Filomat 35:15 (2021), 5195–5207
https://doi.org/10.2298/FIL2115195Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Embedding Besov Type Spaces $B_p(\lambda)$ into Tent Spaces and Volterra Integral operators

Ruishen Qian[a]

[a]*School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, Guangdong, P. R. China*

Abstract. In this paper, the boundedness and compactness of embedding from Besov Type spaces $B_p(\lambda)$ into tent spaces $T_{q,s}(\mu)$ are investigated ($1 \le p \le q < \infty$ and $0 < \lambda, s < \infty$). As an application, the boundedness and compactness of Volterra integral operator T_g and integral operator I_g from Besov Type spaces $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$ spaces are also studied.

1. Introduction

As usual, let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$, $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$ and H^∞ be the set of bounded analytic functions in $\mathbb{D}$. The Hardy space H^p $(0 < p < \infty)$ is the sets of $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Suppose that $0 < p < \infty$ and $\alpha > -1$. Let A_α^p denote the Bergman spaces of function $f \in H(\mathbb{D})$ satisfies

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

Let $1 \le p < \infty$ and $0 < \lambda < 1$. The Besov Type spaces $B_p(\lambda)$ consist of the function $f \in H(\mathbb{D})$ satisfies

$$\|f\|_{B_p(\lambda)}^p = |f(0)|^p + \|f'\|_{A_{p-1-\lambda}^p}^p < \infty.$$

$B_p(\lambda)$ spaces have been studied extensivly, we refer to [6, 14–16] and the paper referinthere.

2020 *Mathematics Subject Classification*. Primary 30D50; Secondary 30H25, 46E15
Keywords. Volterra type operator; Carleson measure; Besov Type spaces $B_p(\lambda)$.
Received: 29 December 2020; Accepted: 17 April 2021
Communicated by Miodrag Spalević
Research supported by NNSF of China (No.11801250) and (No.11871257), Overseas Scholarship Program for Elite Young and Middle-aged Teachers of Lingnan Normal University, Yanling Youqing Program of Lingnan Normal University (No. YL20200202), the Key Subject Program of Lingnan Normal University (No.1171518004) and (No.LZ1905), and Department of Education of Guangdong Province (No. 2018KTSCX133).
Email address: qianruishen@sina.cn (Ruishen Qian)

Suppose that $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. The space $F(p,q,s)$ is defined by those $f \in H(\mathbb{D})$ with

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. This space was first introduced by Zhao in [38]. When $p = 2$ and $q = 0$, it gives Q_s spaces (see [35, 36]). It is well known that $F(p, p-2, s)$ is equivalent to Bloch space for all $s > 1$, where the Bloch space $\mathcal{B}$ is the class of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0$, consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

Let $S(I)$ be the Carleson box based on I with

$$S(I) = \{z \in : 1-|I| \le |z| < 1 \ \ and \ \frac{z}{|z|} \in I\}.$$

If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. For $0 < p < \infty$, we say that a non-negative measure μ on $\mathbb{D}$ is a p-Carleson measure if

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

When $p = 1$, it gives the classical Carleson measure.

Let $0 < q, \lambda < \infty$. $T_{q,\lambda}^{\infty}(\mu)$ is the spaces of function $f \in L^q$ consists of

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$

$T_{2,\lambda}^{\infty}(\mu)$ was first introduced by Xiao in [32]. Xiao proved that the Q_p space ($0 < p < 1$) is continuously contained in $T_{2,\lambda}^{\infty}(\mu)$ if and only if $\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p}(\log\frac{2}{|I|})^2 < \infty$. Pau and Zhao studied Möbius invariant Besov type space $F(p, p-2, s)$ embedding to tent spaces $T_{p,s}^{\infty}(\mu)$ in [23], generalized the main results of [32]. Liu and Lou studied the emdedding from Morrey spaces $\mathcal{L}^{2,\lambda}$ to $T_{2,\lambda}^{\infty}(\mu)$ in [21]. For more information relate to tent spaces, we refer to [21, 23, 31, 32] and the paper referinthere.

For any $g, f \in H(\mathbb{D})$, the integral operator T_g and I_g are defined as

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \ \ I_g f(z) = \int_0^z f'(w)g(w)dw.$$

For $g \in H(\mathbb{D})$, the multiplication operator M_g is defined by $M_g f(z) = f(z)g(z)$. It is easy to see that M_g is related with I_g and T_g by

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z).$$

Aleman, Cima and Pommerenke in [1, 2, 22], showed that T_g is bounded on Hardy spaces if and only if $g \in BMOA$. Aleman and Siskakis in [3] showed that T_g is bounded on the Bergman space A^p if and only if $g \in \mathcal{B}$. Siskakis and Zhao in [26] proved that T_g is bounded on $BMOA$ if and only if $g \in BMOA_{\log}$. For more information related to these operators, we refer to [2], [3], [11], [20], [26] and [32].

In this paper, we prove that identity operator $I : B_p(\lambda) \to T_{q,s}^{\infty}(\mu)$ is bounded (resp. compactly) if and only if μ is a (resp. vanishing) $(s + \frac{q(1-\lambda)}{p})$-Carleson measure, when $1 \le p \le q < \infty$, $0 < \lambda < 1$ and $0 < s < \infty$. As an application, we studying Volterra integral operator T_g acting from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$. The paper is organize as following: Section 2, we give some auxillary results. Section 3, we studied boundedness and compactness of embedding from Besov Type spaces $B_p(\lambda)$ into tent spaces $T_{q,s}(\mu)$, where $1 \leq p < q < \infty$. Section 4, we investigated the boundedness and compactness of integral operator T_g, I_g and M_g acting from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant C such that $f \leq Cg$.

