Filomat 35:15 (2021), 5209–5214
https://doi.org/10.2298/FIL2115209H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Points of Openness of Some Mappings

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Abstract. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. We are interested in finding points of X and Y at which f is open. We will show that if X is developable, the set of points of openness of f in X is a G_δ subset of X. If X is developable and f is closed, then the set of points of openness of f in Y is a G_δ subset of Y. These will extend some results of S. Levi.

1. Introduction

Let X and Y be topological spaces. Let us recall that $f : X \to Y$ is open at $x \in X$ if it maps neighborhoods of x into neighborhoods of $f(x)$ and f is open at $y \in Y$ if for each open A in X, $y \in f(A)$ implies $y \in Int f(A)$. It follows from the definition that $f : X \to Y$ is open at $y \in f(X)$ if and only if it is open at each point of $f^{-1}(y)$.

A continuous mapping f from a topological space X to a topological space Y is called closed at $y \in Y$ [10] if for every open subset $W \subset X$ containing $f^{-1}(y)$, there is a neighborhood V of y such that $f^{-1}(V) \subset W$. f is closed if it is closed at every point of Y.

The investigation of the set of points of Y at which f is open for a continuous closed mapping $f : X \to Y$ has been studied by S. Levi in [7]. In fact, S. Levi [7] proved the following theorem.

Theorem 1.1. *If $f : X \to Y$ is a continuous closed mapping on a metrizable space X, the set of points of Y at which f is open is a G_δ set in Y.*

In this paper, we will generalize Theorem 1.1 for developable topological spaces. We will also prove that if f is a continuous function from a developable topological space X to a topological space Y, then the set of all points of X at which f is open is a G_δ set in X and we present an example of Professor Bouziad, which shows that this is not true if X is a topological space with a base of countable order. The example of Professor Bouziad also shows that Corollary 2.7 in [4] is wrong.

2020 *Mathematics Subject Classification*. Primary 54C10; Secondary 46T20
Keywords. Continuous function, open function, quasicontinuous function, closed function, developable space
Received: 03 January 2021; Revised: 14 April 2021; Accepted: 24 April 2021
Communicated by Ljubiša D.R. Kočinac
Ľ. Holá was supported by Vega 2/0048/21.
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2. Points of openness of mappings in domain

We quote [3] and [6] as basic references. First we remind some notions.

Definition 2.1. Let X and Y be topological spaces and $f : X \to Y$ be a function. The function f is called open (resp. feebly open) at $x \in X$ if $f(x) \in Int(f(U))$ (resp. $f(U)$ has a nonempty interior) for each neighbourhood U of x. f is called open (resp. feebly open) if it is open (feebly open) at each point of X.

Let X be a topological space, $x \in X$ and $\mathcal{G}$ be a collection of subsets of X. Then $st(x, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : x \in G\}$. Let $\mathbb{N}$ be the set of positive integers and $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X.

(1) If for each $x \in X$, the set $\{st(x, \mathcal{G}_n : n \in \mathbb{N}\}$ is a base at x, we say that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a development on X and that the space X is developable. A regular developable space is called a Moore space.

(2) If for every sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$ and for every $x \in \bigcap G_n$, the sequence $\{\cap_{i \le n} G_i : n \in \mathbb{N}\}$ is a base at x, we say that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a weak development on X and that the space X is weakly developable.

The notion of a weak development was introduced by B. Alleche and J. Calbrix in [2].

Let X be a developable topological space, Y be a topological space and $f : X \to Y$ be a function. Let $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a development of X. Of course, without loss of generality, we can suppose that the sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is such that

$$\mathcal{G}_{n+1} \prec \mathcal{G}_n \text{ for every } n \in \mathbb{N}.$$

($\mathcal{G}_{n+1} \prec \mathcal{G}_n$ means that for every $U \in \mathcal{G}_{n+1}$, there is $V \in \mathcal{G}_n$ such that $U \subset V$.)

For every $n \in \mathbb{N}$ we will define sets

(∗) $A_n = \{x \in X : \exists\, O, \exists G \in \mathcal{G}_n, x \in O \subset G, f(O) \text{ is open}\}$ and

$$B_n = \{x \in X : \exists \text{ open } O, \exists G \in \mathcal{G}_n, x \in O \subset G, Int f(O) \neq \emptyset\}.$$

In the definition of sets A_n we used some ideas of Professor Bouziad (private communication).

