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Weakly S-Artinian Modules

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Abstract. Let R be a ring, S a multiplicative subset of R and M a left R-module. We say M is a weakly S-Artinian module if every descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of submodules of M is weakly S-stationary, i.e., there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n N_k \subseteq N_n$ for some $s_n \in S$. One aim of this paper is to study the class of such modules. We show that over an integral domain, weakly S-Artinian forces S to be $R \setminus \{0\}$, whenever S is a saturated multiplicative set. Also we investigate conditions under which weakly S-Artinian implies Artinian. In the second part of this paper, we focus on multiplicative sets with no zero divisors. We show that with such a multiplicative set, a semiprime ring with weakly S-Artinian on left ideals and essential left socle is semisimple Artinian. Finally, we close the paper by showing that over a perfect ring weakly S-Artinian and Artinian are equivalent.

1. introduction

Due to the importance of Noetherian and Artinian rings, there are several attempts to generalize these concepts. Back in 1988, Hamann, Houston and Johnson ([11]) introduced the notion of almost principal ideals over an integral domain. According to [11], an integral domain D is called *almost principal* if there exists an $s \in D \setminus \{0\}$ and an $f \in I$ of positive degree such that $sI \subseteq fD[X]$, for every ideal $I \in D$. The ring $D[X]$ is called an *almost PID* if each ideal of $D[X]$ that extends to a proper ideal of $K[X]$ (where K is the quotient field of D) is almost principal. Note that if D is Noetherian or integrally closed, then $D[X]$ is almost PID. This notion was useful to answer several questions about the divisorial ideals of the ring of polynomials. A few years later, Anderson, Kwak and Zafrullah called a domain D an *agreeable domain* if for each fractional ideal F of $D[X]$ with $F \subseteq K[X]$, there exists an $s \in D \setminus \{0\}$ with $sF \subseteq D[X]$, ([3]). Also, they called an ideal I of $K[X]$ *almost finitely generated* when there is a finite subset $\{f_1, f_2, \cdots, f_n\}$ of I and an element $s \in D \setminus \{0\}$ such that $sI \subseteq \langle f_1, f_2, \cdots, f_n \rangle$. Later, Anderson and Dumitrescu generalized the concept of almost principal and almost finitely generated ideals to modules over commutative rings ([1]). Let R be a (commutative) ring and $S \subseteq R$ be a multiplicative set. An R-module M is said to be *S-finite* (resp., *S-principal*) if $sM \subseteq F$ for some $s \in S$ and some finitely generated (resp., principal) submodule F of M. Also, M is called *S-Noetherian* (resp., *S-PIR*) if each submodule of M is an S-finite (resp., S-principal) module.

In 2016, Ahmed and Sana ([10]) tried to characterize the concept of S-Noetherian modules via a suitable chain condition and a special kind of maximality. An ascending chain $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ of submodules of

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M is called *S-stationary* if there exist a positive integer k and $s \in S$ such that for each $n \geq k$, $sN_n \subseteq N_k$. Let $\mathcal{F}$ be a set of submodules of M. A submodule N of $\mathcal{F}$ is called *S-maximal* if there exists $s \in S$ such that for every $L \in \mathcal{F}$ with $N \subseteq L$, $sL \subseteq N$. They showed that if every nonempty set of ideals of R has an S-maximal element, then R is S-Noetherian and the later implies that every increasing sequence of ideals of R is S-stationary. In 2017, Bilgin , Reyes and Tekir ([4]) characterized S-Noetherian modules over noncommutative rings. They called a family $\mathcal{F}$ of submodules of a right R-module M, *S-saturated* if for every submodule N of M, whenever there exist $s \in S$ and $N_0 \in \mathcal{F}$ such that $Ns \subseteq N_0$, then $N \in F$. They proved that M is S-Noetherian if and only if every increasing sequence of submodules of M is S-stationary if and only if every nonempty set of submodules of M has an S-maximal element if and only if every nonempty S-saturated set of submodules of M has a maximal element. In [16], Sevim, Tekir and Koc studied the duality of the S-Noetherian concept. They introduced the S-Artinian rings and finitely S-cogenerated rings. A ring R is called *S-Artinian* if for each descending chain $\{I_n\}_{n\in\mathbb{N}}$ of ideals of R, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sI_k \subseteq I_n$ for all $n \geq k$.

On the other hand, Ghorbani and colleagues' studies on uniserial dimension and co-uniserial dimension of modules ([7] and [14]) were caused to the definitions of divisibility on chains, [5, 6]. These chain conditions are related to endomorphism ring of the module. Considering a ring R as a module over itself, endomorphisms are just the elements of R. So, divisibility on chains of ideals was defined as follows: a ring R is said to *satisfy divisibility on ascending (descending) chain of right ideals* if, for every ascending (descending) chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ ($I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$) of right ideals of R, there exists $k \in \mathbb{N}$ such that, for each $i \geq k$, $I_i = x_i I_{i+1}$ ($I_{i+1} = x_i I_i$) for some $x_i \in R$.

In this paper, we aim to generalize the concept of S-Artinian modules in a similar way. We call a descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of submodules of M *weakly S-stationary* if there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n N_k \subseteq N_n$ for some $s_n \in S$. We say that M is a *weakly S-Artinian module* whenever every descending chain of submodules of M is weakly S-stationary. Let $\mathcal{F}$ be a set of submodules of M. We say that $N \in \mathcal{F}$ is *weakly S-minimal* in $\mathcal{F}$ if for every $L \in \mathcal{F}$ with $L \subseteq N$, there exists $s \in S$ such that $sN \subseteq L$. A submodule N of M is called *weakly S-minimal* if it is weakly S-minimal in the set of all nonzero submodules of M. In section 2 we present some necessary and preliminary results. Later, we show that over an integral domain, weakly S-Artinian on ideals implies S-Noetherian. Moreover, We study conditions under which weakly S-Artinian implies DCC. In section 3, we consider the case when $S \subseteq R$ is a regular multiplicative set. We obtain several results in this case. We show that if M satisfies weakly S-Artinian on submodules, then M has finite uniform dimension. Moreover, for a commutative ring with weakly S-Artinian on ideals, $N(R)$ is nilpotent. We prove that a semiprime ring R with essential left socle which is a left weakly S-Artinian ring must be semisimple Artinian. Further, we show that over a right perfect ring, weakly S-Artinian and Artinian are equivalent.

Throughout this paper, R denotes a unitary ring with $0_R \neq 1_R$ and modules are unitary left modules. For any undefined notion, we refer the reader to [12].

