that exponential functions are preserved and for such operators, we establish a Voronovskaja type convergence result.

2. Link between Lupaş and its Kantorovich variant

Theorem 2.1. *We have the following link between Lupaş and its Kantorovich variant*

$$(K_nf)=(\nabla\circ L_n\circ F), \tag{3}$$

where $F(x)=n\int_0^x f(t)dt$*, and* ∇ *is the backward difference operator for the function* $f(nx)$ *with unit step length.*

Proof. Obviously, we have

$$\nabla(nx)_k=k(nx)_{k-1}$$

and

$$\nabla 2^{-nx}=2^{-nx}-2^{-nx+1}.$$

Thus using the identity

$$\nabla(fg)=f(\nabla g)+g(\nabla f)-(\nabla f)(\nabla g),$$

we can write

$$\begin{aligned}\nabla((nx)_k2^{-nx}) &= (nx)_k.(2^{-nx}-2^{-nx+1})+k(nx)_{k-1}2^{-nx}-k(nx)_{k-1}(2^{-nx}-2^{-nx+1})\\&= (nx)_k(2^{-nx}-2^{-nx+1})+k(nx)_{k-1}2^{-nx+1}\\&= (nx)_k2^{-nx}-2^{-nx+1}[(nx)_k-k(nx)_{k-1}]\\&= 2^{-nx+1}k(nx)_{k-1}-2^{-nx}(nx)_k,\end{aligned}$$

implying

$$\begin{aligned}\nabla(l_k(nx)) &= l_k(nx)-2l_k(nx)+l_{k-1}(nx)\\&= l_{k-1}(nx)-l_k(nx).\end{aligned}$$

We start with

$$\begin{aligned}
(\nabla \circ L_n \circ F)(x) &= \nabla\left((L_nF)(x)\right) = \nabla\left(\sum_{k=0}^{n} l_k(nx)F\left(\frac{k}{n}\right)\right)\\
&= \sum_{k=0}^{\infty}(\nabla l_k(nx))F\left(\frac{k}{n}\right)\\
&= \sum_{k=0}^{\infty}\left[l_{k-1}(nx) - l_k(nx)\right]F\left(\frac{k}{n}\right)\\
&= \sum_{k=0}^{\infty} l_k(nx)\left(F\left(\frac{k+1}{n}\right) - F\left(\frac{k}{n}\right)\right)\\
&= n\sum_{k=0}^{\infty} l_k(nx)\int_{k/n}^{(k+1)/n} f(t)dt = (K_nf)(x).
\end{aligned}$$

This completes the proof of the theorem. □

Remark 2.2. *Using the link between the backward difference* ∇ *with the unit step length and the differential operator* D*, in our Theorem 2.1,* ∇ *can be replaced by* $1 - e^{-D/n}$.

Remark 2.3. *Here we provide the application of Theorem 2.1, to verify the link between two operators for calculating few moments. By (3) and applying Remark 1.1, we have*

$$\begin{aligned}
(\nabla \circ L_n \circ Fe_0)(x) &= \nabla \circ L_n \circ nx\\
&= \nabla \circ nx = nx - (nx-1) = 1 = (K_ne_0)(x).
\end{aligned}$$

Next, we have

$$\begin{aligned}
(\nabla \circ L_n \circ Fe_1)(x) &= \nabla \circ L_n \circ \frac{nx^2}{2}\\
&= \frac{n}{2}\nabla \circ \left(x^2 + \frac{2x}{n}\right)\\
&= \frac{1}{2n}\left[n^2x^2 - (nx-1)^2\right] + \frac{1}{n}[nx - (nx-1)]\\
&= x - \frac{1}{2n} + \frac{1}{n} = x + \frac{1}{2n} = (K_ne_1)(x).
\end{aligned}$$

Also, we can write

$$\begin{aligned}
(\nabla \circ L_n \circ Fe_2)(x) &= \nabla \circ L_n \circ \frac{nx^3}{3}\\
&= \frac{n}{3}\nabla \circ \left(x^3 + \frac{6x^2}{n} + \frac{6x}{n^2}\right)\\
&= \frac{1}{3n^2}\left[n^3x^3 - (nx-1)^3\right] + \frac{2}{n^2}[n^2x^2 - (nx-1)^2] + \frac{2}{n^2}[nx - (nx-1)]\\
&= x^2 + \frac{3x}{n} + \frac{1}{3n^2} = (K_ne_2)(x).
\end{aligned}$$

Thus by knowing the moments of Lupaş operators, one can find the other moments of Lupaş-Kantorovich operators in a similar manner.