2. Preliminaries

In this section, we will give some auxiliary results.

Lemma 1. *Suppose that* $1 \leq p < \infty$, $0 < \lambda < 1$ *and* $f \in B_p(\lambda)$. *Then*

$$|f(z)| \lesssim \frac{\|f\|_{B_p(\lambda)}}{(1-|z|^2)^{\frac{1-\lambda}{p}}}, \quad z \in \mathbb{D}.$$

Proof. By growth of Bergman spaces $A^p_{p-1-\lambda}$, we have

$$|f'(z)| \lesssim \frac{\|f'\|_{A^p_{p-1-\lambda}}}{(1-|z|^2)^{\frac{p+1-\lambda}{p}}}, \quad f \in B_p(\lambda).$$

Thus, we can get our desire result by integral of z on both side of above. The proof is completed. □

Lemma 2. *([40, Lemma 3.10]) Suppose that* $\alpha > 0$, *then we have*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\bar{a}z|^{2+\alpha}} dA(z) \lesssim \frac{1}{(1-|a|^2)^\alpha}.$$

Lemma 3.*Let* $1 \leq p < \infty$, $0 < \lambda < 1$ *and* $z, w \in \mathbb{D}$. *Then*

$$f_w(z) = \frac{(1-|w|^2)^{\frac{p-1+\lambda}{p}}}{(1-\overline{w}z)} \in B_p(\lambda).$$

$$F_w(z) = \frac{(1-|w|^2)^{\frac{p-1+\lambda}{p}}}{\overline{w}(1-\overline{w}z)} \in B_p(\lambda).$$

Proof. Combine with Lemma 2, we have

$$\begin{aligned}
&\int_{\mathbb{D}} |f_w'(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z) \\
\lesssim &\int_{\mathbb{D}} \frac{(1-|w|^2)^{p-1+\lambda}}{|1-\overline{w}z|^{2p}} (1-|z|^2)^{p-1-\lambda} dA(z) \lesssim 1.
\end{aligned}$$

F_w can be verified in similar way. The proof is completed.
□

Lemma 4.*Let* $1 \le p < \infty$ *and* $0 < \lambda < 1$. *Then*

$$f \circ \varphi_a \in B_p(\lambda), \quad f \in B_p(\lambda).$$

Moreover,

$$\left\| f \circ \varphi_a \right\|_{B_p(\lambda)} \lesssim \frac{\|f\|_{B_p(\lambda)}}{(1-|a|^2)^{\frac{1-\lambda}{p}}}$$

and

$$\left\| f \circ \varphi_a - f(a) \right\|_{B_p(\lambda)} \lesssim \frac{\|f\|_{B_p(\lambda)}}{(1-|a|^2)^{\frac{1-\lambda}{p}}}.$$

Proof. Since

$$\|f \circ \varphi_a\|^p_{B_p(\lambda)} = |f(a)|^p + \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z).$$

Making change of variable $w = \varphi_a(z)$, combine with the well known fact that

$$|\varphi_a'(z)| = \frac{1-|\varphi_a(z)|^2}{1-|z|^2},$$

we have

$$\begin{aligned}
&\int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z) \\
=&\int_{\mathbb{D}} |f'(\varphi_a(z))|^p |\varphi_a'(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z) \\
=&\int_{\mathbb{D}} |f'(\varphi_a(z))|^p \left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2} \right)^{p-2} |\varphi_a'(z)|^2 (1-|z|^2)^{p-1-\lambda} dA(z) \\
=&\int_{\mathbb{D}} |f'(w)|^p \left(\frac{1-|w|^2}{1-|\varphi_a(w)|^2} \right)^{p-2} (1-|\varphi_a(w)|^2)^{p-1-\lambda} dA(w) \\
=&\int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_a(w)|^2)^{1-\lambda} dA(w).
\end{aligned}$$

Note that

$$\frac{1-|z|^2}{|1-\bar{a}z|^2} \lesssim \frac{1}{1-|a|^2}.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\varphi_a(w)|^2)^{1-\lambda} dA(w) \\
\lesssim&\frac{1}{(1-|a|^2)^{1-\lambda}} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-1-\lambda} dA(w)
\end{aligned}$$

The proof is completed. □

3. Carleson embedded

Theorem 1. *Suppose that* $1 \le p < \infty$, $0 < \lambda < 1$ *and* $\lambda \le s < \infty$. *Let* μ *be a nonnegative Borel measure on* $\mathbb{D}$. *The identity operator* $I : B_p(\lambda) \to T^\infty_{p,s}(\mu)$ *is bounded if and only if* μ *is a* $(s+1-\lambda)$*-Carleson measure.*

Proof. Suppose that the identity operator $I : B_p(\lambda) \to T^{\infty}_{p,s}(\mu)$ is bounded. For any given arc $I \subseteq \partial\mathbb{D}$, set

$$f_w(z) = \frac{(1-|w|^2)^{\frac{p-1+\lambda}{p}}}{(1-\overline{w}z)},$$

where $w = (1-|I|)\xi$ and ξ is the center point of I. By Lemma 3, we see that $f_w \in B_p(\lambda)$. In addition, it is easily to see that

$$|1-\overline{w}z| \approx 1-|w|^2 \approx |I|, \quad z \in S(I).$$

Thus,

$$|f_w(z)| \approx |I|^{\frac{\lambda-1}{p}}$$

when $z \in S(I)$. By the boundedness of $I : B_p(\lambda) \to T^{\infty}_{p,s}(\mu)$, we have

$$\|f_w\|^p_{T^{\infty}_{p,s}(\mu)} = \sup_{I\subseteq\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f_w(z)|^p d\mu(z) < \infty,$$

i.e.,

$$\sup_{I\subseteq\mathbb{D}} \frac{\mu(S(I))}{|I|^{s+1-\lambda}} < \infty.$$

Hence μ is a $(s+1-\lambda)$-Carleson measure.