2. Weakly descending chain condition

Let M be a left R-module and $S \subseteq R$ a multiplicative set. In this section we generalize the notion of weakly Artinian modules by introducing the concept of weakly S-Artinian modules. We start this section by introducing the following definitions in order to generalize some known results about weakly Artinian modules.

Definition 2.1. *Let M be a left R-module and S a multiplicative subset of R such that* $\mathrm{ann}(M) \cap S = \emptyset$. *We call a descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ of submodules of M* weakly S-stationary *if there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n N_k \subseteq N_n$ for some $s_n \in S$.*

We say that M is a weakly S-Artinian module *(or* satisfies weakly S-Artinian on submodules*) if every descending chain of submodules of M is weakly S-stationary.*

Definition 2.2. *According to [16], a commutative ring R is said to be an S-Artinian ring if for every descending chain of ideals of the form $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of R, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sI_k \subseteq I_n$ for all $n \geq k$. We can generalize this definition to noncommutative setting in a natural way as above.*

Let M be a left R-module and S a multiplicative subset of R. It is clear that

$$\text{Artinian} \Rightarrow S\text{-Artinian} \Rightarrow \text{weakly } S\text{-Artinian}.$$

Our next Examples (2.3, 2.6 and 2.7) prove that the reverse is not true in general.

Example 2.3. *Let F be a field, let D be an integral Noetherian domain which is not a field and let $R = D \times F$. Let $0 \neq a \in D$ be an element of D which is not a unit. As D is an integral domain, by the Krull Intersection Theorem we have:*

$$\bigcap_{n \in \mathbb{N}} a^n D = 0.$$

Thus

$$aD \times \{0\} \supset a^2 D \times \{0\} \supset \cdots \supset a^n D \times \{0\} \supset \cdots$$

is a strictly descending chain of ideals of R. Thus, R is not an Artinian ring. Let $S = \{(r,t) | \ r \in D, \ 0 \neq t \in F\}$. S is a multiplicative subset of R. Put $s = (0,1)$ then for any descending chain of ideals $(I_n)_{n\in\mathbb{N}}$ of R, there exists n_0 such that

$$sI_{n_0} \subseteq I_n$$

for all $n \geq n_0$. Hence, R is S-Artinian ring.

Definition 2.4. *Let $\mathcal{F}$ be a set of submodules of M and S a multiplicative subset of R such that* $\operatorname{ann}(M) \cap S = \emptyset$. *We say that $N \in \mathcal{F}$ is* weakly S-minimal in $\mathcal{F}$ *if for every $L \in \mathcal{F}$ with $L \subseteq N$, there exists $s \in S$ such that $sN \subseteq L$.*

A submodule N of M is called weakly S-minimal *if it is weakly S-minimal in the set of all nonzero submodules of M.*

Proposition 2.5. *Let M be a uniserial left R-module and $S \subseteq R$ a multiplicative subset of R such that* $\operatorname{ann}(M) \cap S = \emptyset$. *Assume that $\bigcap_{s\in S} RsM = 0$. Then every submodule of M is weakly S-minimal. In particular M is a weakly S-Artinian module.*

Proof. Let A be a submodule of M. We show that A is weakly S-minimal. Let B be a submodule of M such that $0 \subsetneq B \subseteq A$. Since $B \neq 0$, there exists $s \in S$ such that $B \not\subseteq RsM$; so $sA \subseteq RsM \subseteq B$. This shows that every submodule of M is weakly S-minimal, and hence M is a weakly S-Artinian module. □

Example 2.6. *Let D be an integral domain and $S = D\backslash\{0\}$. Since for each ideals $0 \subsetneq I \subseteq J$ of D, $bJ \subseteq I$, for some $0 \neq b \in I$, every ideal of D is weakly S-minimal. Therefore D is a weakly S-Artinian domain.*

Note that the ring $\mathbb{Z}$ does not satisfy the S-Artinian property, where $S = \mathbb{Z}\backslash\{0\}$. Indeed, the chain $2\mathbb{Z} \supseteq 2^2\mathbb{Z} \supseteq 2^3\mathbb{Z} \supseteq \cdots$ is not S-stationary.

Example 2.7. *Let D be a PID, p a prime element of D and $S = \{p^n \mid n \in \mathbb{Z}, n \geq 0\}$. Then S is a multiplicative subset of D_p. We show that $\cap_{n\geq 0} p^n D_p = (0)$. Assume that $\cap_{n\geq 0} p^n D_p \neq (0)$ and let $x \in \cap_{n\geq 0} p^n D_p$. Then for each $n \geq 0$, $x \in p^n D_p$ which implies that for each $n \geq 0$, $p^n D_p = p^{n+1} D_p$; so $p^n \in p^{n+1} D_p$. Write $p^n = p^{n+1}\frac{a}{s}$ for some $a \in D$ and $s \in D\backslash pD$. This implies that $s = pa \in pD$, a contradiction. Hence $\cap_{n\geq 0} p^n D_p = (0)$.*

Now, since D is a Prüfer domain, D_p is a valuation domain; so by Proposition 2.5, D_p is a weakly S-Artinian domain. But D_p does not satisfy the S-Artinian property, because the chain $D_p \supseteq pD_p \supseteq p^2 D_p \supseteq \cdots$ is not S-stationary.

Proposition 2.8. *Let S be a multplicative subset of R such that* $\operatorname{ann}(M) \cap S = \emptyset$. *If M is a weakly S-minimal left R-module, then M is indecomposable. In particular, if M is faithful, then M is indecomposable.*

Proof. Assume $M = M_1 \oplus M_2$ such that $M_1 \neq 0$ and $M_2 \neq 0$. Since $M_1 \subseteq M_1 \oplus M_2$, there exists $s_1 \in S$ such that $s_1(M_1 \oplus M_2) \subseteq M_1$. Then $s_1 M_2 \subseteq M_2 \cap M_1 = 0$, and so $s_1 M_2 = 0$. Similarly, there exists $s_2 \in S$ such that $s_2 M_1 = 0$. Set $s := s_1 s_2$. Then $s_1 s_2(M_1 \oplus M_2) \subseteq s_1 s_2 M_1 + s_1 s_2 M_2 = 0$. This implies that $s_1 s_2 \in S \cap \operatorname{ann}_R(M)$, a contradiction. Hence M is indecomposable. □

Let R be a ring and S a multiplicative subset of R. It is well known that if R is an S-Artinian ring, then $S^{-1}R$ is an Artinian ring ([16]). Our next Proposition is obtained by relaxing the S-Artinian property.

