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Iterated Function System of Generalized Contractions in Partial Metric Spaces

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Abstract. In this paper we aim to obtain the attractors with the assistance of a finite family of generalized contractive mappings, which belong to a special class of mappings defined on a partial metric space. Consequently, a variety of results for iterated function systems satisfying a different set of generalized contractive conditions are acquired. We present some examples in support of the results proved herein. Our results generalize, unify and extend a variety of results which exist in current literature.

1. Introduction

Iterated function system has as a base, the mathematical foundations laid down in 1981 by Hutchinson [15]. He proved that the Hutchinson operator defined on $\mathbb{R}^k$ has as its fixed point, a subset of $\mathbb{R}^k$ which is closed and bounded, known as an attractor of iterated function system [10]. According to [11], it is a generalized version of the celebrated Banach's contraction principle which we state below.

Theorem 1.1. [9, 24] Consider a complete metric space (Y, ρ) and $h : Y \to Y$, a contraction on Y with contraction constant $\kappa \in [0, 1)$, that is, for any $v, w \in Y$, the following condition holds:

$$\rho(hv, hw) \leq \kappa\rho(v, w).$$

Then h has a unique fixed point, say u in Y. Moreover, for any initial guess $v_0 \in Y$, the sequence of simple iterates $\{v_0, hv_0, h^2v_0, h^3v_0, ...\}$ converges to u.

The importance of Banach contraction mapping principle [9] in the study of fixed point theory in metric spaces cannot be overspecialized. Its vast range of applications, which include among others, iterative methods for solving linear and nonlinear difference, differential and integral equations, attracted several researchers to intensify and extend the scope of fixed point theory in metric spaces. Some focused on the expansion of the Banach contraction principle either with the aim of generalizing the domain of the mapping [4, 5, 14, 16, 17, 29, 30] or extending the contractive condition [12, 13, 18, 21, 25, 28]. Others considered cases where the range Y of a mapping is replaced with a collection of sets which possess some special topological structure. Nadler [2, 7, 23, 27] pioneered the research of fixed point theory in metric

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spaces involving multivalued operators. Secelean studied generalized countable iterated function systems on a metric space [26].

Our primary objective in this paper is the construction of a fractal set of generalized iterated function system on a partial metric space. We observe that the Hutchinson operator defined on a finite family of contractive mappings on a complete partial metric space is itself a generalized contractive mapping on a family of compact subsets of Y. By successive application of a generalized Hutchinson operator, a final fractal is obtained and this shall be followed by a presentation of a nontrivial example in support of the proved result.

Notations $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{R}^k$ will denote a set of natural numbers, a set of nonnegative real numbers, a set of real numbers and a set of k-tuples of real numbers respectively. We give the following preliminary definitions and results [8, 22].

Definition 1.2. By a partial metric space is meant a pair (Y, p) consisting of a nonempty set Y and a function $p : Y \times Y \to \mathbb{R}^+$ defined for all $t_1, t_2, t_3 \in Y$ with the following properties:

(p_1) $t_1 = t_2$ if and only if $p(t_1, t_1) = p(t_1, t_2) = p(t_2, t_2)$,
(p_2) $p(t_1, t_1) \leq p(t_1, t_2)$,
(p_3) $p(t_1, t_2) = p(t_2, t_1)$,
(p_4) $p(t_1, t_2) + p(t_3, t_3) \leq p(t_1, t_3) + p(t_3, t_2)$.

The non-empty set Y is the space and p is a partial metric on Y.

From the definition, we see that if $p(t_1, t_2) = 0$, then properties (1) and (2) imply that $t_1 = t_2$ but in general, the converse is not true. An elementary example [8] is given by a partial metric space $(\mathbb{R}^+, p)$, with $p(t_1, t_2) = \max\{t_1, t_2\}$.

Example 1.3. [8, 20] If $Y = \{[\phi_1, \phi_2] : \phi_1, \phi_2 \in \mathbb{R}, \phi_1 \leq \phi_2\}$, then

$$p([\phi_1, \phi_2], [\phi_3, \phi_4]) = \max\{\phi_2, \phi_3\} - \min\{\phi_1, \phi_4\}$$

is a partial metric defined on Y.

Following [1, 8, 20], a T_0 topology τ_p on Y having as a base, a family of open p-balls $\{B_p(t_1, \varepsilon) : t_1 \in Y, \varepsilon > 0\}$, such that $B_p(t_1, \varepsilon) = \{t_2 \in Y : p(t_1, t_2) < p(t_1, t_1) + \varepsilon\}$ for all $t_1 \in Y$ and $\varepsilon > 0$, is generated by each partial metric p on Y.

Let p be a partial metric on Y then $p^s : Y \times Y \to \mathbb{R}^+$ with $p^s(t_1, t_2) = 2p(t_1, t_2) - [p(t_1, t_1) + p(t_2, t_2)]$, is a metric on Y [8, 20].

Moreover, $\{t_k\}$ has as its limit, a point $t \in Y$ if and only if

$$\lim_{k,\eta\to\infty} p(t_k, t_\eta) = \lim_{k\to\infty} p(t_k, t) = p(t, t).$$

Definition 1.4. [20] Consider a partial metric space (Y, p).

(i) $\{t_k\}$ is called a Cauchy sequence in Y if $\lim_{k,\eta\to\infty} p(t_k, t_\eta)$ exists.

(ii) (Y, p) is said to be complete if every Cauchy sequence $\{t_k\}$ in Y converges to a point $t \in Y$ with respect to the topology τ_p such that $p(t, t) = \lim_{k\to\infty} p(t_k, t)$.

Lemma 1.5. [8] Let (Y, p) be a partial metric space. Then,

(i) $\{t_k\}$ is Cauchy in (Y, t) if and only if it is Cauchy in (Y, p^s).

(ii) (Y, p) is complete if and only if (Y, p^s) is a complete metric space.

We shall denote by $\mathcal{CB}^p(Y)$, a collection of all closed and bounded nonempty subsets of the partial metric space (Y, p).
Let $\mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$ and $v \in Y$, define

$$p(v, \mathcal{M}) = \inf\{p(v, \mu) : \mu \in \mathcal{M}\}, \quad \delta_p(\mathcal{M}, \mathcal{N}) = \sup\{p(\mu, \mathcal{N}) : \mu \in \mathcal{M}\}$$

and

$$\delta_p(\mathcal{N}, \mathcal{M}) = \sup\{p(\eta, \mathcal{M}) : \eta \in \mathcal{N}\}.$$

Remark 1.6. For be a partial metric space (Y, p) and any nonempty set $\mathcal{M}$ in (Y, p),

$$p(\mu, \mu) = p(\mu, \mathcal{M}) \text{ if and only if } \mu \in \overline{\mathcal{M}}.$$

Furthermore $\overline{\mathcal{M}} = \mathcal{M}$ if and only if $\mathcal{M}$ is closed in (Y, p).

Now we look at some properties of the mapping $\delta_p : \mathcal{CB}^p(Y) \times \mathcal{CB}^p(Y) \to \mathbb{R}^+$ [8].

Proposition 1.7. Consider a partial metric space (Y, p). Then for any $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$, we have

(a) $\delta_p(\mathcal{L}, \mathcal{L}) = \sup\{p(\ell, \ell) : \ell \in \mathcal{L}\}$;

(b) $\delta_p(\mathcal{L}, \mathcal{L}) \leq \delta_p(\mathcal{L}, \mathcal{M})$;

(c) $\delta_p(\mathcal{L}, \mathcal{M}) = 0$ implies that $\mathcal{L} \subseteq \mathcal{M}$;

(d) $\delta_p(\mathcal{L}, \mathcal{M}) \leq \delta_p(\mathcal{L}, \mathcal{N}) + \delta_p(\mathcal{N}, \mathcal{M}) - \inf_{\eta \in \mathcal{N}} p(\eta, \eta)$.

Let (Y, p) be a partial metric space, then for $\mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$, define

$$H_p(\mathcal{M}, \mathcal{N}) = \max\{\delta_p(\mathcal{M}, \mathcal{N}), \delta_p(\mathcal{N}, \mathcal{M})\}.$$

Proposition 1.8. [8] Consider a partial metric space (Y, p). Then for all $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$,

(a) $H_p(\mathcal{L}, \mathcal{L}) \leq H_p(\mathcal{L}, \mathcal{M})$;

(b) $H_p(\mathcal{L}, \mathcal{M}) = H_p(\mathcal{M}, \mathcal{L})$;

(c) $H_p(\mathcal{L}, \mathcal{M}) \leq H_p(\mathcal{L}, \mathcal{N}) + H_p(\mathcal{N}, \mathcal{M}) - \inf_{\eta \in \mathcal{N}} p(\eta, \eta)$.