Conversely. If μ is a $(s+1-\lambda)$-Carleson measure.

(1). When $s=\lambda$. Then μ is a Carleson measure. For any given $I \subseteq \partial\mathbb{D}$, denote by $w = (1-|I|)\xi$, where ξ is the midpoint of I. For any $f \in B_p(\lambda)$. Lemma 1 gives

$$|f(w)| \lesssim \frac{1}{|I|^{\frac{1-\lambda}{p}}}\|f\|_{B_p(\lambda)}.$$

Since μ is a Carleson measure, by the well known fact that

$$\int_{\mathbb{D}} \left|g(z)\right|^p d\mu(z) \le \|\mu\|^p \|g\|^p_{H^p}, \quad g \in H^p.$$

We obtain

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^p d\mu(z)\\
&\lesssim \frac{1}{|I|^s}\left(\int_{S(I)} |f(z)-f(w)|^p d\mu(z)\right) + |f(w)|^p \frac{\mu(S(I))}{|I|^s}\\
&\le (1-|w|^2)^{2-s}\left(\int_{\mathbb{D}} \left|\frac{f(z)-f(w)}{(1-\overline{w}z)^{2/p}}\right|^p d\mu(z)\right) + \frac{\mu(S(I))}{|I|}\\
&\le (1-|w|^2)^{1-s}\left(\int_{\partial\mathbb{D}} |f(\xi)-f(w)|^p \frac{1-|w|^2}{|1-\overline{w}\xi|^2}|d\xi|\right) + \frac{\mu(S(I))}{|I|}\\
&\le (1-|w|^2)^{1-s}\left(\int_{\partial\mathbb{D}} |(f\circ\varphi_w)(\xi)-f(w)|^p |d\xi|\right) + \frac{\mu(S(I))}{|I|}\\
&\le (1-|w|^2)^{1-s}\|f\circ\varphi_w - f(w)\|^p_{H^p} + \frac{\mu(S(I))}{|I|}.
\end{aligned}$$

By [6, Lemma 2.4], we can deduce that

$$\|f-f(0)\|_{H^p} \lesssim \|f-f(0)\|_{B_p(\lambda)}.$$

Thus,

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)}|f(z)|^p d\mu(z)\\
&\leq (1-|w|^2)^{1-\lambda}\|f\circ\varphi_w - f(w)\|_{H^p}^p + \frac{\mu(S(I))}{|I|}\\
&\lesssim (1-|w|^2)^{1-\lambda}\|f\circ\varphi_w - f(w)\|_{B_p(\lambda)}^p + \frac{\mu(S(I))}{|I|}\\
&\lesssim (1-|w|^2)^{1-\lambda}(1-|w|^2)^{-1+\lambda}\|f\|_{B_p(\lambda)}^p + \frac{\mu(S(I))}{|I|}\\
&\lesssim \|f\|_{B_p(\lambda)}^p + \frac{\mu(S(I))}{|I|}.
\end{aligned}$$

Therefore, $I : B_p(\lambda) \to T_{q,s}^{\infty}(\mu)$ is bounded.

(2). When $s > \lambda$. Checking above proof, we only need to show that

$$A =: \frac{1}{|I|^s}\int_{S(I)}|f(z) - f(w)|^p d\mu(z) < \infty.$$

Since μ is a $(s+1-\lambda)$-Carleson measure, by the well known fact that

$$\int_{\mathbb{D}}\left|g(z)\right|^p d\mu(z) \leq \|\mu\|^p\|g\|_{A_{s-1-\lambda}^p}^p, \quad g \in A_{s-1-\lambda}^p.$$

Note that $B_p(\lambda) \subseteq H^p \subseteq A_{s-1-\lambda}^p$. Now, we consider the case $s-1-\lambda \geq 0$ and $-1 < s-1-\lambda < 0$ separately.

Case 1. $s-1-\lambda \geq 0$. Let $\eta = \varphi_w(z)$. Combine with [7, Lemma 2.1], we have

$$\begin{aligned}
A \approx &(1-|w|^2)^{3-\lambda}\int_{S(I)}\left|\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{s}{p}+\frac{(1-\lambda)}{p}+\frac{2}{p}}}\right|^p d\mu(z)\\
&\lesssim (1-|w|^2)^{3-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{s+(1-\lambda)+2}}(1-|z|^2)^{s-1-\lambda}dA(z)\\
&\leq (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(z)\\
&= (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|f\circ\varphi_w(\eta) - f(w)|^p dA(\eta)
\end{aligned}$$

$$\begin{aligned}
&\lesssim (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|(f\circ\varphi_w)'(\eta)|^p(1-|\eta|^2)^p dA(\eta)\\
&\lesssim (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(\varphi_w(\eta))|^p(1-|\varphi_w(\eta)|^2)^p dA(\eta)\\
&\lesssim (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(z)\\
&= (1-|w|^2)^{1-\lambda}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p\frac{(1-|w|^2)^2}{|1-\overline{w}z|^{3-\lambda+1+\lambda}}dA(z)\\
&\lesssim (1-|w|^2)^{3-\lambda}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1-\lambda}\frac{1}{|1-\overline{w}z|^{3-\lambda}}dA(z).
\end{aligned}$$

Thus, we can deduce that $A \lesssim \|f\|_{B_p(\lambda)}^p$.