Proposition 2.9. *Let R be a ring and S a multiplicative subset of R. If R is a weakly S-Artinian ring, then $S^{-1}R$ is Artinian.*

Proof. Let $S^{-1}I_1 \supseteq S^{-1}I_2 \supseteq S^{-1}I_3 \supseteq \cdots$ be a descending sequence of ideals of $S^{-1}R$. For each $n \geq 1$, put $J = I_1 \cap I_2 \cap \cdots \cap I_n$. Then $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ is a descending sequence of ideals of R. Since R is a weakly S-Artinian ring, there exists a $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n J_k \subseteq J_n$ for some $s_n \in S$; so

$$S^{-1}(s_n J_k) = S^{-1}J_k \subseteq S^{-1}J_n \subseteq S^{-1}J_k,$$

which implies that for each $n \geq k$, $S^{-1}J_n = S^{-1}J_k$. But

$$S^{-1}J_n = S^{-1}(I_1 \cap I_2 \cap \cdots \cap I_n) = S^{-1}I_1 \cap S^{-1}I_2 \cap \cdots \cap S^{-1}I_n = S^{-1}I_n,$$

since the sequence $(S^{-1}I_n)_n$ is descending. Hence for each $n \geq k$, $S^{-1}I_n = S^{-1}I_k$, which indicate that $S^{-1}R$ is an Artinian ring. □

Since every S-Artinian ring is a weakly S-Artinian ring, we regain the following result

Corollary 2.10. *Let R be a ring and S a multiplicative subset of R. If R is an S-Artinian ring, then $S^{-1}R$ is an Artinian ring*

The next proposition collects some properties of the weakly S-Artinian concept.

Proposition 2.11. *Let M be a left R-module. Then the following statements hold.*

1. *Let $S_1 \subseteq S_2$ be tow multiplicative subsets of R. If M is a weakly S_1-Artinian module, then M is also a weakly S_2-Artinian module.*
2. *Let $\overline{S}$ be the saturation of S. Then M is a weakly S-Artinian module if and only if M is weakly $\overline{S}$-Artinian.*
3. *Let S be a finite regular multiplicative subset of R (This is also the case if $S \subseteq \mathrm{U}(R)$). Then M is a weakly S-Artinian module if and only if M is an Artinian module.*

Proof. (1). This assertion is clear.

(2). The only if part is obvious. For if part, let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a chain of submodules of M. Since M is weakly $\overline{S}$-Artinian, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $t_n N_k \subseteq N_n$ for some $t_n \in S$. Let $n \geq k$ and let $s_n \in S$ be a multiple of t_n. Thus

$$s_n N_k \subseteq t_n N_k \subseteq N_n.$$

Hence M is a weakly S-Artinian module.

(3). Follows from [10, Example 3.2]. □

Proposition 2.12. *Let M and M' be left R-modules, S a multiplicative subset of R and $f : M \to M'$ a surjective module homomorphism. Suppose that $\mathrm{ann}(M') \cap S = \emptyset$. If M is a weakly S-Artinian module, then M' is a weakly S-Artinian module.*

Proof. Since $\mathrm{ann}(M') \cap S = \emptyset$ and f is an epimorphism, then $\mathrm{ann}(M) \cap S = \emptyset$. Consider a descending chain of submodules $N_1' \supseteq N_2' \supseteq \cdots \supseteq N_n' \supseteq \cdots$ of M', then $f^{-1}(N_1') \supseteq f^{-1}(N_2') \supseteq \cdots \supseteq f^{-1}(N_n') \supseteq \ldots$ is a descending chain of submodules of M. Since M is a weakly S-Artinian module, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n f^{-1}(N_k') \subseteq f^{-1}(N_n')$. Now, since f is an epimorphism, $f(s_n f^{-1}(N_k')) = s_n(f(f^{-1}(N_k'))) = s_n N_k'$. Hence $s_n N_k' \subseteq N_n'$ for all $n \geq k$, therefore M' is a weakly S-Artinian module. □

Corollary 2.13. *Let M be a left R-module, S a multiplicative subset of R and N a proper submodule of M. Suppose that $S \cap (N :_R M) = \emptyset$. If M is a weakly S-Artinian module, then the quotient module M/N is a weakly S-Artinian module.*

Proof. Consider the canonical epimorphism $\Pi : M \to M/N$. Since $S \cap (N :_R M) = \emptyset$, we obtain $S \cap \text{ann}(M/N) = \emptyset$. Hence, by Proposition 2.12, M/N is a weakly S-Artinian module. □

Theorem 2.14. *Let M be a left R-module, N a proper submodule of M and S a multiplicative subset of R. Suppose that $S \cap (N :_R M) = \emptyset$ and $S \cap \text{ann}(N) = \emptyset$. Then the following assertions are equivalent.*

1. *M is a weakly S-Artinian module.*
2. *N and M/N are both a weakly S-Artinian modules.*

Proof. (1) ⇒ (2) Follows from Corollary 2.13 and the fact that any descending chain of submodules of N is a descending chain of submodules of M.

(2) ⇒ (1) Let $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ be a chain in M. By assumption, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n(L_k + N)/N \subseteq (L_n + N)/N$ and $s_n^{'}(N \cap L_k) \subseteq N \cap L_n$ for some $s_n, s_n^{'} \in S$. We show that for each $n \geq k$, $s_n^{'} s_n L_k \subseteq L_n$. Since $L_k \subseteq L_k + N$, $s_n L_k \subseteq s_n(L_k + N) \subseteq L_n + N$. Let $x \in L_k$. Then $s_n x \in L_n + N$ and so there exists $l \in L_n$ and $y \in N$ such that $s_n x = l + y$. Hence $s_n x - l \in N \cap L_k$ and then $s_n^{'}(s_n x - l) \in N \cap L_n$. Thus $s_n^{'} s_n x \in L_n$, as we wanted. □

Corollary 2.15. *Let S be a multiplicative subset of R. Then R is a weakly S-Artinian ring if and only if for each $n \in \mathbb{N}^*$, R^n is a weakly S-Artinian module.*

Proof. Assume that R is a weakly S-Artinian ring. We will show this via induction. Let $P(n)$ be the property that R^n is a weakly S-Artinian module. For $n = 1$, R is a weakly S-Artinian module if and only if R is a weakly S-Artinian ring. Suppose that the property holds for $1 \leq n$. Let's prove $P(n+1)$. The module R^n is isomorphic to the submodule $N = R^n \times \{0\}$. Hence, by the induction hypothesis and Proposition 2.12, N is weakly S-Artinian. Clearly $R^{n+1}/N \simeq R$. Moreover, $(N :_R R^{n+1}) = \{0\}$. Hence, by Corollary 2.13, R^{n+1}/N is weakly S-Artinian. Thus by Theorem 2.14, R^{n+1} is a weakly S-Artinian module. The other implication is obvious. □

