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Geometric Characteristics of a Manifold With a Symmetric-Type Quarter-Symmetric Projective Conformal Non-metric Connection

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Abstract. We introduce a quarter-symmetric projective conformal non-metric connection family and study its geometrical properties. Further we investigate the geometries of a symmetric-type quarter-symmetric projective conformal non-metric connection satisfying the Schur's theorem.

1. Introduction

The study of geometries and topologies of a manifold associated with a semi-symmetric (metric or non-metric) connection has been an active field over the past seven decades. In particular, the problem of geometric classifications based on the transformation groups has always been the main research topics of classical geometries. From the isometric transformation group, the conformal transformation group and the projective transformation group, the isometric geometry, the conformal geometry and the projective geometry come into being. We have reasons to believe that the study of transformation groups based on semi-symmetric connection is bound to have the important theoretical significance and potential application value for the study of related geometric classification problems.

Since the concept of the semi-symmetric connection was introduced by Friedman and Schouten in [10] for the first time, the relevant research on semi-symmetric connections has been springing up with great success. Many geometricists have devoted themselves to this field and have made great achievements. There are many celebrated works related to semi-symmetric connections. For instance, Hayden in [16] introduced the metric connection with torsion, and Yano in [27] defined a semi-symmetric metric connection and studied its geometric properties. N. Agache and M. Chafle [1] investigated the semi-symmetric non-metric connection. Before long, a quarter-symmetric connection in [12] was defined and studied. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = U(Y)\varphi X - U(X)\varphi Y, \tag{1.1}$$

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where U is a 1-form and φ is a tensor of type $(1, 1)$. Afterwards, several types of a quarter-symmetric metric connection were studied ([8, 15, 21, 25, 28]). In [11, 20, 22, 26, 31, 32], the geometric and physic properties of conformal and projective the semi-symmetric metric recurrent connections were studied. And in [23, 24] a projective conformal quarter-symmetric metric connection and a generalized quarter-symmetric metric recurrent connection were studied. In [9] a curvature copy problem of the symmetric connection was studied. And in [24] the mutual connection of a semi-symmetric connection was studied.

Recently, De, Han and Zhao in [5] introduced and studied the geometrical natures of a manifold with a semi-symmetric non-metric connection; Barman and De [2] investigated the quarter-symmetric non-metric connection on Sasakian manifolds; Chaubey and De [3], Chaubey, Suh and De [4] and De, Zhao, Mandal and Han [6], respectively, considered the geometrical features of a manifold associated with semi-symmetric connections.

Following this concept and the related researches, Zhao, Ho, An [29], Zhao, Jen and Ho [30] further studied the geometrical properties of a manifold with Ricci quarter-symmetric recurrent connections and mutual connections. For this topics, one can also see [2, 8, 15, 23, 24, 28] for details.

On the other hand, the Schur's theorem of a semi-symmetric non-metric connection is well known ([17, 18]) based only on the second Bianchi identity. A semi-symmetric metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied ([7]) and a conjugate symmetry condition of the Amari-Chentsov connection with metric recurrent was also studied.

Especially, Han, Fu and Zhao in [14] further studied the similar topics in sub-Riemannian manifolds. The readers can refer to [13, 19] for further understanding on geometry and analysis on sub-Riemannian spaces in details.

Motivated by the statements above, we define one class of quarter-symmetric non-metric connection family obtained by applying certain rules to the covariance derivative of the metric, and consider the major geometrical properties for this new type of quarter-symmetric non-metric connection family.

The paper is organized as follows. Section 2 studies the property of a quarter-symmetric projective conformal non-metric connection family; Section 3 considers the geometric characteristics of a manifold associated with a symmetric-type quarter-symmetric non-metric connection satisfying the Schur's theorem.

2. Geometries of a manifold with a quarter-symmetric projective conformal non-metric connection family

Let $(\mathcal{M}, g)$ be a Riemannian manifold ($dim\mathcal{M} \geq 3$), g be the Riemannian metric on $\mathcal{M}$ and $\overset{0}{\nabla}$ be the Levi-Civita connection with respect to g. Denote $T(\mathcal{M})$ by the collection of all vector fields on $\mathcal{M}$.

Definition 2.1. *A connection $\overset{n}{\nabla}$ is called a quarter-symmetric non-metric connection if it satisfies the relation*

$$(\overset{0}{\nabla}_Z g)(X, Y) = -\pi(X)U(Y, Z) - \pi(Y)U(X, Z), \ \overset{n}{T}(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y$$

The local expressions of this relations are as follows

$$\overset{n}{\nabla}_k g_{ji} = -\pi_i U_{jk} - \pi_j U_{ik}, \ \overset{n}{T}{}^k_{ji} = \pi_j \varphi^k_i - \pi_i \varphi^k_j,$$

and the connection coefficient is

$$\overset{n}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\} + \pi_j U^k_i - \pi_i V^k_j.$$

Definition 2.2. *A connection family $\overset{t}{\nabla}$ is called a quarter-symmetric non-metric connection family if it satisfies*

$$(\overset{t}{\nabla}_Z g)(X, Y) = (t-1)[\pi(X)U(Y, Z) + \pi(Y)U(X, Z)], \ \overset{t}{T}(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y$$

where $t \in \mathbb{R}$ is a family parameter.