Case 2. $-1 < s-1-\lambda < 0$. Then

$$\begin{aligned}
A &\approx (1-|w|^2)^4 \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{4}{p}+\frac{s}{p}}} \right|^p d\mu(z) \\
&\lesssim (1-|w|^2)^4 \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{4+s}} (1-|z|^2)^{s-1-\lambda} dA(z) \\
&\leq (1-|w|^2)^{2-s} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p (1-|w|^2)^2}{|1-\overline{w}z|^4} (1-|z|^2)^{s-1-\lambda} dA(z) \\
&= (1-|w|^2)^{2-s} \int_{\mathbb{D}} |f\circ\varphi_w(\eta)-f(w)|^p (1-|\varphi_w(\eta)|^2)^{s-1-\lambda} dA(\eta)
\end{aligned}$$

$$\begin{aligned}
&= (1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |f\circ\varphi_w(\eta)-f\circ\varphi_w(0)|^p (1-|\eta|^2)^{s-1-\lambda} dA(\eta) \\
&\lesssim (1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |(f\circ\varphi_w)'(\eta)|^p (1-|\eta|^2)^{p-1-\lambda+s} dA(\eta) \\
&\lesssim (1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(\varphi_w(\eta))|^p (1-|\varphi_w(\eta)|^2)^p (1-|\eta|^2)^{s-1-\lambda} dA(\eta) \\
&\lesssim (1-|w|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p (1-|\varphi_w(z)|^2)^{s-1-\lambda} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} dA(z) \\
&\leq \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-1-\lambda} dA(z) \lesssim \|f\|^p_{B_p(\lambda)}.
\end{aligned}$$

□

Theorem 2. *Suppose that* $1 \leq p < q < \infty$, $0 < \lambda < 1$ *and* $0 < s < \infty$. *Let* μ *be a nonnegative Borel measure on* $\mathbb{D}$. *The identity operator* $I : B_p(\lambda) \to T^\infty_{q,s}(\mu)$ *is bounded if and only if* μ *is a* $s + \frac{q}{p}(1-\lambda)$*-Carleson measure.*

Proof. Suppose that $I : B_p(\lambda) \to T^\infty_{q,s}(\mu)$ is bounded. The proof is similar to Theorem 1, thus we omitted the proof.

On the other hand. Combine with the proof of Theorem 1, we deduce

$$\begin{aligned}
&\frac{1}{|I|^s} \int_{S(I)} |f(z)|^q d\mu(z) \\
&\lesssim \frac{1}{|I|^s} \left(\int_{S(I)} |f(z)-f(w)|^q d\mu(z) \right) + |f(w)|^q \frac{\mu(S(I))}{|I|^s} \\
&\lesssim \frac{1}{|I|^s} \left(\int_{S(I)} |f(z)-f(w)|^q d\mu(z) \right) + \frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}} \\
&\approx (1-|w|^2)^{\frac{q(2-\lambda)}{p}} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{(2-\lambda)}{p}+\frac{s}{q}}} \right|^q d\mu(z) + \frac{\mu(S(I))}{|I|^{s+\frac{q(1-\lambda)}{p}}}.
\end{aligned}$$

If μ is a $s + \frac{q}{p}(1-\lambda)$-Carleson measure, by [16, Theorem 1], we known that $\mathcal{D}^p_{p-1-\lambda+\frac{ps}{q}} \subseteq L^q(d\mu)$. Note that

$B_p(\lambda) \subseteq \mathcal{D}^p_{p-1-\lambda+\frac{ps}{q}}$. Hence,

$$
\begin{aligned}
&(1-|w|^2)^{\frac{q(2-\lambda)}{p}} \int_{S(I)} \left| \frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{(2-\lambda)}{p}+\frac{s}{q}}} \right|^q d\mu(z) \\
\lesssim &(1-|w|^2)^{\frac{q(2-\lambda)}{p}} \left(|f(0)-f(w)|^p + \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{(2-\lambda)}{p}+\frac{s}{q}}} \right)' \right|^p (1-|z|^2)^{p-1-\lambda+\frac{ps}{q}} dA(z) \right)^{q/p} \\
\lesssim &\left((1-|w|^2)^{2-\lambda} |f(0)-f(w)|^p \right)^{q/p} \\
&+ \left((1-|w|^2)^{2-\lambda} \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{(2-\lambda)}{p}+\frac{s}{q}}} \right)' \right|^p (1-|z|^2)^{p-1-\lambda+\frac{ps}{q}} dA(z) \right)^{q/p}.
\end{aligned}
$$

By growth of $B_p(\lambda)$, we have

$$(1-|w|^2)^{1-\lambda} |f(0)-f(w)|^p \lesssim 1.$$

Since

$$\left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{2-\lambda}{p}+\frac{s}{q}}} \right)' = \frac{f'(z)(1-\overline{w}z)^{\frac{2-\lambda}{p}+\frac{s}{q}} + \overline{w}(\frac{2-\lambda}{p}+\frac{s}{q})(f(z)-f(w))(1-\overline{w}z)^{\frac{2-\lambda}{p}+\frac{s}{q}-1}}{(1-\overline{w}z)^{\frac{4-2\lambda}{p}+\frac{2s}{q}}}.$$