Corollary 2.16. *Let R be a ring, S a mutiplicative subset of R and M a finitely generated left R-module. Suppose that* $\text{ann}(M) \cap S = \emptyset$. *If R is a weakly S-Artinian ring, then M is weakly S-Artinian.*

Proof. As M is a finitely generated R-module, there exist $n \in \mathbb{N}^*$ and a surjective module homomorphism $f : R^n \longrightarrow M$, such that $R^n/Ker(f) \simeq M$. Since $\text{ann}(M) \cap S = \emptyset$ and f is an epimorphism, $S \cap (ker(f) : R^n) = \emptyset$. Hence by Corollary 2.13, $R^n/Ker(f)$ is a weakly S-Artinian module. Therefore M is weakly S-Artinian. □

Definition 2.17. *Let M be a left R-module and $s \in R$. We say that s is a* nonzero divisor *for M, if for each $m \in M$, $sm = 0$ implies that $m = 0$. A* regular multiplicative set S over M *is a multiplicative subset of R such that for every $s \in S$, s is a nonzero divisor for M.*

Corollary 2.18. *Let R be a weakly S-Artinian commutative ring, S a mutiplicative subset of R and M an S-finite R-module. Suppose that S is regular over M. Then M is a weakly S-Artinian module.*

Proof. Since M is S-finite, there exist an $s \in S$ and a finitely generated submodule F of M such that $sM \subseteq F$. Regularity of S implies that $M \cong sM \subseteq F$; so M is a weakly S-Artinian module. □

In the next proposition we give an equivalent condition for a module to satisfy weakly S-Artinian on submodules.

Proposition 2.19. *Let R be a ring, S a mutiplicative subset of R and M a left R-module such that* $\text{ann}(M) \cap S = \emptyset$. *Then the following assertions are equivalent:*

1. *M is a weakly S-Artinian module.*
2. *Every nonempty set of submodules of M has a weakly S-minimal element.*

Proof. (1) ⇒ (2) Let $\mathcal{F}$ be a nonempty set of submodules of M with no weakly S-minimal element. Let $N_1 \in \mathcal{F}$. Since N_1 is not weakly S-minimal in $\mathcal{F}$, there exists $N_2 \in \mathcal{F}$ such that $N_2 \subseteq N_1$ and for each $s \in S$, $sN_1 \nsubseteq N_2$. Again $N_2 \in \mathcal{F}$ is not weakly S-minimal; so there exists $N_3 \in \mathcal{F}$ such that $N_3 \subseteq N_2$ and for each $s \in S$, $sN_2 \nsubseteq N_3$. By continuing this way, we obtain a chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ which is not weakly S-stationary. This shows that M does not satisfy weakly S-Artinian on submodules.

(2) ⇒ (1) Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a chain of submodules of M. Set $\mathcal{F} = \{N_i | i \in \mathbb{N}^*\}$. By the assumption, $\mathcal{F}$ has a weakly S-minimal element, say N_k for some $k \in \mathbb{N}$. Clearly for every $n \geq k$, there exists $s_n \in S$ such that $s_n N_k \subseteq N_n$, as we wanted. □

Corollary 2.20. *Let M be a left R-module and S be a multiplicative subset of R such that* $\text{ann}(M) \cap S = \emptyset$. *If M is a weakly S-Artinian module, then M contains an essential submodule which is a direct sum of weakly S-minimal submodules.*

Proof. Let $M \neq 0$. By Proposition 2.19, M contains a weakly S-minimal submodule. Let $\mathcal{F}$ be the set of all independent set of weakly S-minimal submodules of M with $\subseteq$ as an order relation. Clearly $\mathcal{F} \neq 0$ and by Zorn's Lemma, we can find a maximal independent family of weakly S-minimal submodules of M, say $\{M_\alpha\}_{\alpha\in\Lambda}$. Consider the submodule $\bigoplus_{\alpha\in\Lambda} M_\alpha$. Let N be a nonzero submodule of M. Then by Proposition 2.19, N contains a weakly S-minimal submodule, say K. By maximality of $\{M_\alpha\}_{\alpha\in\Lambda}$, $\{M_\alpha\}_{\alpha\in\Lambda} \cup \{K\}$ is not independent; so $K \cap (\bigoplus_{\alpha\in\Lambda} M_\alpha) \neq 0$. Hence $N \cap (\bigoplus_{\alpha\in\Lambda} M_\alpha) \neq 0$. This shows that $\bigoplus_{\alpha\in\Lambda} M_\alpha$ is essential in M. □

According to [4], a proper right ideal P of R is a *completely prime right ideal* if for any $a, b \in R$ satisfying $aP \subseteq P$ and $ab \in P$ we have $a \in P$ or $b \in P$.

Lemma 2.21. [4, Theorem 2.7] *Let S be a multiplicative subset of a ring R, and let M be a nonzero S-Noetherian left R-module. If every element of S is a non-zero-divisor for M, then M has a point annihilator that is a completely prime left ideal. In particular, if R is commutative, then R has an associated prime.*

Corollary 2.22. *Let M be a left R-module satisfying weakly S-Artinian on submodules, where S is a regular multiplicative subset of R. Then M has a point annihilator that is a completely prime left ideal. In particular, if R is commutative, then R has an associated prime.*

Proof. By Proposition 2.19, M that is a completely prime left ideal contains a nonzero submodule K such that every submodule of K is S-principal. Thus K is S-Noetherian and whence by Lemma 2.21, K has a point annihilator. □

Theorem 2.23. *Let $S \subseteq R$ be a multiplicative set. Consider $T = \text{Mat}_n(R)$ and $S' = \{sI_n \mid s \in S\}$ (I_n is the identity matrix). Then S' is a multiplicative subset of T and R is a weakly S-Artinian ring if and only if T is a left weakly S'-Artinian ring.*

Proof. Suppose that ${}_TT$(T as a left T-module) is a weakly S'-Artinian module. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a chain in $R^{(n)}$ as a left R-module. By $R^{(n)}$ we mean the direct sum of n copies of R. Then $A_1^{(n)} \supseteq A_2^{(n)} \supseteq \cdots$ is a chain in ${}_TT$ and so there exists $k \in \mathbb{N}$ such that, for each $j \geq k$ there exists $s_j \in S$ with $s_j I_n A_k^{(n)} \subseteq A_j^{(n)}$. Hence, for each $j \geq k$, $s_j A_k \subseteq A_j$. Thus ${}_RR^{(n)}$ is a weakly S-Artinian module and so does ${}_RR$.