The local expressions of this relations above are below

$$\overset{t}{\nabla}_k g_{ji} = (t-1)[\pi_i U_{jk} + \pi_j U_{ik}], \ \ \overset{t}{T}{}^k_{ji} = \pi_j \varphi^k_i - \pi_i \varphi^k_j,$$

and the connection coefficient is

$$\overset{t}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i - \pi_i V^k_j - tU_{ij}\pi^k.$$

Remark 2.1. *The quarter-symmetric non-metric connection family $\overset{t}{\nabla}$ is a connection homotopy of the quarter-symmetric metric connection $\overset{q}{\nabla}$ in [30] and the quarter-symmetric non-metric connection $\overset{n}{\nabla}$. Namely if $t = 0$, then $\overset{t}{\nabla} = \overset{n}{\nabla}$ and if $t = 1$, then $\overset{t}{\nabla} = \overset{q}{\nabla}$.*

The connection $\overset{c}{\nabla}$ is called a quarter-symmetric conformal non-metric connection family, if $\overset{c}{\nabla}$ is conformally equivalent to a quarter-symmetric non-metric connection family $\overset{t}{\nabla}$. The coefficient of the connection $\overset{c}{\nabla}$ is

$$\overset{c}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \sigma_i\delta^k_j + \sigma_j\delta^k_i - g_{ij}\delta^k - \pi_i V^k_j + \pi_j U^k_i - tU_{ij}\pi^k$$

where the conformal metric $\bar{g}_{ij} = e^{2\sigma} g_{ij}$ and $\sigma_i = \partial_i\sigma$.

The connection $\overset{p}{\nabla}$ is called a quarter-symmetric projective non-metric connection family, if $\overset{p}{\nabla}$ is projectively equivalent to a quarter-symmetric non-metric connection family $\overset{t}{\nabla}$. The coefficient of the connection $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \psi_i\delta^k_j + \psi_j\delta^k_i + \pi_i U^k_j - \pi_i V^k_j - tU_{ij}\pi^k,$$

where ψ_i is a projective component.

Definition 2.3. *A connection ∇ is called a quarter-symmetric projective conformal non-metric connection family, if ∇ is projectively and conformally equivalent to a quarter-symmetric non-metric connection family $\overset{t}{\nabla}$.*

In a Riemannian manifold, a quarter-symmetric projective conformal non-metric connection family ∇ satisfies the relation

$$\begin{cases} \nabla_Z g(X,Y) = -2[\psi(Z) + Z\sigma]g(X,Y) - \psi(X)g(Y,Z) - \psi(Y)g(X,Z) \\ \qquad\qquad +(t-1)[\pi(X)U(Y,Z) + \pi(Y)U(X,Z)], \\ T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \end{cases} \tag{2.1}$$

The local expression of this relation is

$$\begin{cases} \nabla_k g_{ji} = -2(\psi_k + \sigma_k)g_{ij} - \psi_i g_{jk} - \psi_j g_{ik} + (t-1)(\pi_i U_{jk} + \pi_j U_{ik}), \\ T^k_{ji} = \pi_j\varphi^k_i - \pi_i\varphi^k_j \end{cases} \tag{2.2}$$

and its coefficient is

$$\Gamma^k_{ij} \ = \ \{^k_{ij}\} + (\psi_i + \sigma_i)\delta^k_j + (\psi_j + \sigma_j)\delta^k_i - g_{ij}\sigma^k + \pi_j U^k_i - \pi_i V^k_j - tU_{ij}\pi^k \tag{2.3}$$

Remark 2.2. *If $\psi_i = 0$, then $\nabla = \overset{c}{\nabla}$ and if $\sigma_i = 0$, then $\nabla = \overset{p}{\nabla}$. If $\psi_i = \sigma_i = 0$, then $\nabla = \overset{t}{\nabla}$. And if $t = 1$ and $\psi_i = \sigma_i = 0$, then $\nabla = \overset{q}{\nabla}$ and if $\psi_i = \sigma_i = 0$ and $t = 0$, then $\nabla = \overset{n}{\nabla}$. And if $\varphi X = X$, then a quarter-symmetric projective conformal non-metric connection family ∇ is a projective conformal semi-symmetric non-metric connection family.*

From the expression (2.3) we find that the curvature tensor of ∇ is

$$\begin{aligned} R_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_j a_{ik} - \delta^l_i a_{jk} + g_{ik} b^l_j - g_{jk} b^l_i + U^l_j c_{ik} - U^l_i c_{jk} + U_{ik} d^l_j - U_{jk} d^l_i \\ &+ \delta^l_k(\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) - t(\nabla_i U_{jk} - \nabla_j U_{ik})\pi^l + (\nabla_i U^l_j - \nabla_j U^l_i)\pi_k + \pi_i V^l_{jk} - \pi_j V^l_{ik} \\ &- V^l_k(\overset{0}{\nabla}_i \pi_j - \overset{0}{\nabla}_j \pi_i), \end{aligned} \tag{2.4}$$

where $K_{ijk}{}^{l}$ is the curvature tensor of $\overset{0}{\nabla}$ of g_{ij}, and