We deduce that

$$
\begin{aligned}
M &= (1-|w|^2)^{2-\lambda} \int_{\mathbb{D}} \left| \left(\frac{f(z)-f(w)}{(1-\overline{w}z)^{\frac{(1-\lambda)}{p}+\frac{s}{q}}} \right)' \right|^p (1-|z|^2)^{p-1-\lambda+\frac{ps}{q}} dA(z) \\
&\lesssim I_1 + I_2,
\end{aligned}
$$

where

$$I_1 =: (1-|w|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|f'(z)|^p}{|1-\overline{w}z|^{2-\lambda+\frac{sp}{q}}} (1-|z|^2)^{p-1-\lambda+\frac{ps}{q}} dA(z)$$

and

$$I_2 =: (1-|w|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\overline{w}z|^{2-\lambda+\frac{sp}{q}+p}} (1-|z|^2)^{p-1-\lambda+\frac{ps}{q}} dA(z).$$

Clearly $I_1 \lesssim \|f\|^p_{B_p(\lambda)}$. Making change of variable $\eta = \varphi_w(z)$, combine with [7, Lemma 2.1], we have

$$\begin{aligned}
I_2 &= (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} \frac{|(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p}{|1-\overline{w}\varphi_w(\eta)|^{2-\lambda+\frac{sp}{q}+p}}(1-|\varphi_w(\eta)|^2)^{p-1-\lambda+\frac{ps}{q}}\frac{(1-|w|^2)^2}{|1-\overline{w}\eta|^4}dA(\eta)\\
&= (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} |(f\circ\varphi_w)(\eta)-(f\circ\varphi_w)(0)|^p\frac{(1-|\eta|^2)^{p-1-\lambda+\frac{ps}{q}}}{|1-\overline{w}\eta|^{p-\lambda+\frac{ps}{q}}}dA(\eta)\\
&\lesssim (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} |(f\circ\varphi_w)'(\eta)|^p\frac{(1-|\eta|^2)^{2p-1-\lambda+\frac{ps}{q}}}{|1-\overline{w}\eta|^{p-\lambda+\frac{ps}{q}}}dA(\eta)\\
&\lesssim (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} |f'(\varphi_w(\eta))|^p(1-|\varphi_w(\eta)|^2)^p\frac{(1-|\eta|^2)^{p-1-\lambda+\frac{ps}{q}}}{|1-\overline{w}\eta|^{p-\lambda+\frac{ps}{q}}}dA(\eta)\\
&\lesssim (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p\frac{(1-|\varphi_w(z)|^2)^{p-1-\lambda+\frac{ps}{q}}}{|1-\overline{w}\varphi_w(z)|^{p-\lambda+\frac{ps}{q}}}\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(z)\\
&\lesssim (1-|w|^2)^{2-\lambda}\int_{\mathbb{D}} |f'(z)|^p\frac{(1-|z|^2)^{p-1-\lambda+p+\frac{ps}{q}}}{|1-\overline{w}z|^{p+2-\lambda+\frac{ps}{q}}}dA(z)\\
&= \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-1-\lambda}\frac{(1-|z|^2)^{p+\frac{ps}{q}}(1-|w|^2)^{2-\lambda}}{|1-\overline{w}z|^{p+2-\lambda+\frac{ps}{q}}}dA(z) \lesssim \|f\|^p_{B_p(\lambda)}.
\end{aligned}$$

Thus, combine with I_1 and I_2, we get our desire results. The proof is completed. □

Theorem 3. *Suppose that $1 \le p \le q < \infty$, $0 < \lambda < 1$ and $0 < s < \infty$. Let μ be a nonnegative Borel measure on $\mathbb{D}$. The identity operator $I : B_p(\lambda) \to T^\infty_{q,s}(\mu)$ is compacted if and only if μ is a vanishing $s + \frac{q}{p}(1-\lambda)$-Carleson measure.*

Proof. Let identity operator $I : B_p(\lambda) \to T^\infty_{q,s}(\mu)$ is compacted. Let $\{I_n\}$ be a sequence arcs with $\lim_{n\to\infty} |I_n| = 0$. Denote by $w_n = (1-|I_n|)\xi_n$, where ξ_n is the midpoint of arc I_n. Set

$$f_n(z) = \frac{(1-|w_n|^2)^{\frac{p-1+\lambda}{p}}}{(1-\overline{w_n}z)} \in B_p(\lambda),$$

Note that $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\frac{\mu(S(I_n))}{|I_n|^{s+\frac{q(1-\lambda)}{p}}} \lesssim \frac{1}{|I_n|^s}\int_{S(I_n)} |f_n(z)|^q d\mu(z) \to 0,$$

as $n \to \infty$. Since I_n is arbitrary, we see that μ is a vanishing $s + \frac{q(1-\lambda)}{p}$-Carleson measure.