Conversely, assume ${}_RR$ is a weakly S-Artinian module. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a chain in ${}_TT$. For each $i \in \{1, \cdots, n\}$, ${}_RR^{(n)} \supseteq A_1E_{ii} \supseteq A_2E_{ii} \supseteq \cdots$. We can find $k \in \mathbb{N}$ such that, for each $i \in \{1, \cdots, n\}$ and for each $j \geq k$ there exists $s_{ij} \in S$ with $s_{ij}A_kE_{ii} \subseteq A_jE_{ii}$. Let $j \geq k$. Then $s_{1j}A_kE_{11} \subseteq A_jE_{11}$ and similarly $s_{2j}A_kE_{22} \subseteq A_jE_{22}, \cdots, s_{nj}A_kE_{nn} \subseteq A_jE_{nn}$. Set $s_j = s_{1j}s_{2j}\cdots s_{nj}$. Thus, for each $i \in \{1, \cdots, n\}$, $s_jA_kE_{ii} \subseteq A_jE_{ii}$. Hence

$$s_jI_nA_k = s_jA_kI_n = \sum_{i=1}^{n} s_jA_kE_{ii} \subseteq \sum_{i=1}^{n} A_jE_{ii} = A_jI_n = A_j.$$

This shows that ${}_TT$ is a weakly S'-Artinian module. □

Proposition 2.24. *Let A and B be two rings with two multiplicative subsets $S \subseteq A$ and $T \subseteq B$ such that* $\mathrm{ann}_A(A \oplus M) \cap S = \emptyset$ *where M is an (A,B)-bimodule and $A \oplus M$ is a left A-module. Consider the triangular ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ and the multiplicative set $U = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \subseteq R$. Then R is a left weakly U-Artinian ring if and only if $A \oplus M$ is a weakly S-Artinian module and B is a left weakly T-Artinian ring.*

Proof. Suppose ${}_RR$ is a weakly U-Artinian module. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a chain of submodules of $A \oplus M$. Then $I_1 \oplus 0 \supseteq I_2 \oplus 0 \supseteq \cdots$ is a chain in ${}_RR$. So, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $u_n(I_k \oplus 0) \subseteq I_n \oplus 0$ for some $u_n = \begin{pmatrix} s_n & 0 \\ 0 & t_n \end{pmatrix} \in U$. Hence for every $n \geq k$, $s_n I_k \subseteq I_n$. Now, let $J_1 \supseteq J_2 \supseteq \cdots$ be a chain in ${}_BB$. Then $\begin{pmatrix} 0 & M \\ 0 & J_1 \end{pmatrix} \supseteq \begin{pmatrix} 0 & M \\ 0 & J_2 \end{pmatrix} \supseteq \cdots$ is a chain in ${}_RR$. So there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $u_n \begin{pmatrix} 0 & M \\ 0 & J_k \end{pmatrix} \subseteq \begin{pmatrix} 0 & M \\ 0 & J_n \end{pmatrix}$ for some $u_n = \begin{pmatrix} s_n & 0 \\ 0 & t_n \end{pmatrix} \in U$. Thus for each $n \geq k$, $t_n J_k \subseteq J_n$.

Conversely, suppose that $A \oplus M$ is a weakly S-Artinian module and ${}_BB$ is a weakly T-Artinian module. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a chain of submodules of ${}_RR$. For each $i \in \mathbb{N}$, $I_i = I_i^{(1)} \oplus I_i^{(2)}$, where $I_i^{(1)} \leq A \oplus M$ and $I_i^{(2)} \leq {}_BB$ with $MI_i^{(2)} \leq I_i^{(1)}$. So $I_1^{(1)} \supseteq I_2^{(1)} \supseteq \cdots$ is a chain in $A \oplus M$ and $I_1^{(2)} \supseteq I_2^{(2)} \supseteq \cdots$ is a chain in ${}_BB$. By assumption, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n I_k^{(1)} \subseteq I_n^{(1)}$ and $t_n I_k^{(2)} \subseteq I_n^{(2)}$ for some $s_n \in S$ and $t_n \in T$. Hence for each $n \geq k$, $\begin{pmatrix} s_n & 0 \\ 0 & t_n \end{pmatrix} I_k \subseteq I_n$, as we wanted. □

In the following, we investigate weakly S-Artinian on ideals for direct product of rings.

Proposition 2.25. *Let $S_1, S_2, \cdots, S_n$ be multiplicative subsets of rings $R_1, R_2, \cdots, R_n$, respectively. Set $R = \prod_{i=1}^n R_i$ and $S = \prod_{i=1}^n S_i$. Then the following conditions are equivalent.*

1. *R is a weakly S-Artinian ring.*
2. *For each $i \in \{1, \cdots, n\}$, R_i is a weakly S_i-Artinian ring.*

Proof. (1) ⇒ (2) It is straight forward.
(2) ⇒ (1) Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a chain of ideals of R. Since each $I_i = J_1^{(i)} \times J_2^{(i)} \times \cdots \times J_n^{(i)}$ we get a chain $J_i^{(1)} \supseteq J_i^{(2)} \supseteq \cdots \supseteq J_i^{(n)}$. By assumption R_i is an S_i-Artinian ring. So there exist k_i such that for all $m \geq k_i$ there exist $s_{k_i} \in S_i$ such that $s_{k_i} J_i^{(k_i)} \subseteq J_i^{(m)}$. Put $k = \max\{k_1, k_2, \cdots, k_n\}$ and $s_k = (s_{k_1}, s_{k_2}, \cdots, s_{k_n})$. Then for all $m \geq k$, $s_k I_k \subseteq I_m$ for some $s_k \in S$ and R is weakly S-Artinian. □

Example 2.26. *Let $R = \prod_{i=1}^\infty R_i$, where $R_i = \mathbb{Z}$ for all $i \in \mathbb{N}$. Also let $S = \prod_{i=1}^\infty (R_i \setminus \{0\})$ which is a multiplicative set in R. Consider the following chain of left ideals of R*

$$\bigoplus_{i=1}^{\infty} R_i \supseteq \bigoplus_{i=2}^{\infty} R_i \supseteq \bigoplus_{i=3}^{\infty} R_i \supseteq \cdots$$

If there exist $k \in \mathbb{N}$ and $(s_n)_{n=1}^\infty \in S$ with $(s_n)_{n=1}^\infty (\bigoplus_{i=k}^\infty R_i) \subseteq \bigoplus_{i=k+1}^\infty R_i$, then

$$s_k R_k \subseteq R_k \cap (\bigoplus_{i=k+1}^{\infty} R_i) = 0$$

and whence $s_k \mathbb{Z} = 0$ which is a contradiction. So R does not satisfy weakly S-Artinian on ideals.