$$\begin{cases} a_{ik} &= \overset{0}{\nabla}_i(\psi_k + \sigma_k) - (\psi_i + \sigma_i)(\psi_k + \sigma_k) - U_{ip}(\psi^p + \sigma^p)\pi_k + tU_{ik}(\psi_p + \sigma_p)\pi^p + g_{ik}(\psi_p + \sigma_p)\sigma^p, \\ b_{ik} &= \overset{0}{\nabla}_i \sigma_k - \sigma_i \sigma_k + U_{ik}\pi^p \sigma_p - tU^p_i \sigma_p \pi_k, \\ c_{ik} &= \overset{0}{\nabla}_i \pi_k - \pi_i(\sigma_k + \psi_k) - U^p_i \pi_p \pi_k + tU_{ik}\pi_p \pi^p \\ d_{ik} &= t(\overset{0}{\nabla}_i \pi_k - \pi_i \sigma_k - tU^p_i \pi_k \pi_p), \\ V^l_{ik} &= \overset{0}{\nabla}_i V^l_k - V^l_i(\psi_k + \sigma_k) + U^l_i V^p_k \pi_p - U^p_i V^l_p \pi_k + t(U_{ik} V^l_p \pi^p - U_{ip} V^p_k \pi^l) \\ &\quad + V_{ik}\sigma^l + g_{ik} V^l_p \sigma^p + \delta^l_i V^p_k(\psi_p + \sigma_p) \end{cases} \tag{2.5}$$

From the expression (2.2), the mutual connection $\overset{m}{\nabla}$ of the quarter-symmetric projective conformal non-metric connection family ∇ satisfies

$$\begin{cases} \overset{m}{\nabla}_k g_{ji} = -2(\psi_k + \sigma_k)g_{ij} - \psi_i g_{jk} - \psi_j g_{ik} + 2\pi_k U_{ij} + \pi_j(U_{ki} + tU_{ki}) + \pi_j(V_{ki} + tV_{ki}), \\ \overset{m}{T}{}^k_{ji} = \pi_j(U^k_i + V^k_i) - \pi_i(U^k_j + V^k_j), \end{cases} \tag{2.6}$$

and its coefficient is

$$\overset{m}{\Gamma}{}^k_{ij} = \left\{{k \atop ij}\right\} + (\psi_i + \sigma_i)\delta^k_j + (\psi_j + \sigma_j)\delta^k_i - g_{ij}\sigma^k + \pi_i U^k_j - \pi_j V^k_i - tU_{ij}\pi^k \tag{2.7}$$

The curvature tensor of $\overset{m}{\nabla}$ is

$$\begin{aligned} \overset{m}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_j \overset{m}{a}_{ik} - \delta^l_i \overset{m}{a}_{jk} + g_{ik}\overset{m}{b}{}^l_j - g_{jk}\overset{m}{b}{}^l_i + \delta^l_k(\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) \\ &+ (\overset{m}{c}{}^l_j U_{ik} - \overset{m}{c}{}^l_i U_{jk}) + d^l_{ik}\pi_j - d^l_{jk}\pi_i + U^l_k(\overset{0}{\nabla}_i \pi_j - \overset{0}{\nabla}_j \pi_i) \\ &+ t(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l + V^l_i e_{jk} - V^l_j e_{ik} + (V^l_{ji} - V^l_{ij})\pi_k \end{aligned} \tag{2.8}$$

where

$$\begin{cases} \overset{m}{a}_{ik} &= \overset{0}{\nabla}_i(\psi_k + \sigma_k) - (\psi_i + \sigma_i)(\psi_k + \sigma_k) + tU_{ik}(\psi_p + \sigma_p)\pi^p - \pi_i U^p_k(\psi_p + \sigma_p) + g_{ik}(\psi_p + \sigma_p)\sigma^p, \\ \overset{m}{b}_{ik} &= \overset{0}{\nabla}_i \sigma_k - \sigma_i \sigma_k - tU_{ip}\sigma^p \pi_k + \pi_i U_{kp}\pi^p, \\ \overset{m}{c}_{ik} &= t(\overset{0}{\nabla}_i \pi_k - \pi_i \sigma_k + \pi_i U_{kp}\pi^p - tU_{ip}\pi^p \pi_k), \\ d^{\,l}_{ik} &= \overset{0}{\nabla}_i U^l_k - U^l_i(\psi_k + \sigma_k) + U_{ik}\sigma^l - tU_{ip}U^p_k \pi^l, \\ e_{ik} &= \overset{0}{\nabla}_i \pi_k - \pi_i(\psi_k + \sigma_k) - \pi_i U^p_k \pi_p + V^p_i \pi_p \pi_k + tU_{ik}\pi_p \pi^p + g_{ik}\pi_p \sigma^p, \\ V^{\,l}_{ij} &= \overset{0}{\nabla}_i V^l_j + \delta^l_i V^p_j(\psi_p + \sigma_p) + V_{ij}\sigma^l + \pi_i V^p_j U^l_p - tU^p_i V_{jp}\pi^l. \end{cases} \tag{2.9}$$

Let α, β and γ be of 1-forms with its components

$$\alpha_i = U^k_i \pi_k, \;\; \beta_i = U^k_k \pi_i, \;\; \gamma_i = V^k_i \pi_k \tag{2.10}$$

Theorem 2.1. *A Riemannian manifold $(\mathcal{M}, g, \nabla)$ associated with 1-form ψ and α being of closed, then it is Ricci-like flat with respect to the quarter-symmetric projective conformal non-metric connection family ∇, namely, there holds*

$$P_{ij} = 0 \tag{2.11}$$

where $P_{ij} \hat{=} R_{ijkl} g^{kl}$ is a volume (or Ricci-like) curvature tensor of ∇.