On the other hand, suppose that μ is a vanishing Carleson measure. We also assume that $\|f_n\|_{B_p(\lambda)} \lesssim 1$ and $\{f_n\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Note that if μ is a vanishing Carleson measure, by [19, Lemma 2.2], we have

$$\|\mu-\mu_r\|_{s+\frac{q(1-\lambda)}{p}} \to 0, r \to 1,$$

where $\mu_r(z) = \mu(z)$ for $|z| < r$ and $\mu_r(z) = 0$ for $r \le |z| < 1$. Then

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu(z)\\
&\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d(\mu-\mu_r)(z)\\
&\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|^q_{s+\frac{q(1-\lambda)}{p}} \|f_n\|^q_{B_p(\lambda)}\\
&\lesssim \frac{1}{|I|^s}\int_{S(I)} |f_n(z)|^q d\mu_r(z) + \|\mu-\mu_r\|^q_{s+\frac{q(1-\lambda)}{p}}.
\end{aligned}$$

Letting $n \to \infty$ and then $r \to 1$, we have $\lim_{n\to\infty} \|f_n\|_{T^\infty_{q,s}(\mu)} = 0$. Therefore $I : B_p(\lambda) \to T^\infty_{q,s}(\mu)$ is compact. The proof is complete.

□

4. Boundedness and compactness of T_g, I_g and M_g operators

Theorem 4. *Let p, λ, q, s be the same as Theorems 1 and 2. Suppose that $g \in H(\mathbb{D})$, then T_g is bounded (resp. compact) from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$ if and only if $g \in F(q, q-2, s+\frac{q}{p}(1-\lambda))$ (resp. $g \in F_0(q, q-2, s+\frac{q}{p}(1-\lambda))$).*

Proof. Suppose that $f \in B_p(\lambda)$ and $g \in F(q, q-2, s+\frac{q}{p}(1-\lambda))$. Then, $d\mu_g(z) = |g'(z)|^q(1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)}dA(z)$ is a $s+\frac{q}{p}(1-\lambda)$-Carleson measure. Combine with Theorem 1, we deduce that

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |(T_g f)'(z)|^q (1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)} dA(z)\\
=&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q|g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)} dA(z)\\
=&\frac{1}{|I|^s}\int_{S(I)} |f(z)|^q d\mu_g(z)\\
\lesssim&\|f\|^2_{B_p(\lambda)}\|g\|^q_{F(q,q-2,s+\frac{q}{p}(1-\lambda))}.
\end{aligned}$$

On the other hand. For any $I \in \partial\mathbb{D}$, let $w = (1-|I|)\zeta \in \mathbb{D}$, where ζ is the center of I. Then

$$1-|w| \approx |1-\overline{w}z| \approx |I|, \ \ z \in S(I).$$

If T_g is bounded from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$ and f_w is defined as in Lemma 3. We have

$$\begin{aligned}
&\frac{1}{|I|^{s+\frac{q}{p}(1-\lambda)}}\int_{S(I)} |g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)} dA(z)\\
\lesssim&\frac{1}{|I|^s}\int_{S(I)} |f_w(z)|^q|g'(z)|^q (1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)} dA(z)\\
\lesssim&\frac{1}{|I|^s}\int_{S(I)} |(T_g f_w)'(z)|^q (1-|z|^2)^{q-2+s+\frac{q}{p}(1-\lambda)} dA(z)\\
\lesssim&\|T_g F_w\|^q_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} < \infty.
\end{aligned}$$

Thus, $g \in F(q, q-2, s+\frac{q}{p}(1-\lambda))$.

Now, we consider the compactness. To prove T_g is compact if and only if for any bounded sequence $\{f_n\}$ is $B_p(\lambda)$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have

$$\lim_{n\to\infty} \|T_g(f_n)\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} = 0.$$

Hence, similar to above, we get our desire result. The proof is complete. □

Theorem 5. *Let p, λ, q, s be the same as Theorems 1 and 2. Suppose that $g \in H(\mathbb{D})$, then I_g is bounded (resp. compact) from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$ if and only if $g \in H^\infty$ (resp. $g = 0$).*

Proof. Let $f \in B_p(\lambda)$ and $g \in H^\infty$. By the growth of $B_p(\lambda)$, we have

$$|f'(z)| \lesssim \frac{\|f\|_{B_p(\lambda)}}{(1-|z|^2)^{1+\frac{1-\lambda}{p}}}.$$

Then

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^q |g(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
\lesssim&\|g\|_{H^\infty}^q \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
=&\|g\|_{H^\infty}^q \int_{\mathbb{D}} |f'(z)|^{p+(q-p)} (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
\lesssim&\|g\|_{H^\infty}^q \|f\|_{B_p(\lambda)}^{q-p} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)-(q-p)(1+\frac{1-\lambda}{p})} \left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
\lesssim&\|g\|_{H^\infty}^q \|f\|_{B_p(\lambda)}^p.
\end{aligned}$$

On the other hand. If I_g is bounded from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$, using the function F_w as in Lemma 3, subharmonic property of $|g|^q$, we easy to calculate that

$$\begin{aligned}
\infty >&\|I_g F_w\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)}^q\\
\gtrsim&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_a(z)|^2\right)^s dA(z)\\
\gtrsim&\int_{\mathbb{D}} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_w(z)|^2\right)^s dA(z)\\
\gtrsim&\int_{D(w,r)} |F_w'(z)|^q |g(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)} \left(1-|\varphi_w(z)|^2\right)^s dA(z)\\
\gtrsim&\frac{1}{(1-|w|^2)^2} \int_{D(w,r)} |g(z)|^q dA(z) \gtrsim |g(w)|^q.
\end{aligned}$$

Since $w \in \mathbb{D}$ is arbitrary, we have

$$\infty > \|I_g F_w\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)}^q \gtrsim \|g\|_{H^\infty}^q.$$