Theorem 2.27. *Let D be an integral domain and $S \subseteq D$ be a saturated multiplicative set. Then the following assertions are equivalent.*

1. *D is a weakly S-Artinian ring.*
2. $S = D \setminus \{0\}$.

Proof. (1)⇒ (2) Consider the chain $aD \supseteq a^2D \supseteq a^3D \cdots$ of ideals of D, where $a \in D \setminus \{0\}$. Since D is a weakly S-Artinian ring, there exists $k \in \mathbb{N}$ such that for each $n \geqslant k$, $s_n a^k D \subseteq a^n D$ for some $s_n \in S$. Specially $s_{k+1}a^kD \subseteq a^{k+1}D$. Then there exists $d \in D$ such that $s_{k+1}a^k = a^{k+1}d$. Thus $a^k(s_{k+1} - ad) = 0$; so $s_{k+1} = ad$, and then $ad \in S$. By assumption S is saturated which implies that $a \in S$. Therefore $S = D \setminus \{0\}$.

(2) ⇒ (1) Follows from Example 2.6. □

Corollary 2.28. *Let D be an integral domain and S be a saturated multiplicative subset of D. If D is a weakly S-Artinian domain, then D is an S-Noetherian domain.*

Proof. By Theorem 2.27, $S = D \setminus \{0\}$ and it is the same idea as Example 2.5, so D is S-Noetherian. □

The following example shows that the converse of Corollary 2.28 is not true in general.

Example 2.29. *Let* $D = \mathbb{Z}$ *and* $S = \{2^n | n \in \mathbb{N}\}$. *Then D does not satisfy weakly S-Artinian on ideals, since the chain* $3\mathbb{Z} \supseteq 3^2\mathbb{Z} \supseteq 3^3\mathbb{Z} \supseteq \cdots$ *is not weakly S-stationary. But D is Noetherian, and so it is S-Noetherian for all multiplicative subset* $S \subseteq D$, *specially for* $S = \{2^n | n \in \mathbb{N}\}$.

Proposition 2.30. *Let R be a weakly S-Artinian ring, where S is a multiplicative subset of R. If every left ideal I of R contains a left regular element, then* ${}_RR$ *is weakly S-minimal.*

Proof. Let $0 \neq I \leq {}_RR$. Then I contains a left regular element, say x. Consider the chain $I \supseteq Rx \supseteq Ix \supseteq Rx^2 \supseteq Ix^2 \supseteq \cdots$. By assumption there exists $k \in \mathbb{N}$ such that $sRx^k \subseteq Ix^k$ for some $s \in S$. Let $r \in R$. Then $srx^k \in Ix^k$ and so there exists $a \in I$ such that $srx^k = ax^k$. Thus $sr = a \in I$. So $sR \subseteq I$. This shows that ${}_RR$ is weakly S-minimal. □

In Example 2.7 we show that a ring with weakly S-Artinian on ideals need not be S-Artinian. Here we provide conditions under which weakly S-Artinian and S-Artinian are equivalent.

Proposition 2.31. *Let* $S \subseteq R$ *be a finite multiplicative set and M be a left R-module such that* $\mathrm{ann}(M) \cap S = \emptyset$. *Then M satisfies weakly S-Artinian on submodules if and only if M is S-Artinian.*

Proof. (⇐) Is clear.

(⇒) Let $S = \{s_1, s_2, s_3, \cdots, s_t\}$ and set $s = s_1 s_2 \cdots s_t$. Let $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ be a chain of submodules of M. Put $A = \{N_i \mid i \in \mathbb{N}\}$. A is nonempty; so by assumption, it contains a weakly S-minimal element, say N_k. For each $n \geq k$, there exists $s_n \in S$ such that $s_n N_k \subseteq N_n$ which implies $sN_k \subseteq N_n$. So M is S-Artinian. □

In the following, we investigate the relationship between Artinian and weakly S-Artinian properties. First we need the following Lemma.

Lemma 2.32. *Let* $S \subseteq R$ *be a multiplicative set. If I is a weakly S-minimal left ideal of R, then every left ideal of R contained in I is weakly S-minimal.*

Proof. Let $K \subseteq J \subseteq I$ be a chain of left ideals of R. Since I is weakly S-minimal, there exists $s \in S$ such that $sI \subseteq K$, and whence $sJ \subseteq sI \subseteq K$. Therefore, J is weakly S-minimal. □

Theorem 2.33. *Let M be a left R-module and S a multiplicative subset of R such that* $\mathrm{ann}(M) \cap S = \emptyset$. *Then the following assertions are equivalent.*

1. *M is an Artinian module.*
2. *M is weakly S-Artinian and every descending chain of weakly S-minimal submodules stops.*

Proof. (1) ⇒ (2) This implication is clear.

(2) ⇒ (1) Suppose that $\mathcal{A}$ is a nonempty set of submodules of M which has no minimal element. By assumption, $\mathcal{A}$ has a weakly S-minimal element, say N_0. Since N_0 is not minimal, there exists $N_1 \in \mathcal{A}$ with $N_0 \supsetneq N_1$ and clearly N_1 is not minimal. By continuing this way, we have a chain $N_0 \supsetneq N_1 \supsetneq N_2 \supsetneq \cdots$ which is not stationary. Since N_0 is weakly S-minimal, by Lemma 2.32, all submodules in this chain are weakly S-minimal and this is a contradiction. Therefore M is Artinian. □

3. Weakly S-Artinian with regular multiplicative set

In this section we focus on modules with weakly S-Artinian on submodules, where $S \subseteq R$ is a regular multiplicative set over M.

Definition 3.1. *Let M be an R-module. We say that $f \in End_R(M)$ is essential in M if the intersection of all submodules of M with $f(M)$ is nonzero.*

Proposition 3.2. *Let S be a regular multiplicative subset of R and M be a weakly S-minimal left R-module. Then the following holds.*

1. *M is uniform.*
2. *Every nonzero endomorphism of M is an essential monomorphism.*
3. *If M contains a minimal submodule and $S \subseteq \text{cent}(R)$, then M is simple.* ($\text{cent}(R)$ *denotes the center of the ring* R.)

Proof. (1). Let A and B be two nonzero submodules of M. Then there exists $s \in S$ such that $sM \subseteq A$ and so $sB \subseteq A$. Since $sB \neq 0$ and $sB \subseteq A \cap B$, we have $A \cap B \neq 0$. Hence M is an uniform R-module.

(2). Let $0 \neq f \in End_R(M)$. If $Ker(f) \neq 0$, then there exist $s \in S$ such that $sM \leq Ker(f)$. So $sf(M) = f(sM) = 0$. By regularity of S, $f(M) = 0$, a contradiction. Hence $Ker(f) = 0$. Now, by (i), ${}_RM$ is uniform and so $f(M)$ is essential in M.