3. Differences between Lupaş and its Kantorovich variant

Let $B_2[0,\infty)$ be the set of all functions f defined on positive real line with some constant $C(f)$ depending only on f, satisfying the condition $\left|f(x)\right| \le C(f)\left(1+x^2\right)$. Let $C_2[0,\infty) = C[0,\infty) \cap B_2[0,\infty)$ and by $C_2^*[0,\infty)$, we denote subspace of all continuous functions $f \in B_2[0,\infty)$ for which $\lim_{x\to\infty} |f(x)|(1+x^2)^{-1} < \infty$.

Following [7], [11] (see also [8]), the difference between L_n and K_n is given by:

Theorem A. Let $f \in C_2[0,\infty)$ with $f'' \in C_2^*[0,\infty)$. Then

$$\left|(L_n - K_n)(f,x)\right| \le \frac{1}{2}\left\|f''\right\|_2 \beta(x) + 8\Omega(f'',\delta_1)(1+\beta(x)) + 16\Omega(f,\delta_2)(\gamma(x)+1),$$

where

$$\begin{aligned}
\beta(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left\{\left(1+(F_{n,k}(e_1))^2\right)\mu_2^{F_{n,k}} + \left(1+(G_{n,k}(e_1))^2\right)\mu_2^{G_{n,k}}\right\},\\
\gamma(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+((FG)_{n,k}(e_1))^2\right),\\
\delta_1^4(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left\{\left(1+(F_{n,k}(e_1))^2\right)\mu_6^{F_{n,k}} + \left(1+(G_{n,k}(e_1))^2\right)\mu_6^{G_{n,k}}\right\}
\end{aligned}$$

and

$$\delta_2^4(x) = \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+((FG)_{n,k}(e_1))^2\right)(F_{n,k}(e_1) - G_{n,k}(e_1))^4,$$

where $(FG)_{n,k}(e_1) = min\{F_{n,k}(e_1), G_{n,k}(e_1)\}$ we suppose that $\delta_1(x) \le 1, \delta_2(x) \le 1$ and $\Omega(f,\delta)$ is the weighted modulus of continuity given by

$$\Omega(f,\delta) = \sup_{|h|<\delta, x\in[0,\infty)} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)},$$

for each $f \in C_2[0,\infty)$.

Quantitative estimates for differences of Lupaş operators and Lupaş-Kantorovich operators are given as:

Theorem 3.1. *Let $f \in C_2^*[0,\infty)$ with $f'' \in C_2^*[0,\infty)$. Then*

$$\begin{aligned}
\left|(L_n - K_n)(f,x)\right| \le\; & \frac{1}{8}\left(\frac{x^2}{3n^2} + \frac{x}{n^3} + \frac{4n^2+1}{12n^4}\right)\|f''\|_2\\
& +8\left(1 + \frac{x^2}{12n^2} + \frac{x}{4n^3} + \frac{4n^2+1}{48n^4}\right)\Omega\left(f'', \frac{x^2}{448n^6} + \frac{3x}{448n^7} + \frac{4n^2+1}{1792n^8}\right)\\
& +16\left(x^2 + \frac{2x}{n} + 2\right)\Omega\left(f, \frac{(x^2+1)}{16n^4} + \frac{x}{8n^5}\right).
\end{aligned}$$

Proof. Using Remark 1.1 and Remark 1.2, we get

$$\begin{aligned}\beta(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left\{\left(1+\left(b^{F_{n,k}}\right)^2\right)\mu_2^{F_{n,k}}+\left(1+\left(b^{G_{n,k}}\right)^2\right)\mu_2^{G_{n,k}}\right\}\\ &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+\left(\frac{2k+1}{2n}\right)^2\right)\left(\frac{1}{12n^2}\right)\\ &= \frac{x^2}{12n^2}+\frac{x}{4n^3}+\frac{4n^2+1}{48n^4},\end{aligned}$$

and $b^{(FG)_{n,k}} = min\left\{\frac{k}{n}, \frac{2k+1}{2n}\right\} = \frac{k}{n}$, thus

$$\begin{aligned}\gamma(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+\left(b^{(FG)_{n,k}}\right)^2\right)\\ &= x^2+\frac{2x}{n}+1.\end{aligned}$$