In what follows, $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a development of X.

Lemma 2.2. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a function. Then f is open at $x \in X$ if and only if $x \in \bigcap\{A_n : n \in \mathbb{N}\}$.*

Proof. Let $x \in X$. Suppose that f is open at x. Let $n \in \mathbb{N}$ and let $G \in \mathcal{G}_n$ be such that $x \in G$. Then $f(x) \in Int f(G)$. Put $U = Int f(G)$ and put $O = f^{-1}(U) \cap G$. Then $x \in O, O \subset G$ and $f(O)$ is open. Hence $x \in A_n$.

Suppose now that $x \in \bigcap\{A_n : n \in \mathbb{N}\}$. To prove that f is open at x, let U be a neighbourhood of x. There is $n \in \mathbb{N}$ such that

$$st(x, \mathcal{G}_n) \subset U.$$

Since $x \in A_n$, there are a subset O of X and $G \in \mathcal{G}_n$ such that $x \in O \subset G$ and $f(O)$ is open. Thus we have

$$x \in O \subset st(x, \mathcal{G}_n) \subset U \text{ and } f(x) \in f(O) \subset Int f(U).$$

□

Lemma 2.3. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a function. Then f is feebly open at $x \in X$ if and only if $x \in \bigcap\{B_n : n \in \mathbb{N}\}$.*

Proof. Let $x \in X$. If f is feebly open at x, then of course $x \in B_n$ for every $n \in \mathbb{N}$. Suppose now that $x \in \bigcap\{B_n : n \in \mathbb{N}\}$. To prove that f is feebly open at x, let U be a neighbourhood of x. There is $n \in \mathbb{N}$ such that

$$st(x, \mathcal{G}_n) \subset U.$$

Since $x \in B_n$, there is an open set O in X and an open set $G \in \mathcal{G}_n$ such that $x \in O \subset G$ and $Int f(O) \neq \emptyset$. Thus we have

$$x \in O \subset st(x, \mathcal{G}_n) \subset U \text{ and } \emptyset \neq Int f(O) \subset Int f(U).$$

□

Corollary 2.4. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a function. The set of all points of X at which f is feebly open is a G_δ set in X.*

Proof. It is easy to verify that the set B_n as defined in (∗) is open for every $n \in \mathbb{N}$. □

The following example shows that the condition of developability of X in Corollary 2.4 cannot be replaced by the condition of weak developability.

Example 2.5. Let X be Gruenhage's space. We need to remind the definition of this space. Let B be a Bernstein subset of $\mathbb{R}$, i.e., every uncountable closed subset of $\mathbb{R}$ meets both B and $\mathbb{R} \setminus B$. Let $\{B_\alpha : \alpha < 2^\omega\}$ be an enumeration of all countable subsets of B having uncountable closure in $\mathbb{R}$. For each $\alpha < 2^\omega$ choose a point $x_\alpha \in \overline{B_\alpha} \setminus (B \bigcup \{x_\beta : \beta < \alpha\})$, and points $x_{\alpha,n} \in B_\alpha$ such that $x_{\alpha,n}$ converges to x_α. Let

$$X = B \bigcup \{x_\alpha : \alpha < 2^\omega\}.$$

Topologize X by declaring the points of B to be isolated, and $x_\alpha \cup \{x_{\alpha,n} : n \geq m\}$, $m \in \omega$, to be a base at x_α. By Example 3 in [1] X is a weakly developable non developable space.

Put $Y = \mathbb{R}$. Put $A = \{x_{\alpha,2n} : \alpha < 2^\omega, n \in \mathbb{N}\}$. Topologize Y by declaring all points of the set A to be isolated and every $y \in Y \setminus A$ to have neighbourhoods from the usual Euclidean topology on $\mathbb{R}$. Define a function $f : X \to Y$ as follows: $f(x) = x$ for $x \notin A$ and $f(x) = x_{1,1}$ for $x \in A$. Put $H = \{x_\alpha : \alpha < 2^\omega\}$. It is easy to verify that the set of all points of X at which f is feebly open is the set H, which is not a G_δ set in X.