(3). If A is a minimal submodule of M, then there exists $s \in S$ such that $sM \subseteq A$, and so $M \cong sM = A$. □

Corollary 3.3. *Let S be a regular multiplicative subset of R and M be a weakly S-minimal R-module. If $S \subseteq \text{cent}(R)$ and M is injective, then M is simple.*

Proof. Let A be a nonzero submodule of M. Then there exists $s \in S$ such that $sM \subseteq A$ and whence $M \cong sM \subseteq A$. Since M is injective, sM is a direct summand of A. By Proposition 3.2, M is uniform. So $sM = A$. Thus A is injective and then A is a direct summand of M. Again, Proposition 3.2 implies that $A = M$. □

The following example shows that if S is a non-regular multiplicative set in R, then weakly S-Artinian on left ideals does not imply finite uniform dimension for ${}_RR$.

Example 3.4. *Let $T = F[x_1, x_2, \cdots]$ and $I = \langle x_i x_j \mid i \neq j \rangle$, where F is an arbitrary field. Consider the ring $R = T/I$ and the multiplicative subset $S = \{\overline{x_1}^i \mid i \in \mathbb{N} \cup \{0\}\} \subseteq R$. Set $A_i = R\overline{x_i}$ for every $i \in \mathbb{N}$. Let $A = \sum_{i=2}^{\infty} R\overline{x_i}$. With attention to the structure of A_i's we can conclude that $A = \bigoplus_{i=2}^{\infty} A_i$. Now, $\overline{x_1}A = 0$ and whence A is weakly S-minimal.*

But for a regular multiplicative set over M we have following Proposition.

Proposition 3.5. *Let M be a left R-module with weakly S-Artinian on submodules, where $S \subseteq R$ is a regular multiplicative set. Then M has finite uniform dimension.*

Proof. Suppose to the contrary that M does not have finite uniform dimension. Then there exists a family of independent nonzero submodules of M, say $\{N_1, N_2, N_3, \cdots\}$. Consider the following chain of submodules of M:

$$\bigoplus_{i=1}^{\infty} N_i \supseteq \bigoplus_{i=2}^{\infty} N_i \supseteq \bigoplus_{i=3}^{\infty} N_i \supseteq \cdots$$

Since M satisfies weakly S-Artinian on submodules, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n(\bigoplus_{i=k}^{\infty} N_i) \subseteq \bigoplus_{i=n}^{\infty} N_i$ for some $s_n \in S$. So

$$s_{k+1} N_k \subseteq N_k \cap (\bigoplus_{i=k+1}^{\infty} N_i) = 0$$

Since S is regular over M, we must have $N_k = 0$, a contradiction. So M has finite uniform dimension. □

According to [12], an R-module M is said to be *co-hopfian* if every injective endomorphism of M is an isomorphism. It is well-known that Artinian modules are co-hopfian. A module with weakly S-Artinian on submodules need not to be co-hopfian. Indeed, Let $D = \mathbb{Z}$ and $S = D \setminus \{0\}$, then S is a multiplicative subset of D. By Example 2.6, D is a weakly S-Artinian ring. But, clearly D is not co-hopfian. The following corollary shows that in the case when M is quasi-injective, weakly S-Artinian on submodules then that M is co-hopfian.

Corollary 3.6. *Let M be a quasi-injective module with weakly S-Artinian on submodules, where $S \subseteq R$ is a regular multiplicative set. Then M is co-hopfian.*

Proof. By Proposition 3.5, M has finite uniform dimension and By [9, Example 1.8], every R-module with finite uniform dimension is weakly co-hopfian. On the other hand by [9, Proposition 1.4], a quasi-injective weakly co-hopfian module is co-hopfian. Hence M is co-hopfian. □

Theorem 3.7. *Let R be a commutative ring and S be a regular multiplicative subset of R. If R is a weakly S-Artinian ring, then $N(R)$ is nilpotent.*

Proof. Put $N = N(R)$, consider the chain $N \supseteq N^2 \supseteq N^3 \supseteq \cdots$. Since R is a weakly S-Artinian ring, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_nN^k \subseteq N^n$ for some $s_n \in S$. We show that $N^k = 0$. Suppose that $N^k \neq 0$. Consider the set $\mathcal{A} = \{I \subseteq R \mid IN^k \neq 0\}$ of ideals of R. $R \in \mathcal{A}$, so $\mathcal{A}$ is a nonempty set. By assumption, $\mathcal{A}$ contains a weakly S-minimal element, say C. Then $CN^k \neq 0$; so there exists $x \in C$ such that $xN^k = (xR)N^k \neq 0$. Also, $(xN)N^k = xN^{k+1}$. On the other hand $s_{k+1}N^k \subseteq N^{k+1}$ and so $s_{k+1}xN^k \subseteq xN^{k+1}$. If $s_{k+1}xN^k = 0$, then by regularity of S we have $xN^k = 0$ and this is a contradiction. So $s_{k+1}xN^k \neq 0$ and whence $xN^{k+1} \neq 0$. Thus $xN \in \mathcal{A}$. We have $xN \subseteq xR \subseteq C$. Since C is weakly S-minimal, by Lemma 2.32, xR is weakly S-minimal too. So there exists $t \in S$ such that $txR \subseteq xN$. Then there exists $a \in N$ such that $tx = xa$. Since $a \in N$, there exist $m \in \mathbb{N}$ such that $a^m = 0$. Therefore $txa = xa^2$, and so $txa^2 = xa^3, ..., txa^{m-1} = xa^m = 0$. So $txa^{m-1} = 0$. By regularity of S, $xa^{m-1} = 0$ which implies that $txa^{m-2} = 0$. By continuing this way we have $tx = 0$, and then $x = 0$, a contradiction. So the result follows. □

Theorem 3.8. *Let R be a ring with weakly S-Artinian on left ideals, where $S \subseteq R$ is a multiplicative set. Assume $N(R) = 0$ and R has essential left socle. Then R is semisimple Artinian.*