Corollary 1.9. [8] Consider a partial metric space (Y, p), then

$$H_p(\mathcal{M}, \mathcal{N}) = 0 \text{ implies that } \mathcal{M} = \mathcal{N}$$

for $\mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$.

The Example below shows that the converse of Corollary 1.9 is not true, in general.

Example 1.10. [8] Let $Y = [0, 1]$ be equipped with the partial metric $p : Y \times Y \to \mathbb{R}^+$ such that

$$p(t_1, t_2) = \max\{t_1, t_2\}.$$

From (a) of Proposition 1.7, we get

$$H_p(Y, Y) = \delta_p(Y, Y) = \sup\{t_1 : 0 \leq t_1 \leq 1\} = 1 \neq 0.$$

Based on Proposition 1.8 and Corollary 1.9, we shall refer to the mapping

$$H_p : \mathcal{CB}^p(Y) \times \mathcal{CB}^p(Y) \to \mathbb{R}^+,$$

as a partial Hausdorff metric generated by p.

Definition 1.11. Let (Y, p) be a partial metric space and $C^p \subseteq Y$. Then C^p is said to be compact if every sequence $\{v_n\}$ in C^p contains a subsequence $\{v_{n_i}\}$ which converges to a point in C^p.
It is worth noting that closed and bounded subsets of an Euclidean space $\mathbb{R}^k$ are compact. Similarly, every finite set in $\mathbb{R}^k$ is compact. The half-open interval $(0, 1] \subset \mathbb{R}$ is an example of a set which is not compact since $\{1, \frac{1}{2}, \frac{1}{2^2}, ...\} \subset (0, 1]$ does not have any convergent subsequence. Similarly the set of integers, $\mathbb{Z} \subset \mathbb{R}$ is not compact too.

Consider a partial metric space (Y, p) and denote by $C^p(Y)$ the set of all non-empty compact subsets of Y. For $\mathcal{M}, \mathcal{N} \in C^p(Y)$, let

$$H_p(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\eta\in\mathcal{N}} p(\eta, \mathcal{M}), \sup_{\mu\in\mathcal{M}} p(\mu, \mathcal{N})\},$$

where $p(t, \mathcal{M}) = \inf\{p(t, \mu) : \mu \in \mathcal{M}\}$ is a measure of how far a point t is from the set $\mathcal{M}$. Such a mapping H_p is referred to as the Pompeiu-Hausdorff metric induced by the partial metric p. $(C^p(Y), H_p)$ is a complete partial metric space, provided (Y, p) is a complete partial metric space [24].

Lemma 1.12. Let (Y, p) be a partial metric space. Then for all $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N} \in C^p(Y)$, the following conditions are true:

(a) If $\mathcal{L} \subseteq \mathcal{M}$, then $\sup_{k\in\mathcal{K}} p(k, \mathcal{M}) \le \sup_{k\in\mathcal{K}} p(k, \mathcal{L})$.

(b) $\sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t, \mathcal{M}) = \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M}), \sup_{\ell\in\mathcal{L}} p(\mathcal{L}, \mathcal{M})\}$.

(c) $H_p(\mathcal{K} \cup \mathcal{L}, \mathcal{M} \cup \mathcal{N}) \le \max\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\}$.

Proof. (a) Since $\mathcal{L} \subseteq \mathcal{M}$, for all $k \in \mathcal{K}$, we have

$$\begin{aligned} p(k, \mathcal{M}) &= \inf\{p(k, \mu) : \mu \in \mathcal{M}\} \\ &\le \inf\{p(k, \ell) : \ell \in \mathcal{L}\} = p(k, \mathcal{L}), \end{aligned}$$

this implies that

$$\sup_{k\in K} p(k, \mathcal{M}) \le \sup_{k\in K} p(k, \mathcal{L}).$$

(b)

$$\begin{aligned} \sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t, \mathcal{M}) &= \sup\{p(t, \mathcal{M}) : t \in \mathcal{K} \cup \mathcal{L}\} \\ &= \max\{\sup\{p(t, \mathcal{M}) : t \in \mathcal{K}\}, \sup\{p(t, \mathcal{M}) : t \in \mathcal{L}\}\} \\ &= \max\{\sup_{k\in K} p(k, \mathcal{M}), \sup_{\ell\in L} p(\ell, \mathcal{M})\}. \end{aligned}$$

(c) We note that

$$\begin{aligned} & \sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t, \mathcal{M} \cup \mathcal{N}) \\ \le\ & \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M} \cup \mathcal{N}), \sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{L} \cup \mathcal{N})\} \text{ (from (b))} \\ \le\ & \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M}), \sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{N})\} \quad \text{(from (a))} \\ \le\ & \max\left\{\max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M}), \sup_{\mu\in\mathcal{M}} p(\mu, \mathcal{K})\}, \max\{\sup_{\ell\in L} p(\ell, \mathcal{N}), \sup_{\eta\in\mathcal{N}} p(\eta, \mathcal{L})\}\right\} \\ =\ & \max\left\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\right\}. \end{aligned}$$

Similarly,

$$\sup_{v\in\mathcal{N}\cup\mathcal{M}} p(v,\mathcal{L}\cup\mathcal{K}) \leq \max\left\{H_p\left(\mathcal{K},\mathcal{M}\right),H_p\left(\mathcal{L},\mathcal{N}\right)\right\}.$$

Hence it follows that

$$\begin{aligned} H_p(\mathcal{K}\cup\mathcal{L},\mathcal{N}\cup\mathcal{M}) &= \max\left\{\sup_{v\in\mathcal{L}\cup\mathcal{N}} p(v,\mathcal{K}\cup\mathcal{L}), \sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t,\mathcal{M}\cup\mathcal{N})\right\} \\ &\leq \max\left\{H_p\left(\mathcal{K},\mathcal{M}\right),H_p\left(\mathcal{L},\mathcal{N}\right)\right\}. \end{aligned}$$

□

Theorem 1.13. [20] Consider a complete partial metric space (Y,p) and let $h : Y \to Y$ be a contraction mapping such that, for any $\lambda \in [0,1)$,

$$p\left(ht_1,ht_2\right) \leq \lambda p(t_1,t_2)$$

is true for all $t_1,t_2 \in Y$. Then there exists a unique fixed point u of h in Y and for every v_0 in Y (with $v = t_1$) a sequence $\{v_0, hv_0, h^2v_0, ...\}$ converges to the fixed point u of h.

Theorem 1.14. [24] Consider a partial metric space (Y,p) and let $h : Y \to Y$ be a contraction mapping. Then

(a) h maps elements in $C^p(Y)$ to elements in $C^p(Y)$.

(b) If for any $\mathcal{M} \in C^p(Y)$,

$$h(\mathcal{M}) = \{h(t_1) : t_1 \in \mathcal{M}\}, \tag{1}$$

then $h : C^p(Y) \to C^p(Y)$ is a contraction mapping on $(C^p(Y),H_p)$.

Proof. (a) We know that every contraction mapping is continuous. Moreover under every continuous mapping $h : Y \to Y$, the image of a compact subset is also compact, that is, if

$$\mathcal{M} \in C^p(Y) \text{ then } h(\mathcal{M}) \in C^p(Y).$$

(b) Let $\mathcal{M},\mathcal{N} \in C^p(Y)$. Since $h : Y \to Y$ is contraction, we obtain that

$$p\left(ht_1,h\left(\mathcal{N}\right)\right) = \inf_{t_2\in\mathcal{N}} p\left(ht_1,ht_2\right) \leq \lambda \inf_{t_2\in\mathcal{N}} p\left(t_1,t_2\right) = \lambda p\left(t_1,\mathcal{N}\right).$$

Also

$$p\left(ht_2,h\left(\mathcal{M}\right)\right) = \inf_{t_1\in\mathcal{M}} p\left(ht_2,ht_1\right) < \lambda \inf_{t_1\in\mathcal{M}} p\left(t_2,t_1\right) = \lambda p\left(t_2,\mathcal{M}\right).$$

Now

$$\begin{aligned} H_p\left(h\left(\mathcal{M}\right),h\left(\mathcal{N}\right)\right) &= \max\{\sup_{t_1\in\mathcal{M}} p(ht_1,h\left(\mathcal{N}\right)),\sup_{t_2\in\mathcal{N}} p(ht_2,h\left(\mathcal{M}\right))\} \\ &\leq \max\{\lambda\sup_{t_1\in\mathcal{M}} p(t_1,\mathcal{N}),\lambda\sup_{t_2\in\mathcal{N}} p(t_2,\mathcal{M})\} = \lambda H_p\left(\mathcal{M},\mathcal{N}\right). \end{aligned}$$

Thus h satisfies

$$H_p\left(h\left(\mathcal{M}\right),h\left(\mathcal{N}\right)\right) \leq \lambda H_p\left(\mathcal{M},\mathcal{N}\right)$$

for all $t_1,t_2 \in C^p(Y)$, and so $h : C^p(Y) \to C^p(Y)$ is a contraction mapping.