Lemma 2.6. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a function. Then $C(f) \cap A_n \subset IntA_n$ for every $n \in \mathbb{N}$, where $C(f)$ is the set of all points of continuity of f.*

Proof. Let $x \in C(f) \cap A_n$, $n \in \mathbb{N}$. There is a set O in X and an open set $G \in \mathcal{G}_n$ such that $x \in O \subset G$ and $f(O)$ is open. Let V be an open set in X such that $x \in V$, $V \subset G$ and $f(V) \subset f(O)$. Then $V \subset A_n$, since $V \cup O \subset G$ and the set $f(V \cup O) = f(O)$ is open. □

Corollary 2.7. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a continuous function. Then the set A_n is open for every $n \in \mathbb{N}$.*

Proof. By Lemma 2.6, $C(f) \cap A_n \subset IntA_n$ for every $n \in \mathbb{N}$. Thus $A_n \subset IntA_n$ for every $n \in \mathbb{N}$, i.e. A_n is open for every $n \in \mathbb{N}$. □

Corollary 2.8. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a continuous function. The set of all points of X at which f is open is a G_δ set in X.*

The following example shows that the condition of developability of X in Corollary 2.8 cannot be replaced by the condition of weak developability.

Example 2.9. To define X and Y we will use the notions of Example 2.5. For each $\alpha < 2^\omega$ choose different points $x_\alpha, y_\alpha \in \overline{B_\alpha} \setminus (B \cup \{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\})$, and points $x_{\alpha,n} \in B_\alpha$ such that $x_{\alpha,n}$ converges to x_α. Let $X = B \cup \{x_\alpha : \alpha < 2^\omega\}$ be Gruenhage space. Put $Y = B \cup \{x_\alpha : \alpha < 2^\omega\} \cup \{y_\alpha : \alpha < 2^\omega\}$.

Topologize Y by declaring the sets $\{x_\alpha, y_\alpha\}$ open, $\alpha < 2^\omega$ and every $y \in B$ to have neighbourhoods induced from the usual Euclidean topology on B.

Define a function $f : X \to Y$ as follows: $f(x_{\alpha,n}) = y_\alpha$, $n \in \mathbb{N}, \alpha < 2^\omega$ and $f(x) = x$ otherwise. It is easy to verify that f is continuous and the set of all points of X at which f is open is the set H, which is not a G_δ set in X.

In our paper [4] we wrongly proved Corollary 2.8 for a topological space X with a base of countable order. Professor A. Bouziad sent us an example which showed that it is not true.

We say that a topological space X has a base of countable order [6] if there is a sequence $\mathfrak{B}_n$ of bases for X such that if $x \in B_n \in \mathfrak{B}_n$ and $B_{n+1} \subset B_n$ for each $n \geq 1$, then $\{B_n\}_n$ is a base at x.

Since every weakly developable space has a base of countable order [1], Example 2.9 shows that Corollary 2.8 cannot work for a topological space X with a base of countable order. However we will also present an example of Professor A. Bouziad, since he pointed out that Corollary 2.7 in [4] does not work for a topological space X with a base of countable order.

Example 2.10. Let $X = [0, \omega_1)$ with the usual order topology τ. Then X is a topological space with a base of countable order [8]. Let us define a new topology $\mathcal{S}$ on X as follows: if x is not a limit ordinal, let $V(x) = \{x + n : n < \omega\}$ and take $\{V(x)\}$ as a basis of neighbourhoods of x for $\mathcal{S}$. Limit ordinals conserve their neighbourdoods from τ. Then $\mathcal{S}$ is a well defined topology on X. Clearly, the identity map f from (X, τ) to $(X, \mathcal{S})$ is continuous and the set of points at which f is open is the set of limit ordinals L. It is well known that L is not a G_δ set in X.