Also

$$\begin{aligned}\delta_1^4(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left\{\left(1+\left(b^{F_{n,k}}\right)^2\right)\mu_6^{F_{n,k}}+\left(1+\left(b^{G_{n,k}}\right)^2\right)\mu_6^{G_{n,k}}\right\}\\ &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+\left(\frac{2k+1}{2n}\right)^2\right)\left(\frac{1}{448n^6}\right)\\ &= \frac{x^2}{448n^6}+\frac{3x}{448n^7}+\frac{4n^2+1}{1792n^8}\end{aligned}$$

and

$$\begin{aligned}\delta_2^4(x) &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+\left(b^{(FG)_{n,k}}\right)^2\right)\left(b^{F_{n,k}}-b^{G_{n,k}}\right)^4\\ &= \sum_{k=0}^{\infty} l_{n,k}(x)\left(1+\left(\frac{k}{n}\right)^2\right)\left(\frac{k}{n}-\frac{2k+1}{2n}\right)^4\\ &= \frac{(x^2+1)}{16n^4}+\frac{x}{8n^5}.\end{aligned}$$

Arrange these terms as in Theorem A, we get the proof of the theorem. □

Example 1. The following graph represents the difference between two operators.

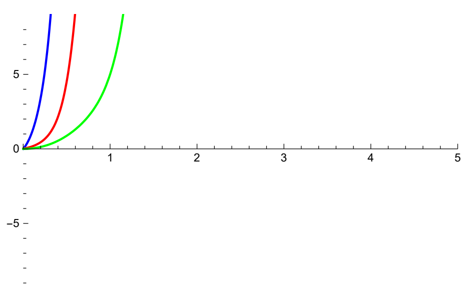

Figure 1: The difference between $(L_n f)$ (Blue) and $(K_n f)$ (Red) for the function $f = x^9 + x^3 + 3x^2$. (Green) and $n = 10$.

4. Preservation of $e^{\nu jx}$, $j = 1, 2$

In order to preserve exponential function, we start with the following form of Lupaş-Kantorovich operators

$$(\widehat{K}_n f)(x) = n \sum_{k=0}^{\infty} \frac{(nb_n(x))_k}{2^k \cdot k!} \cdot 2^{-nb_n(x)} \int_{k/n}^{(k+1)/n} f(t)dt \tag{4}$$

Suppose these operators preserve $e^{\nu x}$, then

$$\begin{aligned}(\widehat{K}_n e^{\nu t})(x) = e^{\nu x} &= n \sum_{k=0}^{\infty} \frac{(nb_n(x))_k}{2^k \cdot k!} \cdot 2^{-nb_n(x)} \int_{k/n}^{(k+1)/n} e^{\nu t}dt \\ &= \frac{n(e^{\nu/n}-1)}{\nu}\left(2-e^{\nu/n}\right)^{-nb_n(x)},\end{aligned}$$

implying

$$b_n(x) = \frac{\log n(e^{\nu/n}-1) - \log \nu - \nu x}{n \log(2-e^{\nu/n})}.$$

Obviously $\lim_{n\to\infty} b_n(x) = x$. Now based on this we define the following operators

$$(\widetilde{K}_n f)(x) = n \sum_{k=0}^{\infty} e^{\nu x} \frac{(nb_n(x))_k}{2^k \cdot k!} \cdot 2^{-nb_n(x)} \int_{k/n}^{(k+1)/n} e^{-\nu t} f(t)dt. \tag{5}$$

These operators (5) reproduce $e^{\nu x}$ and $e^{2\nu x}$, but loose to preserve the constant function.