Theorem 1.15. [24] Consider a partial metric space (Y, p). Let $\{h_k : k = 1, 2, ..., r\}$ be a finite collection of contraction mappings on Y with contraction constants $\lambda_1, \lambda_2, ..., \lambda_r$, respectively. Define $\Psi : C^p(Y) \to C^p(Y)$ by

$$\begin{aligned}\Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_r(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}),\end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. Then Ψ is said to be a contraction mapping on $C^p(Y)$ with contraction constant $\lambda = \max\{\lambda_1, \lambda_2, ..., \lambda_r\}$.

Proof. We illustrate the claim for $r = 2$. Let $h_1, h_2 : Y \to Y$ be two contractions. We take $\mathcal{M}, \mathcal{N} \in C^p(Y)$. Using the result from Lemma 1.12 (c), we have that

$$\begin{aligned}H_p(\Psi(\mathcal{M}), \Psi(\mathcal{N})) &= H_p(h_1(\mathcal{M}) \cup h_2(\mathcal{N}), h_1(\mathcal{N}) \cup h_2(\mathcal{N})) \\ &\le \max\{H_p(h_1(\mathcal{M}), h_1(\mathcal{N})), H_p(h_2(\mathcal{M}), h_2(\mathcal{N}))\} \\ &\le \max\{\lambda_1 H_p(\mathcal{M}, \mathcal{N})), \lambda_2 H_p(\mathcal{M}, \mathcal{N}))\} \\ &\le \lambda H_p(\mathcal{M}, \mathcal{N}),\end{aligned}$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. □

Theorem 1.16. [24] Consider a complete partial metric space (Y, p) and let $\{h_k : k = 1, 2, ..., r\}$ be a finite collection of contraction mappings on Y. Let a mapping on $C^p(Y)$ be defined by

$$\begin{aligned}\Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_r(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}),\end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. Then

(i) $\Psi : C^p(Y) \to C^p(Y)$;

(ii) Ψ has a distinct fixed point $U_1 \in C^p(Y)$, this means that $U_1 = \Psi(U_1) = \cup_{k=1}^{r} h_k(U_1)$;

(iii) for any initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), ...\}$$

of compact sets is convergent and has a fixed point of Ψ as a limit.

Proof. (i) From the definition of Ψ and Theorem 1.14 the conclusion follows immediately since each h_k is a contraction. (ii) $\Psi : C^p(Y) \to C^p(Y)$ is a contraction too, by Theorem 1.15. Thus if (Y, p) is a complete partial metric space, then $(C^p(Y), H_p)$ is complete. As a consequence, (ii) and (iii) may be deduced from Theorem 1.14. □

Definition 1.17. Consider a partial metric space (Y, p). A mapping $\Psi : C^p(Y) \to C^p(Y)$ is called a generalized Hutchinson contraction operator if a constant $\lambda \in [0, 1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in C^p(Y)$

$$H_p(\Psi(\mathcal{M}), \Psi(\mathcal{N})) \le \lambda Z_\Psi(\mathcal{M}, \mathcal{N}),$$

where

$$\begin{aligned}Z_\Psi(\mathcal{M}, \mathcal{N}) = \; & \max\{H_p(\mathcal{M}, \mathcal{N}), H_p(\mathcal{M}, \Psi(\mathcal{M})), H_p(\mathcal{N}, \Psi(\mathcal{N})), \\ & \frac{H_p(\mathcal{M}, \Psi(\mathcal{N})) + H_p(\mathcal{N}, \Psi(\mathcal{M}))}{2}, H_p(\Psi^2(\mathcal{M}), \Psi(\mathcal{M})), \\ & H_p(\Psi^2(\mathcal{M}), \mathcal{N}), H_p(\Psi^2(\mathcal{M}), \Psi(\mathcal{N}))\}.\end{aligned}$$

Note that if Ψ defined in Theorem 1.15 is a contraction, then it is a generalized Hutchinson contraction operator but the converse is not true.

Definition 1.18. Let (Y,p) be a partial metric space, then a mapping $\Psi : C^p(Y) \to C^p(Y)$ is called a generalized rational Hutchinson contraction operator if there exists $\lambda_* \in [0,1)$ such that for any $\mathcal{M}$, $\mathcal{N} \in C^p(Y)$, the following holds:

$$H_p(\Psi(\mathcal{M}), \Psi(\mathcal{N})) \leq \lambda_* R_\Psi(\mathcal{M},\mathcal{N}),$$

where

$$\begin{aligned} R_\Psi(\mathcal{M},\mathcal{N}) &= \max\left\{\frac{H_p(\mathcal{M},\Psi(\mathcal{N}))[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{2(1+H_p(\mathcal{M},\mathcal{N}))},\right. \\ &\quad \frac{H_p(\mathcal{N},\Psi(\mathcal{N}))[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{1+H_p(\mathcal{M},\mathcal{N})}, \\ &\quad \left.\frac{H_p(\mathcal{N},\Psi(\mathcal{M}))[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{1+H_p(\mathcal{M},\mathcal{N})}\right\}. \end{aligned}$$

Definition 1.19. Let (Y,p) be a partial metric space. If $h_k : Y \to Y$, $k = 1,2,...,r$ are contraction mappings, then $\{Y; h_k, k = 1,2,\cdots,r\}$ is called iterated function system (IFS).

It follows that the generalized iterated function system consists of a partial metric space and a finite family of contraction mappings on Y.

Definition 1.20. [24] Let $\mathcal{M} \subseteq Y$ be a nonempty compact set, then $\mathcal{M}$ is an attractor of the IFS if

(i) $\Psi(\mathcal{M}) = \mathcal{M}$ and

(ii) there exist an open set $V_1 \subseteq Y$ such that $\mathcal{M} \subseteq V_1$ and $\lim_{k\to\infty} \Psi^k(\mathcal{N}) = \mathcal{M}$ for any compact set $\mathcal{N} \subseteq V_1$, where the limit is taken with respect to the partial Hausdorff metric.

Thus the maximal open set V_1 such that (ii) is satisfied is referred to as a basin of attraction.

2. Main Results

We state and prove some results on the existence and uniqueness of a fixed point of generalized Hutchinson contraction operator Ψ.

Theorem 2.1. Consider a complete partial metric space (Y,p) and an iterated function system, $\{Y; h_k, k = 1,2,\cdots,r\}$. Let $\Psi : C^p(Y) \to C^p(Y)$ be defined by

$$\begin{aligned} \Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_k(\mathcal{M}) \\ &= \cup_{k=1}^r h_k(\mathcal{M}), \end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. If Ψ is a generalized Hutchinson contraction operator, then Ψ has a unique attractor $U_1 \in C^p(Y)$, that is

$$U_1 = \Psi(U_1) = \cup_{k=1}^r h_k(U_1).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), ...\}$$

of compact sets have for a limit, an attractor of Ψ.