Now, we prove the compactness of I_g. It is clear that if $g = 0$, I_g is compact. Conversely. Suppose that $I_g : B_p(\lambda) \to F(q, q-2+\frac{q}{p}(1-\lambda), s)$ is compact. From above, we know that g is bounded on $\mathbb{D}$. If $g \neq 0$. Follow the maximum principle, we have $g|_{\partial\mathbb{D}} \neq 0$. Thus, there exists a constant $\delta > 0$ and a sequence $\{z_k\} \subseteq \mathbb{D}$ such that $z_k \to b \in \partial\mathbb{D}$ and $|g(z_k)| > \delta$. Using Schwarz's lemma for H^∞, we have

$$|g(z_1) - g(z_2)| \leq 2\|g\|_{H^\infty} |\varphi_{z_1}(z_2)|, \ \ z_1, z_2 \in \mathbb{D}.$$

The inequality shows that there is a sufficiently small number $\epsilon > 0$ such that $|g(z)| \geq \frac{\delta}{2}$ holds for all k and z with $|\varphi_{z_k}(z)| < \epsilon$. Notice the fact that each pseudo-hyperbolic ball $\{z \in \mathbb{D} : |\varphi_{z_k}(z)| < r\}$ is contained in a Carleson box $S(I_j)$ with $|I_k| \approx 1 - |z_k|^2$. Let

$$F_k(z) = \frac{(1-|w_k|^2)^{\frac{p-1+\lambda}{p}}}{\overline{w_k}(1-\overline{w_k}z)}.$$

Thus, we have

$$\begin{aligned}
&\|I_g F_k\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} \\
\geq &\frac{1}{|I_k|^s}\int_{S(I_k)} |F_k'(z)|^q |g(z)|^q (1-|z|^2)^{q-2+\frac{q}{p}(1-\lambda)+s} dA(z) \\
\gtrsim &\frac{1}{|I_k|^s}\int_{\{z\in\mathbb{D}:\ |\varphi_{z_k}(z)|<r\}} |g(z)|^q (1-|z|^2)^{-2+s} dA(z) \\
\approx &\delta^q.
\end{aligned}$$

The compactness of I_g gives that $\|I_g(F_k)\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} \to 0$. That is a contradiction with $\delta > 0$. Thus, $g \equiv 0$. The proof is completed. □

Theorem 6. *Let p, λ, q, s be the same as Theorems 1 and 2. Suppose that $g \in H(\mathbb{D})$, then M_g is bounded (resp. compact) from $B_p(\lambda)$ to $F(q, q-2+\frac{q}{p}(1-\lambda), s)$ if and only if $g \in F(q, q-2+\frac{q}{p}(1-\lambda), s) \cap H^\infty$ (resp. $g = 0$).*

Proof. Given $g \in F(q, q-2+\frac{q}{p}(1-\lambda), s) \cap H^\infty$. It follows from Theorems Theorems 4 and 5 that both integral operators

$$T_g : B_p(\lambda) \to F(q, q-2+\frac{q}{p}(1-\lambda), s), \quad I_g : B_p(\lambda) \to F(q, q-2+\frac{q}{p}(1-\lambda), s)$$

are bounded. So $M_g : B_p(\lambda) \to F(q, q-2+\frac{q}{p}(1-\lambda), s)$ is bounded.

On the other hand. If $f \in F(q, q-2+\frac{q}{p}(1-\lambda), s)$. It easily to deduce that

$$(1-|a|^2)^{q+\frac{q(1-\lambda)}{p}} |f'(a)|^q \lesssim \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2+\frac{q(1-\lambda)}{p}} (1-|\varphi_a(z)|^2)^s dA(z).$$

Thus,

$$|f(z)|(1-|z|^2)^{\frac{1-\lambda}{p}} \lesssim \|f\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)}.$$

Using the boundedness of M_g, we have

$$|(M_g f_w)(z)|(1-|z|^2)^{\frac{1-\lambda}{p}} \lesssim \|M_g f_w\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)}.$$

Let $z = w$. Hence, we have

$$|g(w)| \lesssim \|M_g f_w\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)}.$$

That is, $g \in H^\infty$. Note that

$$T_g f = M_g f - f(0)g(0) - I_g f.$$

It gives the boundedness of T_g, that is, $g \in F(q, q-2+\frac{q}{p}(1-\lambda), s)$.

Now, let us consider the compactness. If $g = 0$, it is clearly that M_g is compact. On the other hand. Suppose that $M_g : B_p(\lambda) \to F(q, q-2+\frac{q}{p}(1-\lambda), s)$ is compact. Let $f_n(z) = \frac{(1-|w_n|^2)^{\frac{p-1+\lambda}{p}}}{(1-\overline{w_n}z)}$ and $|w_n| \to 1$. Then $\|f_n\|_{B_p(\lambda)} \lesssim 1$ and $f_n \to 0$ uniformly on any compact of $\mathbb{D}$. Thus, $\|M_g f_n\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} \to 0$. Follow the some proof as above, we have

$$|g(w_n)| \lesssim \|M_g f_n\|_{F(q,q-2+\frac{q}{p}(1-\lambda),s)} \to 0.$$

Since g is bounded analytic function on $\mathbb{D}$, we deduce that $g = 0$. The proof is completed. □