Lemma 4.1. *Let $\nu > 0$, for each $n \in N$ and $x \in [0, \infty)$, the following identities hold*

$$\begin{aligned}\widetilde{K}_n(e_0, x) &= e^{\nu x} \frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}, \\ \widetilde{K}_n(e^{3\nu t}, x) &= e^{\nu x} \frac{n(e^{2\nu/n}-1)}{2\nu}\left(2-e^{2\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}, \\ \widetilde{K}_n(e^{4\nu t}, x) &= e^{\nu x} \frac{n(e^{3\nu/n}-1)}{3\nu}\left(2-e^{3\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}.\end{aligned}$$

Proof. By using Remark 1.3, we obtain

$$\begin{aligned}\widetilde{K}_n(e_0, x) &= e^{\nu x} \frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{-nb_n(x)} \\ &= e^{\nu x} \frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}.\end{aligned}$$

Next

$$\begin{aligned}\widetilde{K}_n(e^{3\nu t}, x) &= e^{\nu x} \frac{n(e^{2\nu/n}-1)}{2\nu}\left(2-e^{2\nu/n}\right)^{-nb_n(x)} \\ &= e^{\nu x} \frac{n(e^{2\nu/n}-1)}{2\nu}\left(2-e^{2\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}.\end{aligned}$$

Finally Next

$$\begin{aligned}\widetilde{K}_n(e^{4\nu t}, x) &= e^{\nu x} \frac{n(e^{3\nu/n}-1)}{3\nu}\left(2-e^{3\nu/n}\right)^{-nb_n(x)} \\ &= e^{\nu x} \frac{n(e^{3\nu/n}-1)}{3\nu}\left(2-e^{3\nu/n}\right)^{\frac{\log \nu + \nu x - \log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}.\end{aligned}$$

□

Remark 4.2. *Let us consider* $\exp_{\nu,x}(t) = e^{\nu t} - e^{\nu x}$, *then*

$$\widetilde{K}_n(\exp^1_{\nu,x}(t), x) = e^{\nu x}\left[1 - e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\right].$$

$$\begin{aligned}\widetilde{K}_n(\exp^2_{\nu,x}(t), x) &= \widetilde{K}_n(e^{2\nu t}, x) - 2e^{\nu x}\widetilde{K}_n(e^{\nu t}, x) + e^{2\nu x}\widetilde{K}_n(e_0, x)\\ &= e^{2\nu x}\left[e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}} - 1\right].\end{aligned}$$

and

$$\begin{aligned}\widetilde{K}_n(\exp^4_{\nu,x}(t), x) &= \widetilde{K}_n(e^{4\nu t}, x) - 4e^{\nu x}\widetilde{K}_n(e^{3\nu t}, x)\\ &\quad +6e^{2\nu x}\widetilde{K}_n(e^{2\nu t}, x) - 4e^{3\nu x}\widetilde{K}_n(e^{\nu t}, x) + e^{4\nu x}\widetilde{K}_n(e_0, x)\\ &= e^{\nu x}\Bigg[\frac{n(e^{3\nu/n}-1)}{3\nu}\left(2-e^{3\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\\ &\quad -4e^{\nu x}\frac{n(e^{2\nu/n}-1)}{2\nu}\left(2-e^{2\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\\ &\quad +2e^{3\nu x} + e^{4\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\Bigg].\end{aligned}$$

Remark 4.3. *By simple computation using the mathematical software, we have*

$$\lim_{n\to\infty} n\left(e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}} - 1\right) = 2\nu^2 x$$

and

$$\begin{aligned}&\lim_{n\to\infty} n^2\Bigg\{e^{\nu x}\Bigg[\frac{n(e^{3\nu/n}-1)}{3\nu}\left(2-e^{3\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\\ &-4e^{\nu x}\frac{n(e^{2\nu/n}-1)}{2\nu}\left(2-e^{2\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\\ &+2e^{3\nu x} + e^{4\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\Bigg]\Bigg\} = 0.\end{aligned}$$

The space $C^*[0,\infty)$ is a small subspace of $C[0,\infty)$ of real-valued continuous functions on $x \geq 0$ where $\lim_{x\to\infty} f(x)$ exists and is finite.

We consider for a fixed $\nu > 0$ the exponential function $\exp_\nu(t) = e^{\nu t}$ and its inverse $\log_\nu$ is the logarithmic function with base e^ν.