Proof. Choose an element $\mathcal{M}_0$ randomly in $C^p(Y)$. If $\mathcal{M}_0 = \Psi(\mathcal{M}_0)$, then there is nothing further to show. So suppose that $\mathcal{M}_0 \neq \Psi(\mathcal{M}_0)$. Define

$$\mathcal{M}_1 = \Psi(\mathcal{M}_0),\ \mathcal{M}_2 = \Psi(\mathcal{M}_1), ..., \mathcal{M}_{k+1} = \Psi(\mathcal{M}_k)$$

for $k \in \mathbb{N}$.
We assume that $\mathcal{M}_k \neq \mathcal{M}_{k+1}$ for all $k \in \mathbb{N}$. If not, then $\mathcal{M}_k = \mathcal{M}_{k+1}$ for some k implies $\mathcal{M}_k = \Psi(\mathcal{M}_k)$ and this completes the proof. Now take $\mathcal{M}_k \neq \mathcal{M}_{k+1}$ for all $k \in \mathbb{N}$. From Definition 1.17, we have

$$\begin{aligned} H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) &= H_p(\Psi(\mathcal{M}_k), \Psi(\mathcal{M}_{k+1})) \\ &\leq \lambda Z_\Psi(\mathcal{M}_k, \mathcal{M}_{k+1}), \end{aligned}$$

where

$$\begin{aligned} Z_\Psi(\mathcal{M}_k, \mathcal{M}_{k+1}) &= \max\{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k)), H_p(\mathcal{M}_{k+1}, \Psi(\mathcal{M}_{k+1})), \\ &\quad \frac{H_p(\mathcal{M}_k, \Psi(\mathcal{M}_{k+1})) + H_p(\mathcal{M}_{k+1}, \Psi(\mathcal{M}_k))}{2}, H_p(\Psi^2(\mathcal{M}_k), \Psi(\mathcal{M}_k)), \\ &\quad H_p\left(\Psi^2(\mathcal{M}_k), \mathcal{M}_{k+1}\right), H_p\left(\Psi^2(\mathcal{M}_k), \Psi(\mathcal{M}_{k+1})\right)\} \\ &= \max\{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}), \\ &\quad \frac{H_p(\mathcal{M}_k, \mathcal{M}_{k+2}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+1})}{2}, \\ &\quad H(\mathcal{M}_{k+2}, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+2}, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+2}, \mathcal{M}_{k+2})\} \\ &\leq \max\{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}), \\ &\quad \left.\frac{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, M_{k+2})}{2}\right\} \\ &= \max\{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) &\leq \lambda \max\{H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2})\} \\ &= \lambda H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), \end{aligned}$$

for all $k \in \mathbb{N}$. Now with $k, n \in \mathbb{N}$ and $n > k$, we get

$$\begin{aligned} H_p(\mathcal{M}_k, \mathcal{M}_n) &\leq H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) + ... + H_p(\mathcal{M}_{n-1}, \mathcal{M}_n) \\ &\quad - \inf_{\mu_{k+1} \in \mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) - \inf_{\mu_{k+2} \in \mathcal{M}_{k+2}} p(\mu_{k+2}, \mu_{k+2}) - \\ &\quad \cdots - \inf_{\mu_{n-1} \in \mathcal{M}_{n-1}} p(\mu_{n-1}, \mu_{n-1}) \\ &\leq [\lambda^k + \lambda^{k+1} + ... + \lambda^{n-1}] H_p(\mathcal{M}_0, \mathcal{M}_1) \\ &= \lambda^k [1 + \lambda + \lambda^2 + \cdots + \lambda^{n-k-1}] H_p(\mathcal{M}_0, \mathcal{M}_1)] \\ &\leq \frac{\lambda^k}{1-\lambda} H_p(\mathcal{M}_0, \mathcal{M}_1). \end{aligned}$$

and so $\lim_{k,n\to\infty} H_p(\mathcal{M}_k, \mathcal{M}_n) = 0$. Thus $\{\mathcal{M}_k\}$ is a Cauchy sequence in Y. Since $(C^p(Y), H_p)$ is a complete partial metric space, we have that $\mathcal{M}_k \to U_1$ as $k \to \infty$ for some $U_1 \in C^p(Y)$, that is, $\lim_{k\to\infty} H_p(\mathcal{M}_k, U_1) = \lim_{k\to\infty} H_p(M_k, \mathcal{M}_{k+1}) = H_p(U_1, U_1)$.

Now for some $U_1 \in C^p(Y)$, $\mathcal{M}_k \to U_1$ as $k \to \infty$ that is, $\lim_{k\to\infty} H_p(\mathcal{M}_k, U_1) = 0$.
To show that U_1 is the fixed point of Ψ, we assume in the contrary that $H_p(U_1, \Psi(U_1)) > 0$. So

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq H_p(U_1, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \Psi(U_1)) - \inf_{\mu_{k+1} \in \mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &= H_p(U_1, \mathcal{M}_{k+1}) + H_p(\Psi(\mathcal{M}_k), \Psi(U_1)) - \inf_{\mu_{k+1} \in \mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda Z_\Psi(\mathcal{M}_k, U_1) - \inf_{\mu_{k+1} \in \mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \end{aligned}$$

where

$$\begin{aligned} Z_{\Psi}(\mathcal{M}_k, U_1) &= \max\{H_p(\mathcal{M}_k, U_1), H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k)), H_p(U_1, \Psi(U_1)), \\ &\quad \frac{H_p(\mathcal{M}_k, \Psi(U_1)) + H_p(U_1, \Psi(\mathcal{M}_k))}{2}, H_p(\Psi^2(\mathcal{M}_k), \Psi(\mathcal{M}_k)), \\ &\quad H_p(\Psi^2(\mathcal{M}_k), U_1), H_p(\Psi^2(\mathcal{M}_k), \Psi(U_1))\} \\ &= \max\{H_p(\mathcal{M}_k, U_1), H_p(\mathcal{M}_k, \mathcal{M}_{k+1}), H_p(U_1, \Psi(U_1)), \\ &\quad \frac{H_p(\mathcal{M}_k, \Psi(U_1)) + H_p(U_1, \mathcal{M}_{k+1})}{2}, \\ &\quad H_p(\mathcal{M}_{k+2}, \mathcal{M}_{k+1}), H_p(\mathcal{M}_{k+2}, U_1), H_p(\mathcal{M}_{k+2}, \Psi(U_1))\}. \end{aligned}$$

Now we examine the following seven cases:
(1) If $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(\mathcal{M}_k, U_1)$, then

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(\mathcal{M}_k, U_1)$$

taking the limit as $k \to \infty$, gives

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(U_1, U_1)$$

which implies that $H_p(U_1, \Psi(U_1)) = 0$, and so $U_1 = \Psi(U_1)$.
(2) For $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(\mathcal{M}_k, \mathcal{M}_{k+1})$, then

$$H_p(\Psi(U_1), U_1) \leq \lambda H_p(\mathcal{M}_k, \mathcal{M}_{k+1})$$

and taking the limit as $k \to \infty$

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(U_1, U_1)$$

which implies that, $U_1 = \Psi(U_1)$.
(3) In case $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(U_1, \Psi(U_1))$, we get

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(U_1, \Psi(U_1))$$

which gives $U_1 = \Psi(U_1)$.
(4) If $Z_{\Psi}(\mathcal{M}_k, U_1) = \dfrac{H_p(\mathcal{M}_k, \Psi(U_1)) + H_p(U_1, \mathcal{M}_{k+1})}{2}$, then

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq \frac{\lambda}{2}[H_p(\mathcal{M}_k, \Psi(U_1)) + H_p(U_1, \mathcal{M}_{k+1})] \\ &\leq \frac{\lambda}{2}[H_p(\mathcal{M}_k, U_1) + H_p(U_1, \Psi(U_1)) - \inf_{u\in U_1} p(u,u) + H_p(U_1, \mathcal{M}_{k+1})] \end{aligned}$$

and taking the limit as $k \to \infty$,

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq \frac{\lambda}{2}[H_p(U_1, U_1) + H_p(U_1, \Psi(U_1)) - \inf_{u\in U_1} p(u,u) + H_p(U_1, U_1)] \\ &= \lambda\{H_p(U_1, U_1) + \frac{1}{2}[H_p(U_1, \Psi(U_1)) - \inf_{u\in U_1} p(u,u)]\}, \end{aligned}$$

that is,

$$H_p(U_1, \Psi(U_1)) \leq \frac{2\lambda}{2-\lambda}[H_p(U_1, U_1) - \inf_{u\in U_1} p(u,u)]$$

giving us $H_p(U_1, \Psi(U_1)) = 0$ and so $U_1 = \Psi(U_1)$.