References

[1] A. Aleman and J. Cima, An integral operator on H^p and Hardy's inequality, *J. Anal. Math.* **85** (2001), 157–176.
[2] A. Aleman and A. Siskakis, An integral operator on H^p, Complex Var. Theory Appl. 28 (1955), 149-158.
[3] A. Aleman and A. Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math. J. 46 (1997), 337-356.
[4] A. Aleman, Hilbert spaces of analytic functions between the Hardy space and the Dirichlet space, Proc. Amer. Math. Soc. 115 (1992), 97-104.
[5] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson measures for analytic Besov spaces, *Rev. Mat. Iberoamericana.*, **18** (2002), 443-510.
[6] N. Arcozzi, D. Blasi and J. Pau, Interpolating sequences on analytic Besov type spaces, Indiana Univ. Math. J. 58(2009), 1281-1318.
[7] D. Blasi and J. Pau, A charaterization of Besov type spaces and applications on Hankel type operators, Michigan Math. J. 56(2008), 401-417.
[8] L. Carleson, On a class of meromorphic functions and its associated exceptional sets, Uppsala, 1950.
[9] L. Carleson, On the zeros of functions with bounded Dirichlet integrals, Math. Z. 56 (1952), 289-295.
[10] J. Caughran, Two results concerning the zeros of functions with finite Dirichlet integral, Canad. J. Math. 21 (1969), 312-316.
[11] O. Constantin, A Volterra-type integration operator on Fock spaces, Pro. Amer. Math. Soc. 140 (2012), 4247-4257.
[12] P. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.
[13] J. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.
[14] D. Girela and J. Peláez, Growth properties and sequences of zeros of analytic functions in spaces of Dirichlet type, *J. Aust. Math. Soc.* **80** (2006), 397-418.
[15] D. Girela and J. Peláez, Carleson measures for spaces of Dirichlet type, *Integr. Equ. Oper. Theory* **55** (2006), 415-427.
[16] D. Girela and J. Peláez, Carleson measures, multipliers and integration operators for spaces of Dirichlet type, *J. Funct. Anal.* **241** (2006), 334-358.
[17] R. Kerman and E. Sawyer, Carleson measures and multipliers of Dirichlet-type spaces, Trans. Amer. Math. Soc. 309 (1988), 87-98.
[18] J. Laitila S. Miihkinen and J. Nieminen, Essential norms and weak compactness of integration operators, Arch. Math. 85 (2011), 39-48.
[19] S. Li, J. Liu and C. Yuan, Embedding theorems for Dirichlet type spaces, *Canad. Math. Bull.* http://dx.doi.org/10.4153/S0008439519000201.
[20] P. Li; J. Liu and Z. Lou, Integral operators on analytic Morrey spaces. Sci. China Math. 57(2014), 1961-1974.
[21] J. Liu and Z. Lou, Zengjian Carleson measure for analytic Morrey spaces. Nonlinear Anal. 125(2015), 423-432.
[22] C. Pommerenke, Schlichte funktionen und analytische funktionen von beschränkten mittlerer Oszillation, *Comm. Math. Helv.* **52** (1977), 591–602.
[23] J. Pau and R. Zhao, Carleson measures, Riemann-Stieltjes and multiplication operators on general family of function spaces. Integr. Equ. Oper. Theory. 78(2014), 483-514.
[24] R. Rochberg and Z. Wu, A new characterization of Dirichlet type spaces and applications, Illinois J. Math. 37 (1993), 101-122.
[25] H. Shapiro, A. Shields, On the zeros of functions with finite Dirichlet integral and some related function spaces, Math. Z. 80 (1962), 217-229.
[26] A. Siskakis and R. Zhao, A Volterra type operator on spaces of analytic functions. (Edwardsville, IL, 1998), Contemp. Math. American Mathematical Society. Rhode Island. 232 (1999), 299–311.
[27] D. Stegenga, Multipliers of the Dirichlet space, Illinois J. Math. 24 (1980), 113-139.
[28] G. Taylor, Multipliers on D_α, Trans. Amer. Math. Soc. 123 (1966), 229-240.
[29] B. Taylor and D. Williams, Zeros of Lipschitz functions analytic in the unit disc, Michigan Math. J. 18 (1971), 129-139.
[30] Z. Wu and C. Xie, Q spaces and Morrey spaces, J. Funct. Anal. 201(2003), 282-297.
[31] J. Wang, The Carleson measure problem between analytic Morrey spaces . Canad. Math. Bull. 59(2016), 878-890.
[32] J. Xiao, The Q_p Carleson measure problem. Adv Math. 217 (2008), 2075-2088.
[33] J. Xiao and W. Xu, Composition operators between analytic Campanato spaces, J. Geom. Anal. 24(2014), 649-666.
[34] J. Xiao and C. Yuan, Analytic Campanato spaces and their compositions. Indiana Univ. Math. J. 64(2015), 1001-1025.
[35] J. Xiao, Holomorphic Q Classes, Springer, LNM 1767, Berlin, 2001.
[36] J. Xiao, Geometric Q_p functions, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.
[37] S. Ye and Z. Lou, Cyclic vectors and cellular indecomposable operators on Qp spaces, Acta Math. Sci.Ser.B Engl.Ed. 31 (2011), 434-440.
[38] R. Zhao, On a general family of function spaces, *Ann. Acad. Sci. Fenn. Math. Diss.* No. **105** (1996), 56 pp.
[39] R. Zhao, Distances from Bloch functions to some Möbius invariant spaces. Ann. Acad. Sci. Fenn. Math. 33 (2008), 303-313.
[40] K. Zhu, Operator Theory in Function Spaces, American Mathematical Society, Providence, RI, 2007.