Proof. Contracting the indices k and l of the expression (2.4), then we obtain

$$\begin{aligned} P_{ij} &= \overset{0}{P}_{ij} + a_{ij} - a_{ji} + b_{ji} - b_{ij} - U_i^k c_{jk} + U_j^k c_{ik} - U_{jk} d_i^k - U_{ik} d_j^k + (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^k \\ &- t(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik}) + n(\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) + \pi_i V_{jk}^k - \pi_j V_{ik}^k - V_k^k(\overset{0}{\nabla}_i \pi_j - \overset{0}{\nabla}_j \pi_i), \end{aligned} \tag{2.12}$$

where $\overset{0}{P}_{ij}$ is the volume (Ricci-like) curvature tensor of the Levi-Civita connection $\overset{0}{\nabla}$ of g_{ij}. At the same time, $\overset{0}{P}_{ij} = 0$, $V_k^k = 0$. Using (2.5), we have

$$\begin{aligned} a_{ij} - a_{ji} + b_{ji} - b_{ij} &= (\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) + [\pi_i U_{jp}(\psi^p + \sigma^p) - \pi_j U_{ip}(\psi^p + \sigma^p)] - t(\pi_i U_j^p \sigma_p - \pi_j U_i^p \sigma_p), \\ U_j^k c_{ik} - U_i^k c_{jk} &= U_{jk} d_i^k - U_{ik} d_j^k + U_j^k \overset{0}{\nabla}_i \pi_k - U_i^k \overset{0}{\nabla}_j \pi_k - [\pi_i U_j^k(\psi_k + \sigma_k) - \pi_j U_i^k(\psi_k + \sigma_k)] \\ &\quad + t(U_{ik} \overset{0}{\nabla}_j \pi^k - U_{jk} \overset{0}{\nabla}_i \pi^k) + t(\pi_i U_{jk} \sigma^k - \pi_j U_{ik} \sigma^k), \\ V_{ik}^k &= 0. \end{aligned}$$

Substituting these statements above into (2.12) and using (2.10) we have

$$P_{ij} = (n+1)(\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) + (1-t)(\overset{0}{\nabla}_i \alpha_j - \overset{0}{\nabla}_j \alpha_i) \tag{2.13}$$

If a 1-form ψ and a 1-form α are of closed, then $\overset{0}{\nabla}_j \psi_i = \overset{0}{\nabla}_i \psi_j$ and $\overset{0}{\nabla}_j \alpha_i = \overset{0}{\nabla}_i \alpha_j$. From (2.13), we know that (2.11) is tenable. □

Remark 2.3. *Formula (2.13) implies that if $t = 1$ and ψ is a closed form, then $\overset{c}{P}_{ij} = \overset{n}{P}_{ij} = 0$.*

Theorem 2.2. *For a Riemannian manifold $(\mathcal{M}, g, \overset{m}{\nabla})$, if 1-form ψ, α, β and γ are of closed, then the Riemannian manifold $(\mathcal{M}, g, \overset{m}{\nabla})$ is Ricci-like flat with resect to the mutual connection $\overset{m}{\nabla}$, namely*

$$\overset{m}{P}_{ij} = 0, \tag{2.14}$$

where is a volume (Ricci-like) curvature tensor of $\overset{m}{\nabla}$.

Proof. Contracting the indices k and l of the expression (2.8), then we have

$$\begin{aligned} \overset{m}{P}_{ij} &= \overset{0}{P}_{ij} + \overset{m}{a}_{ij} - \overset{m}{a}_{ji} + \overset{m}{b}_{ji} - \overset{m}{b}_{ij} + n(\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) + (U_{ik} \overset{m}{c}{}_j^k - U_{jk} \overset{m}{c}{}_i^k) \\ &+ U_k^k(\overset{0}{\nabla}_i \pi_j - \overset{0}{\nabla}_j \pi_i) + (d_{ik}^k \pi_j - d_{jk}^k \pi_i) - t(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^k \\ &- t(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^k + V_i^k e_{jk} - V_j^k e_{ik} + (V_{ji}^k - V_{ij}^k)\pi_k \end{aligned} \tag{2.15}$$