Lemma 2.11. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a feebly open function. Then the set A_n is dense in X for every $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Let U be a nonempty open set in X. Since $\mathcal{G}_n$ is an open cover of X, let $G \in \mathcal{G}_n$ be such that $U \cap G \neq \emptyset$. Since f is a feebly open function, we have

$$Int f(U \cap G) \neq \emptyset.$$

Put $L = Int f(U \cap G)$ and $H = f^{-1}(L) \cap U \cap G$. Then $H \neq \emptyset$, $f(H) = L$ is open and $H \subset A_n \cap U$. Thus A_n is dense in X. □

Theorem 2.12. *Let X be a Baire developable space and Y be a topological space. Let $f : X \to Y$ be a feebly open continuous function. The set of all points of X at which f is open is a dense G_δ set in X.*

Proof. By Lemma 2.2 the set $\bigcap\{A_n : n \in \mathbb{N}\}$ is the set of all points of X at which f is open. By Corollary 2.7 the set A_n is open for every $n \in \mathbb{N}$. By Lemma 2.11 the set A_n is dense for every $n \in \mathbb{N}$. Since X is a Baire space, the set $\bigcap\{A_n : n \in \mathbb{N}\}$ is dense in X. □

Let X and Y be topological spaces. A function $f : X \to Y$ is quasicontinuous [9] at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$ and every open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

We say that a set $A \subset X$ is quasi-open [9] if $A \subset \overline{IntA}$. A function f from a topological space X into a topological space Y is quasicontinuous if for every open set U in Y the set $f^{-1}(U)$ is quasi-open in X.

Lemma 2.13. *Let X and Y be topological spaces and let X be developable. Let $f : X \to Y$ be a quasicontinuous function. Then the set A_n is quasi-open for every $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Let $x \in A_n$ and let U be an open neighbourhood of x. Since $x \in A_n$, there is a set O and $G \in \mathcal{G}_n$ such that $x \in O \subset G$ and $f(O)$ is open. The quasicontinuity of f at x implies that there is a nonempty open set $V \subset G \cap U$ such that $f(V) \subset f(O)$. Then $V \subset IntA_n$, since $V \cup O \subset G$ and $f(V \cup O) = f(O)$ is open. □

The proof of the following theorem is very similar to the proof of Theorem 2.12.

Theorem 2.14. *Let X be a Baire developable space and Y be a topological space. Let $f : X \to Y$ be a feebly open quasicontinuous function. The set of all points of X at which f is open is a residual set in X (i.e. it contains a dense G_δ set in X).*

We will finish this part with an interesting observation. We need the following theorem from the paper [5].

Theorem 2.15. *Let X be a topological space and Y be a weakly developable space. Let $f : X \to Y$ be a function. Then the set $C(f)$ of the points of continuity of f is a G_δ set.*

Theorem 2.16. *Let X be a developable space and Y be a weakly developable space. Let $f : X \to Y$ be a function. Then the set $C(f) \cap O(f)$ is a G_δ set, where $O(f)$ is the set of $x \in X$ such that f is open at x.*

Proof. By Lemma 2.2 we have $O(f) = \bigcap\{A_n : n \in \mathbb{N}\}$. Thus $C(f) \cap O(f) = C(f) \cap \bigcap\{A_n : n \in \mathbb{N}\} = \bigcap\{C(f) \cap A_n : n \in \mathbb{N}\}$. By Lemma 2.6 we have $\bigcap\{C(f) \cap A_n : n \in \mathbb{N}\} = \bigcap\{C(f) \cap IntA_n : n \in \mathbb{N}\}$. Since by Theorem 2.15, $C(f)$ is a G_δ set, we are done. □

3. Points of openness of mappings in range space

Let X be a developable topological space, Y be a topological space and $f : X \to Y$ be a function. Let $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be a development for X. Using the above definition (∗) of the sets A_n we define a function $O_f : X \to [0, \infty]$ by:

$$O_f(x) = \inf\{1/n : x \in A_n\}, \text{ if } x \in A_n \text{ for some } n \in \mathbb{N} \text{ and } O_f(x) = \infty, \text{ otherwise.}$$

Lemma 3.1. *Let X be a developable space and Y be a topological space. A function $f : X \to Y$ is open at $x \in X$ if and only if $O_f(x) = 0$.*

Proof. If $f : X \to Y$ is open at $x \in X$, then by Lemma 2.2 $x \in A_n$ for every $n \in \mathbb{N}$. Thus $O_f(x) = 0$.