(5) When $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(\mathcal{M}_{k+2}, \mathcal{M}_{k+1})$, then as $k \to \infty$, we get

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(U_1, U_1),$$

which gives $U_1 = \Psi(U_1)$.
(6) In case $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(\mathcal{M}_{k+2}, U_1)$, then as $k \to \infty$, we have

$$H_p(U_1, \Psi(U_1)) \leq \lambda H_p(U_1, U_1),$$

and so $U_1 = \Psi(U_1)$.
(7) Finally if $Z_{\Psi}(\mathcal{M}_k, U_1) = H_p(\mathcal{M}_{k+2}, \Psi(U_1))$, we have

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq \lambda H_p(\mathcal{M}_{k+2}, \Psi(U_1)) \\ &\leq \lambda[H_p(\mathcal{M}_{k+2}, U_1) + H_p(U_1, \Psi(U_1)) - \inf_{u \in U_1} p(u, u)] \end{aligned}$$

and on taking limit as $k \to \infty$, yields

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq \lambda[H_p(U_1, U_1) + H_p(U_1, \Psi(U_1)) - \inf_{u \in U_1} p(u, u)] \\ (1-\lambda) H_p(U_1, \Psi(U_1)) &\leq \lambda[H_p(U_1, U_1) - \inf_{u \in U_1} p(u, u)] \end{aligned}$$

impling that $H_p(U_1, \Psi(U_1)) \leq 0$ and so $U_1 = \Psi(U_1)$. Thus in all cases, U_1 is the attractor of Ψ. We establish the uniqueness of Ψ by assuming that U_1 and U_2 are two attractors of Ψ with $H_p(U_1, U_2) > 0$. Since Ψ is a generalized Hutchinson contraction, we have that

$$\begin{aligned} H_p(U_1, U_2) &= H_p(\Psi(U_1), \Psi(U_2)). \\ &\leq \lambda \max\{H_p(U_1, U_2), H_p(U_1, \Psi(U_1)), H_p(U_2, \Psi(U_2)), \\ &\quad \frac{H_p(U_1, \Psi(U_2)) + H_p(U_2, \Psi(U_1))}{2}, \\ &\quad H_p(\Psi^2(U_1), U_1), H_p(\Psi^2(U_1), U_2), H_p(\Psi^2(U_1), \Psi(U_2))\} \\ &= \lambda \max\{H_p(U_1, U_2), H_p(U_1, U_1), H_p(U_2, U_2), \frac{H_p(U_1, U_2) + H_p(U_2, U_1)}{2}, \\ &\quad H(U_1, U_1), H_p(U_1, U_2), H_p(U_1, U_2)\} \\ &= \lambda H_p(U_1, U_2) \end{aligned}$$

and so $(1-\lambda)H_p(U_1, U_2) \leq 0$, that is $H_p(U_1, U_2) = 0$ and hence $U_1 = U_2$. Thus $U_1 \in C^p(Y)$ is a unique attractor of Ψ. □

Remark 2.2. If in Theorem 2.1 we take $\mathcal{S}^p(Y)$, the family of all singleton subsets of the given space Y, then $\mathcal{S}^p(Y) \subseteq C^p(Y)$. Furthermore, if we take $h_k = h$ for each k, where $h = h_1$ then the operator Ψ becomes

$$\Psi(y_1) = h(y_1).$$

Consequently the following fixed point result is obtained.

Corollary 2.3. Suppose $\{Y; h_k, k = 1, 2, \cdots, r\}$ is a generalized iterated function system defined in a complete partial metric space (Y, p), define a mapping $h : Y \to Y$ as in Remark 2.2. If some $\lambda \in [0, 1)$ exists such that for any $y_1, y_2 \in C^p(Y)$ with $p(hy_1, hy_2) \neq 0$, the following holds:

$$p(hy_1, hy_2) \leq \lambda Z_h(y_1, y_2),$$

where

$$\begin{aligned} Z_h(y_1, y_2) &= \max\{p(y_1, y_2), p(y_1, hy_1), p(y_2, hy_2), \frac{p(y_1, hy_2) + p(y_2, hy_1)}{2}, \\ &\quad p(h^2y_1, y_2), p(h^2y_1, hy_1), p(h^2y_1, hy_2)\} \end{aligned}$$

then h has a unique fixed point $u \in Y$. Furthermore, for any $u_0 \in Y$, the sequence $\{u_0, hu_0, h^2u_0, ...\}$ has as a limit, a fixed point u of h.

Corollary 2.4. Let $\{Y; h_k, k = 1, 2, \cdots, r\}$ be an IFS defined in a complete partial metric space (Y, p) and each h_k for $k = 1, 2, ..., r$ be a contractive self-mapping on Y. Then $\Psi : C^p(Y) \to C^p(Y)$ defined in Theorem 2.1 has a distinct fixed point in $C^p(Y)$. Furthermore, for any initial set $\mathcal{M}_0 \in C^p(X)$, the sequence $\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), \cdots\}$ of compact sets has for a limit, a fixed point of Ψ.

Example 2.5. [8] Let $Y = [0, 10]$ be endowed with the partial metric $p : Y \times Y \to \mathbb{R}^+$ defined by

$$p(y_1, y_2) = \frac{1}{2}\max\{y_1, y_2\} + \frac{1}{4}|y_1 - y_2|$$

for all $y_1, y_2 \in Y$.

Define $h_1, h_2 : Y \to Y$ as

$$\begin{aligned} h_1(y_1) &= \frac{10 - y_1}{2} \text{ for all } y_1 \in Y \text{ and} \\ h_2(y_1) &= \frac{y_1 + 4}{4} \text{ for all } y_1 \in Y. \end{aligned}$$

Now for $y_1, y_2 \in Y$, we have

$$\begin{aligned} p(h_1(y_1), h_1(y_2)) &= \frac{1}{2}\max\left\{\frac{10 - y_1}{2}, \frac{10 - y_2}{2}\right\} + \frac{1}{4}\left|\frac{10 - y_1}{2} - \frac{10 - y_2}{2}\right| \\ &= \frac{1}{2}\left[\frac{1}{2}\max\{10 - y_1, 10 - y_2\} + \frac{1}{4}\left|y_1 - y_2\right|\right] \\ &\leq \lambda_1 p(y_1, y_2), \end{aligned}$$

where $\lambda_1 = \frac{1}{2}$.
Also for $y_1, y_2 \in Y$, we have

$$\begin{aligned} p(h_2(y_1), h_2(y_2)) &= \frac{1}{2}\max\left\{\frac{y_1 + 4}{4}, \frac{y_2 + 4}{4}\right\} + \frac{1}{4}\left|\frac{y_1 + 4}{4} - \frac{y_2 + 4}{4}\right| \\ &= \frac{1}{4}\left[\frac{1}{2}\max\{y_1 + 4, y_2 + 4\} + \frac{1}{4}\left|y_1 - y_2\right|\right] \\ &\leq \lambda_2 p(y_1, y_2), \end{aligned}$$

where $\lambda_2 = \frac{1}{4}$.
Consider the iterated function system $\{Y; h_1, h_2\}$ with the mapping $\Psi : C^p(Y) \to C^p(Y)$ defined by

$$U = \Psi(U) = h_1(U) \cup h_2(U) \text{ for all } U \in C^p(Y)$$

then for $\mathcal{M}, \mathcal{N} \in C^p(Y)$, we have by Theorem 1.15,

$$H_p(\Psi(\mathcal{M}), \Psi(\mathcal{N})) \leq \lambda^* H_p(\mathcal{M}, \mathcal{N}),$$

where $\lambda^* = \max\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{2}$.
Thus all conditions of Theorem 2.1 are satisfied. Moreover, for any initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), ...\}$$

of compact sets is convergent and has for a limit which is the attractor of Ψ. □

Theorem 2.6. Consider a complete partial metric space (Y, p) and an iterated function system, $\{Y; h_k, k = 1, 2, \cdots, r\}$. Let $\Psi : C^p(Y) \to C^p(Y)$ be defined by

$$\begin{aligned} \Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_k(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}), \end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. If Ψ is a generalized rational Hutchinson contraction operator, then Ψ has a unique attractor $U_1 \in C^p(Y)$, that is

$$U_1 = \Psi(U_1) = \cup_{k=1}^{r} h_k(U_1).$$

Furthermore, for any arbitrarily chosen initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence of compact sets

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), ...\}$$

is convergent and has for a limit, the attractor U_1 of Ψ.

Proof. Choose an arbitrary element $\mathcal{M}_0$ in $C^p(Y)$. If $\mathcal{M}_0 = \Psi(\mathcal{M}_0)$, then the proof is finished. Suppose $\mathcal{M}_0 \neq \Psi(\mathcal{M}_0)$ and define

$$\mathcal{M}_1 = \Psi(\mathcal{M}_0),\ \mathcal{M}_2 = \Psi(\mathcal{M}_1), ..., \mathcal{M}_{k+1} = \Psi(\mathcal{M}_k)$$

for $k \in \mathbb{N}$.