From (2.9) we arrive at

$$\overset{m}{a}_{ij} - \overset{m}{a}_{ji} + \overset{m}{b}_{ji} - \overset{m}{b}_{ij} = (\overset{0}{\nabla}_i \psi_j - \overset{0}{\nabla}_j \psi_i) - [\pi_i U_j^p(\psi_p + \sigma_p) - \pi_j U_i^p(\psi_p + \sigma_p)] - (t-1)(\pi_i U_{jp} \sigma^p - \pi_j U_{ip} \sigma^p),$$

$$\begin{aligned} U_{ik}\overset{mk}{c}_j - U_{jk}\overset{mk}{c}_i &= t(U_{ik}\overset{0}{\nabla}_j\pi^k - U_{jk}\overset{0}{\nabla}_i\pi^k) + t(\pi_i U_{jk}\sigma^k - \pi_j U_{ik}\sigma^k) - t(\pi_i U_{jk}U_p^k\pi^p - \pi_j U_{ik}U_p^k\pi^p), \\ d_{ik}^k\pi_j - d_{jk}^k\pi_i &= [\pi_i U_j^k(\psi_k + \sigma_k) - \pi_j U_i^k(\psi_k + \sigma_k)] + (\pi_i U_{jk}\sigma^k - \pi_j U_{ik}\sigma^k) \\ &+ t(\pi_i U_{jp}U_k^p\pi^k - \pi_j U_{ip}U_k^p\pi^k) + (\overset{0}{\nabla}_i U_k^k\pi_j - \overset{0}{\nabla}_j U_k^k\pi_i), \\ V_i^k e_{jk} - V_j^k e_{ik} &= (V_{ij}^k - V_{ji}^k)\pi_k - [\overset{0}{\nabla}_i(V_j^k\pi_k) - \overset{0}{\nabla}_j(V_i^k\pi_k)]. \end{aligned}$$

Substituting these formulaes above into (2.15) and using (2.10), one gets

$$\overset{m}{P}_{ij} = (n+1)(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - t(\overset{0}{\nabla}_i\alpha_j - \overset{0}{\nabla}_j\alpha_i) + (\overset{0}{\nabla}_i\beta_j - \overset{0}{\nabla}_j\beta_i) - (\overset{0}{\nabla}_i\gamma_j - \overset{0}{\nabla}_j\gamma_i). \tag{2.16}$$

If a 1-form ψ, α, β and γ are of closed, then there holds $\overset{0}{\nabla}_i\alpha_j - \overset{0}{\nabla}_j\alpha_i = 0$, $\overset{0}{\nabla}_i\beta_j - \overset{0}{\nabla}_j\beta_i = 0$ and $\overset{0}{\nabla}_i\gamma_j - \overset{0}{\nabla}_j\gamma_i = 0$. From (2.16), it is not hard to see that (2.14) is tenable. □

Set by

$$A_{ijk}{}^l \hat{=} \delta_j^l a_{ik} + g_{ik}b_j^l + (U_j^l c_{ik} + U_{ik}d_j^l) + \overset{0}{\nabla}_i U_j^l\pi_k - t\overset{0}{\nabla}_i U_{jk} + \delta_k^l\overset{0}{\nabla}_i\psi_j + \pi_i V_{jk}^l - V_k^l\overset{0}{\nabla}_i\pi_j,$$

and

$$\overset{m}{A}_{ijk}{}^l \hat{=} \delta_j^l\overset{m}{a}_{ik} + g_{ik}\overset{ml}{b}_j + U_{ik}\overset{m}{c}{}^l_j + d_{ik}^l\pi_j + \delta_k^l\overset{0}{\nabla}_i\psi_j + U_k^l\overset{0}{\nabla}_i\pi_j - t\overset{0}{\nabla}_i U_{jk}\pi^l + V_i^l e_{jk} + V_{ij}^l\pi_k,$$

From these formulaes, we know that (2.4) and (2.8) are exactly as follows

$$R_{ijk}{}^l = K_{ijk}{}^l + A_{ijk}{}^l - A_{jik}{}^l,$$

and

$$\overset{m}{R}_{ijk}{}^l = \overset{m}{K}_{ijk}{}^l + \overset{m}{A}_{ijk}{}^l - \overset{m}{A}_{jik}{}^l,$$

respectively.

Thus we get the following.

Theorem 2.3. *When $A_{ijk}{}^l = A_{jik}{}^l$, then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to \nabla$ and when $\overset{m}{A}_{ijk}{}^l = \overset{m}{A}_{jik}{}^l$, then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to \overset{m}{\nabla}$.*

From (2.2) and (2.3), the connection coefficient of dual connection $\overset{*}{\nabla}$ of the quarter-symmetric projective conformal non-metric connection family ∇ is

$$\overset{*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - (\psi_i + \sigma_i)\delta_j^k + \sigma_j\delta_i^k - g_{ij}(\psi^k + \sigma^k) + t\pi_j U_i^k - \pi_i V_j^k - U_{ij}\pi^k,$$

and the curvature tensor of $\overset{*}{\nabla}$ is

$$\begin{aligned} \overset{*}{R}_{ijk}{}^l &= K_{ijk}{}^l + \delta_j^l b_{ik} - \delta_i^l b_{jk} + g_{ik}a_j^l - g_{jk}a_i^l + (U_j^l d_{ik} - U_i^l d_{jk} - U_{jk}c_i^l + U_{ik}c_j^l) \\ &- \delta_k^l(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + t(\overset{0}{\nabla}_i U_j^l - \overset{0}{\nabla}_j U_i^l)\pi^k - (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l \\ &+ \pi_i\overset{*}{V}{}^l_{jk} - \pi_j\overset{*}{V}{}^l_{ik} - V_k^l(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i), \end{aligned} \tag{2.17}$$