Suppose now that $O_f(x) = 0$. Then $x \in \bigcap\{A_n : n \in \mathbb{N}\}$ (since $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for every $n \in \mathbb{N}$, we have $A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$). Thus by Lemma 2.2 f is open at $x \in X$. □

Lemma 3.2. *Let $f : X \to Y$ be a continuous function, where X is a developable space and Y is a topological space. Then $O_f : X \to [0, \infty]$ is upper semicontinuous.*

Proof. For every $n \in \mathbb{N}$ define $f_n : X \to [0, 1]$ by $f_n(x) = 1/n$, if $x \in A_n$ and $f_n(x) = \infty$, if $x \notin A_n$. Since by Corollary 2.7 the set A_n is open, f_n is upper semicontinuous. Therefore, as $O_f(x) = inf_{n\in\mathbb{N}} f_n(x)$ for every $x \in X$, O_f is upper semicontinuous. □

Definition 3.3. ([7]) Let f be a function from a topological space X to a topological space Y. f is open at $y \in Y$ if for each open set A in X, $y \in f(A)$ implies that $y \in Int f(A)$.

It follows from the definition that $f : X \to Y$ is open at $y \in f(X)$ if and only if it is open at each point of $f^{-1}(y)$.

Definition 3.4. Let X be a developable space, Y be a topological space and $f : X \to Y$ be a function. Define $\Theta_f : Y \to [0, 1]$ by $\Theta_f(y) = sup\{O_f(x) : x \in f^{-1}(y)\}$ if $y \in f(X)$ and $\Theta_f(y) = 0$ otherwise.

The following lemma follows immediately from the definition and Lemma 3.1.

Lemma 3.5. *Let X be a developable space and Y be a topological space. Then a continuous function $f : X \to Y$ is open at $y \in Y$ if and only if $\Theta_f(y) = 0$.*

Following [10] a continuous mapping f from a topological space X to a topological space Y is called closed at $y \in Y$ if for every open subset $W \subset X$ containing $f^{-1}(y)$, there is a neighborhood V of y such that $f^{-1}(V) \subset W$. f is closed if it is closed at every point of Y.

Proposition 3.6. *Let X be a developable space, Y be a topological space and $f : X \to Y$ be a continuous function which is closed at $y \in Y$. Then Θ_f is upper semicontinuous at y.*

Proof. If $y \notin f(X)$, then $y \notin \overline{f(X)}$ since f is not closed at each point of $\overline{f(X)} \setminus f(X)$. Hence $Y \setminus \overline{f(X)}$ is a neighbourhood of y and $\Theta_f(z) = 0$ for every $z \in Y \setminus \overline{f(X)}$. If $y \in f(X)$ and $\Theta_f(y) = \infty$, then there is nothing to prove. Suppose that $\Theta_f(y) < \epsilon$ and choose $\epsilon' > 0$ such that $\Theta_f(y) < \epsilon' < \epsilon$. Then for every $x \in f^{-1}(y)$, we have $O_f(x) < \epsilon'$. Since O_f is upper semicontinuous, for every $x \in f^{-1}(y)$, we can find an open neighbourhood V_x of x such that $O_f(t) < \epsilon'$ for each $t \in V_x$. Let $V = \bigcup_{x\in f^{-1}(y)} V_x$. Then V is an open set which contains $f^{-1}(y)$. Since f is closed in y, there is a neighbourhood W of y such that $f^{-1}(W) \subset V$. If $z \in f(X) \cap W$, then

$$\Theta_f(z) = sup\{O_f(t) : t \in f^{-1}(z)\} \leq \epsilon' < \epsilon.$$

If $z \in W \cap (Y \setminus f(X))$, then $\Theta_f(z) = 0 < \epsilon$. Hence Θ_f is upper semicontinuous. □

The following theorem generalizes S. Levi's result proved for a metrizable space X see [7].

Theorem 3.7. *Let X be a developable space and Y be a topological space. If a continuous function $f : X \to Y$ is closed, then the set of points of Y at which f is open is a G_δ set.*

Proof. Let E denote the set of points of Y at which f is open. Thanks to Lemma 3.5

$$E = \{y \in Y : \Theta_f(y) = 0\} = \bigcap_{n\in\mathbb{N}}\{y \in Y : \Theta_f(y) < 1/n\}.$$

According to Proposition 3.6, the latter set is a G_δ subset of Y . □

Acknowledgement

The authors are thankful to Professor Bouziad for pointing out their mistake in Corollary 2.7 in [4] and also for some ideas which led to an improvement of the paper.

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