Assumed that $\mathcal{M}_k \neq \mathcal{M}_{k+1}$ for all $k \in \mathbb{N}$, else $\mathcal{M}_k = \Psi(\mathcal{M}_k)$ for some k and there is nothing further to show. Consider $\mathcal{M}_k \neq \mathcal{M}_{k+1}$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned} H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) &= H_p(\Psi(\mathcal{M}_k), \Psi(\mathcal{M}_{k+1})) \\ &\leq \lambda_* R_\Psi(\mathcal{M}_k, \mathcal{M}_{k+1}), \end{aligned}$$

where

$$\begin{aligned} R_\Psi(\mathcal{M}_k, \mathcal{M}_{k+1}) &= \max\left\{ \frac{H_p(\mathcal{M}_k, \Psi(\mathcal{M}_{k+1}))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{2(1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1}))}, \right. \\ &\quad \frac{H_p(\mathcal{M}_{k+1}, \Psi(\mathcal{M}_{k+1}))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})}, \\ &\quad \left. \frac{H_p(\mathcal{M}_{k+1}, \Psi(\mathcal{M}_k))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})} \right\} \\ &= \max\left\{ \frac{H_p(\mathcal{M}_k, \mathcal{M}_{k+2})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{2(1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1}))}, \right. \\ &\quad \frac{H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})}, \\ &\quad \left. \frac{H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+1})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})} \right\} \\ &= \max\{\frac{H_p(\mathcal{M}_k, \mathcal{M}_{k+2})}{2}, H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}), \\ &\quad H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+1})\} \\ &= \frac{H_p(\mathcal{M}_k, \mathcal{M}_{k+2})}{2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) &\leq \frac{\lambda_*}{2}[H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) \\ &\quad - \inf_{\xi_{k+1} \in \mathcal{M}_{k+1}} p(\xi_{k+1}, \xi_{k+1})] \\ &\leq \frac{\lambda_*}{2}[H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2})], \end{aligned}$$

$$2H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) - \lambda_* H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) \leq \lambda_*[H_p(\mathcal{M}_k, \mathcal{M}_{k+1})],$$

$$H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) \leq \frac{\lambda_*}{2-\lambda_*} H_p(\mathcal{M}_k, \mathcal{M}_{k+1}),$$

that is, for $\eta_* = \dfrac{\lambda_*}{2-\lambda_*} < 1$, we have

$$H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) \leq \eta_* H_p(\mathcal{M}_k, \mathcal{M}_{k+1})$$

for all $k \in \mathbb{N}$. Thus for $k, n \in \mathbb{N}$ with $k < n$,

$$\begin{aligned} H_p(\mathcal{M}_k, \mathcal{M}_n) &\leq H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) + \cdots + H_p(\mathcal{M}_{n-1}, \mathcal{M}_n) \\ &\quad - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) - \inf_{\mu_{k+2}\in\mathcal{M}_{k+2}} p(\mu_{k+2}, \mu_{k+2}) - \\ &\quad \cdots - \inf_{\mu_{n-1}\in\mathcal{M}_{n-1}} p(\mu_{n-1}, \mu_{n-1}) \\ &\leq \eta_*^k H_p(\mathcal{M}_0, \mathcal{M}_1) + \eta_*^{k+1} H_p(\mathcal{M}_0, \mathcal{M}_1) + \cdots + \eta_*^{n-1} H_p(\mathcal{M}_0, \mathcal{M}_1) \\ &\leq [\eta_*^k + \eta_*^{k+1} + \cdots + \eta_*^{n-1}] H_p(\mathcal{M}_0, M_1) \\ &\leq \eta_*^k [1 + \eta_* + \eta_*^2 + \cdots + \eta_*^{n-k-1}] H_p(\mathcal{M}_0, \mathcal{M}_1) \\ &\leq \frac{\eta_*^k}{1-\eta_*} H_p(\mathcal{M}_0, \mathcal{M}_1). \end{aligned}$$

By the convergence towards 0 from right hand side, we get $H_p(\mathcal{M}_k, \mathcal{M}_n) \to 0$ as $k, n \to \infty$. Therefore $\{\mathcal{M}_k\}$ is a Cauchy sequence in Y. But $(C^p(Y), H_p)$ is complete, so we have $\mathcal{M}_k \to U_1$ as $k \to \infty$ for some $U_1 \in C^p(Y)$, in other words, $\lim_{k\to\infty} H_p(\mathcal{M}_k, U_1) = \lim_{k\to\infty} H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) = H_p(U_1, U_1)$.

To prove that U_1 is the fixed point of Ψ, we assume in the contrary that $H_p(U_1, \Psi(U_1)) > 0$. This implies that

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq H_p(U_1, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \Psi(U_1)) - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &= H_p(U_1, \mathcal{M}_{k+1}) + H_p(\Psi(\mathcal{M}_k), \Psi(U_1)) - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda R_\Psi(\mathcal{M}_k, U_1) - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \end{aligned}$$

where

$$\begin{aligned} R_\Psi(\mathcal{M}_k, U_1) &= \max\left\{ \frac{H_p(\mathcal{M}_k, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{2(1 + H_p(\mathcal{M}_k, U_1))}, \right. \\ &\quad \frac{H_p(U_1, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{1 + H_p(\mathcal{M}_k, U_1)}, \\ &\quad \left. \frac{H_p(U_1, \Psi(\mathcal{M}_k))[1 + H_p(\mathcal{M}_k, \Psi(\mathcal{M}_k))]}{1 + H_p(\mathcal{M}_k, U_1)} \right\} \\ &= \max\left\{ \frac{H_p(\mathcal{M}_k, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{2(1 + H_p(\mathcal{M}_k, U_1))}, \right. \\ &\quad \frac{H_p(U_1, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U_1)}, \\ &\quad \left. \frac{H_p(U_1, \mathcal{M}_{k+1})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U_1)} \right\}. \end{aligned}$$

Consider the following three cases:

(1) If $R_{\Psi}(\mathcal{M}_k, U_1) = \dfrac{H_p(\mathcal{M}_k, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{2(1 + H_p(\mathcal{M}_k, U_1))}$, then we have

$$\begin{aligned} H_p(U_1, \Psi(U_1) &\leq H_p(U_1, \mathcal{M}_{k+1}) \\ &\quad + \frac{\lambda_*[H_p(\mathcal{M}_k, U_1) + H_p(U_1, \Psi(U_1) - \inf_{u\in U_1} p(u,u)][1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{2(1 + H_p(\mathcal{M}_k, U_1))} \\ &\quad - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{k+1}) + \frac{\lambda_*[H_p(\mathcal{M}_k, U_1) + H_p(U_1, \Psi(U_1)][1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{2(1 + H_p(\mathcal{M}_k, U_1))} \end{aligned}$$

and on taking limit as $k \to \infty$, we get

$$H_p(U_1, \Psi(U_1) \leq H_p(U_1, U_1) + \frac{\lambda_*[H_p(U_1, U_1) + H_p(U_1, \Psi(U_1) - \inf_{u_1\in U_1} p(u_1, u_1)][1 + H_p(U_1, U_1)]}{2(1 + H_p(U_1, U_1))}$$

$$(1 - \lambda_*)H_p(U_1, \Psi(U_1)) \leq \left(1 + \frac{\lambda_*}{2}\right) H_p(U_1, U_1)$$

which gives us $H_p(U_1, \Psi(U_1)) = 0$ and so $U_1 = \Psi(U_1)$.

(2) When $R_{\Psi}(\mathcal{M}_k, U_1) = \dfrac{H_p(U_1, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U)}$, we have

$$\begin{aligned} H_p(U_1, \Psi(U_1) &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda_* \left\{ \frac{H_p(U_1, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U_1)} \right\} \\ &\quad - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda_* \left\{ \frac{H_p(U_1, \Psi(U_1))[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U_1)} \right\} \end{aligned}$$

and taking the limit as $k \to \infty$, yields

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq H_p(U_1, U_1) + \lambda_* \left\{ \frac{H_p(U_1, \Psi(U_1))[1 + H_p(U_1, U_1)]}{1 + H_p(U_1, U_1)} \right\} \\ H_p(U_1, \Psi(U_1)) &\leq \frac{1}{1 - \lambda_*} H_p(U_1, U_1) \end{aligned}$$

and so $U_1 = \Psi(U_1)$.