Proof. We show that every left ideal I of R is generated by an idempotent. Let I be a nonzero left ideal of R. Then I contains a minimal left ideal A and since $N(R) = 0$, $A^2 \neq 0$. By [17, Proposition 2.7], every minimal left ideal is either nilpotent or generated by an idempotent. So there exists a nonzero idempotent in I and the following set is nonempty

$$F = \{\mathrm{ann}_l(e) \mid e^2 = e \in I \setminus \{0\}\}$$

By assumption F has a weakly S-minimal element, say $\mathrm{ann}_l(f)$ with $f^2 = f \in I \setminus \{0\}$. Suppose that $I \cap \mathrm{ann}_l(f) \neq 0$. Then there is a minimal left ideal in this intersection which again is generated by an idempotent g, i.e. $0 \neq g \in I \cap \mathrm{ann}_l(f)$. Putting $h = f + g - fg$ we get $h^2 = h \in I \setminus \{0\}$ and $hf = f$. This means $\mathrm{ann}_l(h) \subseteq \mathrm{ann}_l(f)$. By weakly S-minimality of $\mathrm{ann}_l(f)$, there is an $s \in S$ such that $s(\mathrm{ann}_l(f)) \subseteq \mathrm{ann}_l(h)$. Then $sg \in \mathrm{ann}_l(h)$ and so $sgh = 0$. Since S is right regular, we have $gh = 0$ which is a contradiction. So $I \cap \mathrm{ann}_l(f) = 0$. Now, for every $a \in I$, we have $(a - af)f = 0$ and then $a - af \in I \cap \mathrm{ann}_l(f) = 0$, which means $I = Rf$. □

The following example shows that "weakly S-Artinian" is necessary in Theorem 3.8.

Example 3.9. *Consider the ring $R = \prod_{i=1}^{\infty} F_i$, where F_i is a field for all $i \in \mathbb{N}$. Then R is a semiprime ring with essential socle which is not semisimple Artinian. Using an argument similar to Example 2.26, we can show that R does not satisfy weakly S-Artinian on ideals for the multiplicative set $S = \prod_{i=1}^{\infty}(F_i \setminus \{0\}) \subseteq R$.*

Lemma 3.10. *Let $S \subseteq R$ be a regular multiplicative set. Assume R satisfies weakly S-Artinian on left ideals. Then R satisfies ACC on right annihilators.*

Proof. Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a chain in R such that for every $j \in \mathbb{N}$, $I_j = \mathrm{ann}_l(A_j)$ for some $A_j \subseteq R$. Since R is a weakly S-Artinian ring, there exists $k \in \mathbb{N}$ such that for each $n \geq k$, $s_n I_k \subseteq I_n$ for some $s_n \in S$. Let $n \geq k$. Then $s_n I_k A_n = 0$, and by regularity of S, we have $I_k A_n = 0$ which implies $I_k \subseteq I_n$. Therefore $I_n = I_k$. Thus R satisfies DCC on left annihilators. But according to [13, Remark 6.57], DCC on left annihilators is equivalent to ACC on right annihilators. Thus R satisfies ACC on right annihilators. □

According to [12], a ring is called *left Goldie* if it has finite uniform dimension as a left module and satisfies ACC on left annihilators.

Corollary 3.11. *Let S be a regular multiplicative subset of R. If R satisfies weakly S-Artinian on left and right ideals, then R is left and right Goldie.*

Proof. Since R satisfies weakly S-Artinian on left and right ideals, in the situation of Lemma 3.10, we get that R satisfies ACC on left and right annihilators. Then by Proposition 3.5, R is left and right Goldie. □

The following example shows that the converse of Corollary 3.11 does not true in general.

Example 3.12. *By Theorem 2.27, every integral domain D with a saturated multiplicative subset $S \neq D \setminus \{0\}$ does not satisfy weakly S-Artinian on ideals. But, D is a Goldie ring.*

Corollary 3.13. *Assume that R is a weakly S-Artinian ring, where $S \subseteq R$ is a regular multiplicative set. Then $Z(R_R)$ is a nilpotent ideal. ($Z(R_R)$ is the right singular ideal of R.)*

Proof. By Lemma 3.10, R satisfies ACC on right annihilators and by [13, Theorem 7.15], in a ring with ACC on right annihilators, the right singular ideal is nilpotent. □

Corollary 3.14. *Let R be a semiprime ring with weakly S-Artinian on left and right ideals, where S is a multiplicative subset of R. Then the maximal right quotient ring of R is semisimple Artinian.*

Proof. By Corollary 3.13, R is a right nonsingular ring and by Proposition 3.5, R_R has finite uniform dimension. Now, by [8, Theorem 3.17], the maximal right quotient ring of finite dimensional right nonsingular ring is semisimple Artinian. □

Recall from [15] that a ring R is *quasi-Frobenius* if it is left and right Artinian and left and right self-injective.

Corollary 3.15. *Let R be a left self-injective ring with weakly S-Artinian on left ideals, where S is a regular multiplicative subset of R. Then R is quasi-Frobenius.*

Proof. By Lemma 3.10, R satisfies ACC on right annihilators. But by [15, Theorem 1.50], every right or left self-injective ring with ACC on right or left annihilators is quasi-Frobinius. □

Recall from [12] that a ring R is called *right perfect* if $R/J(R)$ is semisimple Artinian and $J(R)$ is a right T-nilpotent ideal (a *T-nilpotent* ideal I, is an ideal where for any sequence of elements $\{a_1, a_2, a_3, ...\} \subseteq I$, there exists an integer $n \geq 1$ such that $a_n \cdots a_2 a_1 = 0$).

Lemma 3.16. [2, Proposition 29.1] *Let R have the maximum condition for right annihilators. If I is a right T-nilpotent one sided ideal, then I is nilpotent.*

Theorem 3.17. *Let R be a right perfect ring and $S \subseteq R$ be a regular multiplicative set. The following assertions are equivalent.*

1. *R is a left weakly S-Artinian ring.*
2. *R is left Artinian.*

Proof. (2) ⇒ (1) Obvious.

(1) ⇒ (2) Lemma 3.10 implies that R satisfies ACC on right annihilators. By Lemma 3.16, $J := \mathrm{J}(R)$ is nilpotent; so there exists $n \in \mathbb{N}$ such that $J^n = 0$. Consider the chain

$$R \supseteq J \supseteq J^2 \supseteq \cdots \supseteq J^{n-1} \supseteq J^n = 0$$

Let $i \in \{0, 1, 2, \cdots, n-1\}$. Then $\frac{R}{J^{i+1}}$ is a left R-module with weakly S-Artinian on submodules. Hence by Proposition 3.5, $\frac{R}{J^{i+1}}$ has finite uniform dimension as an R-module. $\frac{J^i}{J^{i+1}}$ is an $\frac{R}{J}$−module. Thus $\frac{J^i}{J^{i+1}}$ is a semisimple $\frac{R}{J}$−module and so a semisimple R-module. We conclude that $\frac{J^i}{J^{i+1}}$ is a finitely generated semisimple R-module, since $\frac{R}{J^{i+1}}$ has finite uniform dimension. Therefore $\frac{J^i}{J^{i+1}}$ is an Artinian left R-module. Now, going above in the chain we get that R/J is Artinian and since J is nilpotent, R is left Artinian. □

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