where

$$\overset{*}{V}{}^l_{jk} = \overset{0}{\nabla}_i V_k^l - V_i^l\sigma_k + U_{ik}V_p^l\pi^p - U_{ip}V_k^p\pi^l + \delta_i^l V_k^p\sigma_p + tU_i^l V_k^p\pi_p - tU_i^p V_p^l\pi_k + V_{ik}(\psi^l + \sigma^l)$$

$$+ \quad g_{ik}V^l_p(\psi^p + \sigma^p),$$

From (2.4) and (2.17), we obtain

$$\begin{aligned}\overset{*}{R}_{ijk}{}^l &= K_{ijk}{}^l + \delta^l_i(a_{jk} - b_{jk}) - \delta^l_j(a_{ik} - b_{ik}) + g_{ik}(a^l_j - b^l_j) - g_{jk}(a^l_i - b^l_i) + U^l_i(c_{jk} - d_{jk}) - U^l_j(c_{ik} - d_{ik}) \\ &+ U_{ik}(c^l_j - d^l_j) - U_{jk}(c^l_i - d^l_i) + (t-1)[(\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k - (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l] \\ &+ \pi_i(\overset{*}{V}{}^l_{jk} - V^l_{jk}) - 2\delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - \pi_j(\overset{*}{V}{}^l_{ik} - V^l_{ik}).\end{aligned} \tag{2.18}$$

From (2.6) and (2.7), the connection coefficient of dual connection $\overset{m*}{\nabla}$ of the mutual connection $\overset{m}{\nabla}$ of the quarter-symmetric projective conformal non-metric connection family ∇ is

$$\overset{m*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - (\psi_i + \sigma_i)\delta^k_j + \sigma_j\delta^k_i - g_{ij}(\psi^k + \sigma^k) - \pi_i U^k_j + t\pi_j U^k_i + V_{ij}\pi^k$$

and the curvature tensor of $\overset{m*}{\nabla}$ is

$$\begin{aligned}\overset{m*}{R}_{ijk}{}^l &= K_{ijk}{}^l + \delta^l_j\overset{m}{b}_{ik} - \delta^l_i\overset{m}{b}_{jk} + g_{ik}\overset{m}{a}{}^l_j - g_{jk}\overset{m}{a}{}^l_i - \delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + U^l_j\overset{m}{c}_{ik} - U^l_i\overset{m}{c}_{jk} \\ &- U^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + (\overset{*}{d}{}^l_{jk}\pi_i - \overset{*}{d}{}^l_{ik}\pi_j) + t(\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k \\ &+ (V_{jk}e^l_i - V_{ik}el_j) + (\overset{*}{V}_{ijk} - \overset{*}{V}_{jik})\pi^l,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\overset{*}{d}{}^l_{jk} &= \overset{0}{\nabla}_i U^l_k - U^l_i\sigma_k - U^p_i U^l_p\pi_k - U_{ik}(\psi^l + \sigma^l), \\ \overset{*}{V}_{ijk} &= \overset{0}{\nabla}_i V_{jk} + V_{ij}\sigma_k - V_{ip}\pi_j U^p_k + g_{ik}V_{jp}(\psi^p + \sigma^p) - tV_{ip}U^p_j\pi_k.\end{aligned}$$

From (2.8) and (2.19), we obtain

$$\begin{aligned}\overset{m*}{R}_{ijk}{}^l &= \overset{m}{R}_{ijk}{}^l + \delta^l_i(\overset{m}{a}_{jk} - \overset{m}{b}_{jk}) - \delta^l_j(\overset{m}{a}_{ik} - \overset{m}{b}_{ik}) + g_{ik}(\overset{m}{a}{}^l_j - \overset{m}{b}{}^l_j) - g_{jk}(\overset{m}{a}{}^l_i - \overset{m}{b}{}^l_i) \\ &+ (\overset{m}{c}_{ik}U^l_j - \overset{m}{c}_{jk}U^l_i + U_{jk}\overset{m}{c}{}^l_i - U_{ik}\overset{m}{c}{}^l_j) + (\overset{*}{d}_{jk}{}^l + d_{jk}{}^l)\pi_i - (\overset{*}{d}_{ik}{}^l + d_{ik}{}^l)\pi_j \\ &- 2\delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + t[(\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k + (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l] \\ &- 2U^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) + V_{jk}e^l_i - V_{ik}e^l_j + V^l_j e_{ik} - V^l_i e_{jk} + (\overset{m}{V}_{ijk} - \overset{m}{V}_{jik})\pi^l - (V^l_{ij} - V^l_{ji})\pi_k,\end{aligned} \tag{2.20}$$