(3) In case $R_{\Psi}(\mathcal{M}_k, U_1) = \dfrac{H_p(U_1, M_{k+1})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_n, U_1)}$, we obtain

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda_* \left\{ \frac{H_p(U_1, \mathcal{M}_{k+1})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_n, U_1)} \right\} \\ &\quad - \inf_{\mu_{k+1}\in\mathcal{M}_{k+1}} p(\mu_{k+1}, \mu_{k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{k+1}) + \lambda_* \left\{ \frac{H_p(U_1, \mathcal{M}_{k+1})[1 + H_p(\mathcal{M}_k, \mathcal{M}_{k+1})]}{1 + H_p(\mathcal{M}_k, U_1)} \right\} \end{aligned}$$

Taking the limit as $k \to \infty$,

$$\begin{aligned} H_p(U_1, \Psi(U_1)) &\leq H_p(U_1, U_1) + \lambda_* \left\{ \frac{H_p(U_1, U_1)[1 + H_p(U_1, U_1)]}{1 + H_p(U_1, U_1)} \right\} \\ H_p(U_1, \Psi(U_1)) &\leq (1 + \lambda_*) H_p(U_1, U_1) \end{aligned}$$

that is $U_1 = \Psi(U_1)$.
Thus in all three cases it was shown that U_1 is an attractor of the mapping Ψ.
For the uniqueness of attractor of Ψ, assume that U_1 and U_2 are attractors of Ψ with $H_p(U_1, U_2)$ not equal to zero. Since Ψ is a generalized rational contraction, we obtain that

$$\begin{aligned} H_p(U_1,U_2) &= H_p(\Psi(U_1),\Psi(U_2)) \\ &\leq \lambda_* \max\left\{\frac{H_p(U_1,\Psi(U_2))[1+H_p(U_1,\Psi(U_1))]}{2(1+H_p(U_1,U_2))},\right. \\ &\left.\frac{H_p(U_2,\Psi(U_2))[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,U_2)}, \frac{H_p(U_2,\Psi(U_1))[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,U_2)}\right\} \\ &= \lambda_* \max\left\{\frac{H_p(U_1,U_2)[1+H_p(U_1,U_1)]}{2(1+H_p(U_1,U_2))},\right. \\ &\left.\frac{H_p(U_2,U_2)[1+H_p(U_1,U_1)]}{1+H_p(U_1,U_2)}, \frac{H_p(U_2,U_1)[1+H_p(U_1,U_1)]}{1+H_p(U_1,U_2)}\right\} \\ &\leq \lambda_* H_p(U_1,U_2), \end{aligned}$$

and so $(1-\lambda_*)H_p(U_1,U_2) \leq 0$, which implies that $H_p(U_1,U_2) = 0$ and hence $U_1 = U_2$. Thus $U_1 \in C^p(Y)$ is a unique attractor of Ψ. □

Corollary 2.7. Consider a generalized iterated function system $\{Y; h_k, k = 1, 2, \cdots, r\}$ on a complete partial metric space (Y, p) and define a mapping $h : Y \to Y$ as in Remark 2.2. If there exists some $\lambda_* \in [0, 1)$ such that for any $y_1, y_2 \in C^p(Y)$ with $p(h(y_1), h(y_2)) \neq 0$, the following holds:

$$p(hy_1, hy_2) \leq \lambda_* R_h(y_1, y_2),$$

where

$$\begin{aligned} R_h(y_1,y_2) &= \max\left\{\frac{p(y_1,hy_2)[1+p(y_1,hy_1)]}{2(1+p(y_1,y_2))}, \frac{p(y_2,hy_2)[1+p(y_1,hy_1)]}{1+p(y_1,y_2)},\right. \\ &\left.\frac{p(y_2,hy_1)[1+p(y_1,hy_1)]}{1+p(y_1,y_2)}\right\}. \end{aligned}$$

Then h has a unique fixed point $y_1 \in Y$. Furthermore, for any initial choice of $u \in Y$, the sequence $\{u_0, hu_0, h^2u_0, ...\}$ converges to a fixed point of h.

3. Well-posedness of Iterated Function System

Lastly, we investigate the well-posedness of attractor based problems of generalized contractive operator and generalized rational contractive operator given in Definition 1.17 and 1.18, respectively, in the framework of Hausdorff partial metric spaces. Some useful results of well-posedness of fixed point problems are appearing in [3, 19].

Definition 3.1. An attractor based problem of a mapping $\Psi : C^p(Y) \to C^p(Y)$ is said to be well-posed if Ψ has a unique attractor $\Lambda^* \in C^p(Y)$ and for any sequence $\{\Lambda_k\}$ in $C^p(Y)$, $\lim_{k\to\infty} H_p(\Psi(\Lambda_k), \Lambda_k) = 0$ implies that $\lim_{k\to\infty} H_p(\Lambda_k, \Lambda^*) = H_p(\Lambda^*, \Lambda^*)$, that is, $\lim_{k\to\infty} \Lambda_k = \Lambda^*$.

Theorem 3.2. Let (Y, p) be a complete partial metric space and $\Psi : C^p(Y) \to C^p(Y)$ be defined as in Theorem 2.1. Then Ψ has a well-posed attractor based problem.
Proof. From Theorem 2.1, it follows that map Ψ has a unique attractor B_*, say.

Let a sequence $\{B_k\}$ in $C^p(Y)$ be such that $\lim_{k\to\infty} H(\Psi(B_k), B_k) = 0$. We want to show that $B_* = \lim_{k\to\infty} B_k$ for every positive integer k. As Ψ is generalized contractive Hutchinson operator, then

$$\begin{aligned} H_p(B_*, B_k) &\leq H_p(\Psi(B_*), \Psi(B_k)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda Z_\Psi(B_*, B_k) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

where

$$\begin{aligned} Z_\Psi(B_*, B_k) &= \max\Big\{H_p(B_*, B_k), H_p(B_*, \Psi(B_*)), H_p(B_k, \Psi(B_k)), \\ &\quad \frac{H_p(B_*, \Psi(B_k) + H_p(B_k, \Psi(B_*))}{2}, H_p(\Psi^2(B_*), \Psi(B_*)), \\ &\quad H_p(\Psi^2(B_*), B_k), H_p(\Psi^2(B_*), \Psi(B_k))\Big\} \\ &= \max\Big\{H_p(B_*, B_k), H_p(B_k, \Psi(B_k)), \\ &\quad \frac{H_p(B_*, \Psi(B_k) + H_p(B_k, B_*)}{2}, H_p(B_*, \Psi(B_k))\Big\}. \end{aligned}$$

Then we have the following cases:

(i) If $Z_\Psi(B_k, B_*) = H_p(B_*, B_k)$, then

$$H_p(B_*, B_k) \leq \lambda H_p(B_*, B_k) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)$$

$$H_p(B_*, B_k) - \lambda H_p(B_k, B_*) \leq H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)$$

$$H_p(B_*, B_k) \leq \frac{1}{1-\lambda}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)]$$

and as $k \to \infty$ we have

$$\lim_{k\to\infty} H_p(B_*, B_k) \leq \frac{1}{1-\lambda}[\lim_{k\to\infty} H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} \lim_{k\to\infty} p(\beta_k, \beta_k)]$$

thus $\lim_{k\to\infty} B_k = B_*$.

(ii) If $Z_\Psi(B_k, B_*) = H_p(B_k, \Psi(B_k))$, then

$$H_p(B_*, B_k) \leq \lambda H_p(B_k, \Psi(B_k)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} \lim_{k\to\infty} p(\beta_k, \beta_k)]$$

and as $k \to \infty$ we have

$$\begin{aligned} \lim_{k\to\infty} H_p(B_*, B_k) &\leq \lambda \lim_{k\to\infty} H_p(\Psi(B_k), B_k) + \lim_{k\to\infty} H_p(\Psi(B_k), B_k) \\ &\quad - \inf_{\beta_k\in\Psi(B_k)} \lim_{k\to\infty} p(\beta_k, \beta_k), \end{aligned}$$

thus $\lim_{k\to\infty} B_k = B_*$.

(iii) If $Z_\Psi(B_k, B_*) = \dfrac{H_p(B_*, \Psi(B_k) + H_p(B_k, B_*)}{2}$, then

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{2}[H_p(B_*, \Psi(B_k) + H_p(B_k, B_*)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \frac{\lambda}{2}[H_p(B_*, B_k) + H_p(B_k, \Psi(B_k)) - \inf_{b_k\in B_k} p(b_k, b_k) + H_p(B_k, B_*)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

$$\begin{aligned} H_p(B_*, B_k) - \lambda H_p(B_*, B_k) &\leq \frac{\lambda}{2}[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{2(1-\lambda)}[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + \frac{1}{1-\lambda}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)], \end{aligned}$$

and as $k \to \infty$ we have

$$\begin{aligned} \lim_{k\to\infty} H_p(B_*, B_k) &\leq \frac{\lambda}{2(1-\lambda)}[\lim_{k\to\infty} H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} \lim_{k\to\infty} p(b_k, b_k)] \\ &\quad + \frac{1}{1-\lambda}[\lim_{k\to\infty} H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} \lim_{k\to\infty} p(\beta_k, \beta_k)], \end{aligned}$$

which implies that $\lim_{k\to\infty} B_k = B_*$.