Theorem 4.4. *If* $f \in C^*[0,\infty)$ *has a second derivative at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} n\left((\widetilde{K}_n f)(x) - f(x)\right) = 2\nu^2 x f(x) - 3\nu x f'(x) + x f''(x).$$

Proof. By Taylor's theorem, we obtain

$$\begin{aligned}f(t) &= \left(f\circ\log_\nu\right)\left(e^{\nu t}\right) = \left(f\circ\log_\nu\right)(e^{\nu x}) + \left(f\circ\log_\nu\right)'(e^{\nu x})\exp_{\nu,x}(t)\\ &\quad +\frac{\left(f\circ\log_\nu\right)''(e^{\nu x})}{2}\exp^2_{\nu,x}(t) + h_x(t)\exp^2_{\nu,x}(t),\end{aligned}$$

where $h_x(t) := h(e^{\nu t} - e^{\nu x})$ and h continuous function, which vanishes at 0. Applying the operator $\widetilde{L}_n$ and then evaluating at the point x, we get

$$\begin{aligned}(\widetilde{K}_n f)(x) &= f(x)(\widetilde{K}_n e_0)(x) + \left(f\circ \log_\nu\right)'(e^{\nu x})(\widetilde{K}_n \exp^1_{\nu,x}(t))(x)\\ &+\frac{\left(f\circ\log_\nu\right)''(e^{\nu x})}{2}\widetilde{K}_n\left(\exp^2_{\nu,x}(t);x\right)+\widetilde{K}_n\left(h_x(t)\exp^2_{\nu,x}(t);x\right).\end{aligned}$$

Since

$$\left(f\circ\log_\nu\right)'(e^{\nu x}) = e^{-\nu x}\nu^{-1}f'(x)$$

and

$$\left(f\circ\log_\nu\right)''(e^{\nu x}) = e^{-2\nu x}\left(\nu^{-2}f''(x)-\nu^{-1}f'(x)\right),$$

we have

$$\begin{aligned}(\widetilde{K}_n f)(x) - f(x) &= f(x)\left((\widetilde{K}_n e_0)(x)-1\right) + e^{-\nu x}\nu^{-1}f'(x)(\widetilde{K}_n\exp^1_{\nu,x}(t))(x)\\ &+\frac{e^{-2\nu x}\left(\nu^{-2}f''(x)-\nu^{-1}f'(x)\right)}{2}(\widetilde{K}_n\exp^2_{\nu,x}(t))(x) + (\widetilde{K}_n h_x(t)\exp^2_{\nu,x}(t))(x)\\ &= f(x)\left(e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}-1\right)\\ &+e^{-\nu x}\nu^{-1}f'(x)e^{\nu x}\left[1-e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}\right]\\ &+\frac{e^{-2\nu x}\left(\nu^{-2}f''(x)-\nu^{-1}f'(x)\right)}{2}e^{2\nu x}\left[e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}-1\right]\\ &+(\widetilde{K}_n h_x(t)\exp^2_{\nu,x}(t))(x)\\ &= \left(e^{\nu x}\frac{n(1-e^{-\nu/n})}{\nu}\left(2-e^{-\nu/n}\right)^{\frac{\log\nu+\nu x-\log n(e^{\nu/n}-1)}{\log(2-e^{\nu/n})}}-1\right)\left[f(x)-\frac{3}{2\nu}f'(x)+\frac{1}{2\nu^2}f''(x)\right]\\ &+(\widetilde{K}_n h_x(t)\exp^2_{\nu,x}(t))(x).\end{aligned}$$

Using Cauchy-Schwarz inequality, we can write

$$n\left|(\widetilde{K}_n h_x(t)\exp^2_{\nu,x}(t))(x)\right| \le \sqrt{(\widetilde{K}_n h_x^2(t))(x)}\sqrt{n^2(\widetilde{K}_n\exp^4_{\nu,x}(t))(x)}.$$

Also, we have $\lim_{n\to\infty}(\widetilde{K}_n h_x^2(t))(x) = h_x^2(x) = 0$, by simple computations, it follows that:

$$\lim_{n\to\infty} n\left|(\widetilde{K}_n h_x(t)\exp^2_{\nu,x}(t))(x)\right| = 0.$$

This completes the proof of the theorem. □

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