Denote by

$$\begin{aligned}B_{ijk}{}^l &= \delta^l_i(a_{jk} - b_{jk}) + g_{ik}(a^l_j - b^l_j) + U^l_i(c_{jk} - d_{jk}) + U_{ik}(c^l_j - d^l_j) + (t-1)(\overset{0}{\nabla}_i U^l_j\pi_k + \overset{0}{\nabla}_i U_{jk}\pi^l) \\ &+ \pi_i(\overset{*}{V}{}^l_{jk} - V^l_{jk}) - 2\delta^l_k\overset{0}{\nabla}_i\psi_j\end{aligned}$$

and

$$\begin{aligned}\overset{m}{B}_{ijk}{}^l &= \delta^l_i(\overset{m}{a}_{jk} - \overset{m}{b}_{jk}) + g_{ik}(\overset{m}{a}{}^l_j - \overset{m}{b}{}^l_j) + (U^l_j\overset{m}{c}_{ik} + U_{jk}\overset{m}{c}{}^l_i) + \pi_i(\overset{*}{d}{}^l_{jk} + d^l_{jk}) - 2\delta^l_k\overset{0}{\nabla}_i\pi_j \\ &+ t(\overset{0}{\nabla}_i U^l_j\pi_k + \overset{0}{\nabla}_i U_{jk}\pi^l) + V^k_j e^l_i + V^l_j e_{ik} + \overset{*}{V}_{ijk}\pi^l + V_{ij}{}^l\pi_k - 2U^l_k\overset{0}{\nabla}_i\pi_j\end{aligned}$$

From these formulaes above, we get (2.18) and (2.20) are given below

$$\overset{*}{R}_{ijk}{}^l = R_{ijk}{}^l + B_{ijk}{}^l - B_{jik}{}^l,$$

and

$$\overset{m*}{R}_{ijk}{}^{l} = \overset{m}{R}_{ijk}{}^{l} + \overset{m}{B}_{ijk}{}^{l} - \overset{m}{B}_{jik}{}^{l},$$

respectively.

So there exists the following.

Theorem 2.4. *When $B_{ijk}{}^{l} = B_{jik}{}^{l}$, then the quarter-symmetric projective conformal non-metric connection family ∇ is a conjugate symmetry and when $\overset{m}{B}_{ijk}{}^{l} = \overset{m}{B}_{jik}{}^{l}$, then the mutual connection $\overset{m}{\nabla}$ of the quarter-symmetric projective conformal non-metric connection family ∇ is a conjugate symmetric.*

3. Geometries of a manifold with symmetric-type quarter-symmetric non-metric connections

For convenience, we assume in this subsection ψ_{ij} is a symmetric tensor, and $\psi_{ij} = U_{ij}$.

Definition 3.1. *A connection $\overset{s}{\nabla}$ is called a symmetric-typequarter-symmetric non-metric connection, if the connection $\overset{s}{\nabla}$ satisfies*

$$\begin{cases} \overset{s}{\nabla}_Z g(X,Y) = -\pi(Y)U(X,Z) - \pi(X)U(Y,Z), \\ \overset{s}{T}(X,Y) = \pi(Y)UX - \pi(X)UY. \end{cases} \tag{3.1}$$

where U is a symmetric tensor of type (1,1) and π is a 1-form.

Remark 3.1. *If $U(X,Y) = g(X,Y)$, then this connection is the semi-symmetric non-metric connection that was considered in [1].*

The local expression of (3.1) is

$$\overset{s}{\nabla}_k g_{ij} = -\pi_i U_{jk} - \pi_j U_{ik}, \ \overset{s}{T}{}^{k}_{ij} = \pi_j U^k_i - \pi_i U^k_j, \tag{3.2}$$

and the connection coefficient is

$$\overset{s}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \pi_j U^k_i, \tag{3.3}$$

and the curvature tensor is

$$\overset{s}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + U^l_j \pi_{ik} - U^l_i \pi_{jk} + (\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k, \tag{3.4}$$

where $\pi_{ik} = \overset{0}{\nabla}_i \pi_k - U^p_i \pi_p \pi_k$.

Theorem 3.1. *A Riemannian manifold $(\mathcal{M}, g, \overset{s}{\nabla})$ associated with a form α being closed, then the manifold $(\mathcal{M}, g, \overset{s}{\nabla})$ is Ricci-like falt, that is, the volume curvature tensor of $\overset{s}{\nabla}$ is zero, namely*

$$\overset{s}{P}_{ij} = 0, \tag{3.5}$$

where $\overset{s}{P}_{ij}$ is a (Ricci-like) volume curvature tensor of $\overset{s}{\nabla}$.

Proof. Contracting the indices k and l of the expression (3.4) and using $\overset{0}{P}_{ij} = 0$ and (2.10), we have

$$\begin{aligned} \overset{s}{P}_{ij} &= \overset{0}{P}_{ij} + U^k_j \pi_{ik} - U^k_i \pi_{jk} + (\overset{0}{\nabla}_i U^k_j - \overset{0}{\nabla}_j U^k_i)\pi_k \\ &= \overset{0}{P}_{ij} + U^k_j(\overset{0}{\nabla}_i \pi_k - U^p_i \pi_p \pi_k) - U^k_i(\overset{0}{\nabla}_j \pi_k - U^p_j \pi_p \pi_k) + (\overset{0}{\nabla}_i U^k_j - \overset{0}{\nabla}_j U^k_i)\pi_k \\ &= \overset{0}{\nabla}_i(U^k_j \pi_k - \overset{0}{\nabla}_j(U^k_i \pi_k) = \overset{0}{\nabla}_i \alpha_j - \overset{0}{\nabla}_j \alpha_i. \end{aligned}$$