(iv) If $Z_\Psi(B_k, B_*) = H_p(B_*, \Psi(B_k))$, then

$$\begin{aligned} H_p(B_*, B_k) &\leq \lambda H_p(B_*, \Psi(B_k)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda[H_p(B_*, B_k) + H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

$$\begin{aligned} H_p(B_*, B_k) - \lambda H_p(B_*, B_k) &\leq \lambda[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{1-\lambda}[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + \frac{1}{1-\lambda}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)], \end{aligned}$$

and as $k \to \infty$ we have

$$\begin{aligned} \lim_{k\to\infty} H_p(B_*, B_k) &\leq \lambda \lim_{k\to\infty} H_p(\Psi(B_n), B_n) + \lim_{k\to\infty} H_p(\Psi(B_k), B_k) \\ &\quad - \inf_{\beta_k \in \Psi(B_k)} \lim_{k\to\infty} p(\beta_k, \beta_k), \end{aligned}$$

giving us that $\lim_{k\to\infty} B_k = B_*$. □

Theorem 3.3. Consider a complete partial metric space (Y, p) with $\Psi : C^p(Y) \to C^p(Y)$ defined as in Theorem 2.6. Then Ψ has a well-posed attractor based problem.

Proof. It follows from Theorem 2.6, that map Ψ has a unique attractor say B_*. Let $\{B_k\}$ be the sequence in $C^p(X)$ and $\lim_{k\to\infty} H_p(\Psi(B_k), B_k) = 0$. We want to show that $B_* = \lim_{k\to\infty} B_k$ for every $k \in \mathbb{N}$. As Ψ is a generalized rational contractive Hutchinson operator, then

$$\begin{aligned} H_p(B_k, B_*) &\leq H_p(\Psi(B_k), \Psi(B_*)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda_* R_\Psi(B_k, B_*) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

where

$$\begin{aligned} R_\Psi(B_k, B_*) = \max\Bigg\{ & \frac{H_p(B_k, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))}, \\ & \frac{H_p(B_*, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}, \\ & \frac{H_p(B_*, \Psi(B_k))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)} \Bigg\}. \end{aligned}$$

We consider the following three cases:

(i) For $R_\Psi(B_k, B_*) = \dfrac{H_p(B_k, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))}$, we have

$$\begin{aligned} H_p(B_*, B_k) &\leq \lambda_* \frac{H_p(B_k, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))} \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda_* H_p(B_k, B_*)[1 + H_p(B_k, \Psi(B_k))] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

$$H_p(B_*, B_k) - \lambda_* H_p(B_k, B_*)[1 + H_p(B_k, \Psi(B_k))] \leq H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k),$$

$$H_p(B_*, B_k) \leq \frac{1}{1 - \lambda_*[1 + H_p(B_k, \Psi(B_k))]}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)]$$

and by taking the limit as $k \to \infty$ gives

$$\lim_{k\to\infty} H_p(B_*, B_k) \leq 0,$$

which implies that $\lim_{k\to\infty} B_k = B_*$.

(ii) If $R_\Psi(B_k, B_*) = \dfrac{H_p(B_*, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}$, then

$$\begin{aligned} H_p(B_*, B_k) &\leq \lambda_* \left(\frac{H_p(B_*, \Psi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)} \right) \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &= H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

and by applying the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} H_p(B_*, B_k) \leq 0,$$

which implies that $\lim_{k\to\infty} B_k = B_*$.

(iii) And if $R_\Psi(B_k, B_*) = \dfrac{H_p(B_*, \Psi(B_k))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}$, then

$$\begin{aligned} H_p(B_*, B_k) &\leq \lambda_* \frac{H_p(B_*, \Psi(B_k))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)} \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda_*[H_p(B_*, B_k) + H_p(B_k, \Psi(B_k)) - \inf_{\eta_k \in B_k} p(\eta_k, \eta_k)][1 + H_p(B_k, \Psi(B_k))] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

so

$$\begin{aligned} & H_p(B_*, B_k) - \lambda_* H_p(B_*, B_k)[1 + H_p(B_k, \Psi(B_k))] \\ \leq\ & \lambda_*[H_p(B_k, \Psi(B_k)) - \inf_{\eta_k \in B_k} p(\eta_k, \eta_k)][1 + H_p(B_k, \Psi(B_k))] \\ & + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \end{aligned}$$

therefore

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{1}{(1 - \lambda_*)\,[1 + H_p(B_k, \Psi(B_k))]}[\lambda_*[H_p(B_k, \Psi(B_k)) - \inf_{\eta_k \in B_k} p(\eta_k, \eta_k)] \\ &\quad \times [1 + H_p(B_k, \Psi(B_k))] + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

which implies that $\lim_{k \to \infty} B_k = B_*$. Thus the proof is complete. □

References

[1] M. Abbas, T. Nazir, Fixed point of generalized weakly contractive mappings in partial metric spaces, Fixed Point Theory and Applications 1 (2012) 1–19.
[2] M. Abbas, M. R. Alfuraidan, T. Nazir, Common fixed points of multivalued *F*-contractions on metric spaces with a directed graph, Carpathian J. Math. 32 (2016) 1–12.
[3] M. Abbas, B. Fisher, T. Nazir, Well-posedness and periodic point property of mappings satisfying a rational inequality in an ordered complex valued metric space, Sci. Stud. Res. Series Mathematics and Informatics 22 (2012) 5–24.
[4] T. Abdeljawad, Fixed points for generalized weakly contractive mappings in partial metric spaces, Math. Comput. Modelling 54 (2011) 2923–2927.
[5] I. Arandjelović, Z. Kadelburg, S. Radenović, Boyd-Wong-type common fixed point results in cone metric spaces, Appl. Math. Comput. 217 (2011) 7167–7171.
[6] I. Altum, H. Simsek, Some fixed point theorems on dualistic partial metric spaces, J. Adv. Math. Stud. (2012) 1–8.
[7] N. A. Assad, W. A. Kirk, Fixed point theorems for setvalued mappings of contractive type, Pacific J. Math. 43 (1972) 533–562.
[8] H. Aydi, M. Abbas, C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, Topology and its Applications 159 (2012) 3234–3242.
[9] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math. 3 (1922) 133–181.
[10] M. F. Barnsley, Fractals Everywhere, 2nd ed., Academic Press, San Diego, CA (1993).
[11] M. Barnsley, A. Vince, Developments in Fractal Geometry, Bull. Math. Sci. 3 (2013) 299–348.
[12] D. W. Boyd, J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.
[13] M. Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc. 12 (1961) 07–10.
[14] L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive maps, J. Math. Anal. Appl. 332 (2007) 1467–1475.
[15] J. Hutchinson, Fractals and self-similarity, Indiana Univ. J. Math. 30 (5) (1981) 713–747.
[16] D. Illic, M. Abbas, T. Nazir, Iterative approximation of fixed points of Presić operators on partial metric spaces. Math. Nachr. 288 (14-15) (2015) 1634–1646.
[17] Z. Kadelburg, S. Radenović, V. Rakočević, Remarks on "Quasi-contraction on a cone metric space", Appl. Math. Lett. 22 (2009) 1674–1679.

[18] W. A. Kirk, Fixed points of asymptotic contractions, J. Math. Anal. Appl. 277 (2003) 645–650.
[19] M. A. Kutbi, A. Latif, T. Nazir, Generalized rational contractions in semi metric spaces via iterated function system, RACSAM 114:187 (2020) 1–16.
[20] S. G. Matthews, Partial metric topology, In: Proceedings of the 8th Summer Conference on General Topology and Applications. Annals of New York Academy of Sciences, 728 (1994) 183–197.
[21] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969) 326–329.
[22] V. Mykhaylyuk, V. Myronyk, Compactness and completeness in partial metric spaces, Topology and its applications 270 (2020) 106925 1–14.
[23] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488.
[24] T. Nazir, S. Silverstrov, M. Abbas, Fractals of generalized F-Hutchinson operator, Waves Wavelets Fractals Adv. Anal. 2 (2016) 29–40.
[25] E. Rakotch, A note on contractive mappings, Proc. Amer. Math. Soc. 13 (1962) 459–465.
[26] N. A. Secelean, Generalized countable iterated function systems, Filomat 25:1 (2011) 21–36.
[27] M. Sgro, C. Vetro, Multi-valued $F-$Contractions and the Solution of certain functional and integral equations, Filomat 27 (7) (2013), 1259–1268.
[28] S. Shukla, S. Radenovic, Z. Kadelburg, Some fixed point theorems for ordered F-generalized contractions in 0-orbitally complete partial spaces, Theor. Appl. Math. Comput. Sci. 4 (11) (2014) 87–98.
[29] S. Shukla, S. Radenovic, Presic-Boyd-Wong type results in ordered metric spaces, Inter. J. Anal. Appl. 5 (2) (2014), 154–166
[30] E. Tarafdar, An approach to fixed-point theorems on uniform spaces, Trans. Am. Math. Soc. 191 (1974) 209–225.