If a 1-form α is a closed form, then $\overset{0}{\nabla}_i \alpha_j - \overset{0}{\nabla}_j \alpha_i = 0$. Hence we obtain (3.5) is tenable. □

From (3.2) and (3.3) the connection coefficient of dual connection $\overset{s*}{\nabla}$ of $\overset{s}{\nabla}$ is

$$\overset{s*}{\Gamma}{}_{ij}^{k} = \{^{k}_{ij}\} - U_{ij}\pi^{k}, \tag{3.6}$$

and the curvature tensor of is

$$\overset{s*}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + U_{ik}\pi^{l}_{j} - U_{jk}\pi^{l}_{i} - (\overset{0}{\nabla}_{i}U_{jk} - \overset{0}{\nabla}_{j}U_{ik})\pi^{l}. \tag{3.7}$$

Thus we have the following.

Theorem 3.2. *In a Riemannian manifold $(\mathcal{M}, g)$, the first Bianchi identity and the second Bianchi identity of the curvature tensor of dual connection $\overset{s*}{\nabla}$ of $\overset{s}{\nabla}$ are*

$$\begin{cases} \overset{s*}{R}_{ijk}{}^{l} + \overset{s*}{R}_{jik}{}^{l} + \overset{s*}{R}_{kij}{}^{l} = 0, \\ \overset{s*}{\nabla}_{h}\overset{s*}{R}_{ijk}{}^{l} + \overset{s*}{\nabla}_{i}\overset{s*}{R}_{jhk}{}^{l} + \overset{s*}{\nabla}_{j}\overset{s*}{R}_{hik}{}^{l} = 0. \end{cases} \tag{3.8}$$

From (3.2) and (3.3), the connection coefficient of mutual connection $\overset{sm}{\nabla}$ of the connection $\overset{s}{\nabla}$ is

$$\overset{sm}{\Gamma}{}_{ij}^{k} = \{^{k}_{ij}\} + \pi_{i}U^{k}_{j}, \tag{3.9}$$

where the mutual connection $\overset{sm}{\nabla}$ satisfies

$$\overset{sm}{\nabla}_{k}g_{ij} = -2\pi_{k}U_{ij}, \quad \overset{sm}{T}_{ij}{}^{k} = \pi_{i}U^{k}_{j} - \pi_{j}U^{k}_{i}, \tag{3.10}$$

and the curvature tensor of $\overset{sm}{\nabla}$ is

$$\overset{sm}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \overset{0}{\nabla}_{i}(\pi_{j}U^{l}_{k}) - \overset{0}{\nabla}_{j}(\pi_{i}U^{l}_{k}), \tag{3.11}$$

From (3.9) and (3.10), the connection coefficient of dual connection $\overset{sm*}{\nabla}$ of $\overset{sm}{\nabla}$ is

$$\overset{sm*}{\Gamma}{}_{ij}^{k} = \{^{k}_{ij}\} - \pi_{i}U^{k}_{j}, \tag{3.12}$$

and the curvature tensor of $\overset{sm*}{\nabla}$ is

$$\overset{sm*}{R}{}_{ijk}{}^{l} = K_{ijk}{}^{l} + \overset{0}{\nabla}_{i}(\pi_{j}U^{l}_{k}) - \overset{0}{\nabla}_{j}(\pi_{i}U^{l}_{k}), \tag{3.13}$$

So there exists the following.

Theorem 3.3. *In a Riemannian manifold $(\mathcal{M}, g)$, if there holds*

$$\overset{0}{\nabla}_{i}(\pi_{j}U^{l}_{k}) = \overset{0}{\nabla}_{j}(\pi_{i}U^{l}_{k}) \tag{3.14}$$

then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to \overset{sm}{\nabla}$ and the mutual connection $\overset{sm}{\nabla}$ of $\overset{s}{\nabla}$ is a conjugate symmetric connection.

Theorem 3.4. *If a Riemannian manifold $(\mathcal{M}, g)$ associated with a symmetric-type quarter-symmetric non-metric connection $\overset{s}{\nabla}$ admits zero curvature tensor with respect to the mutual connection $\overset{sm}{\nabla}$, then the Riemannian manifold $(\mathcal{M}, g, \overset{sm}{\nabla})$ is flat.*

Proof. Considering (3.11) and (3.13), we obtain

$$\overset{sm}{R}_{ijk}{}^{l} + \overset{sm*}{R}_{ijk}{}^{l} = 2K_{ijk}{}^{l}, \tag{3.15}$$

If $\overset{sm}{R}_{ijk}{}^{l} = 0$, then $\overset{sm*}{R}_{ijk}{}^{l} = 0$. From (3.15) we have $K_{ijk}{}^{l} = 0$. Hence the Riemannian metric g is flat. □

It is well known that if a sectional curvature at a point p in a Riemannian manifold $(\mathcal{M}, g)$ is independent of E(2-dimensional subspace of $T_p(\mathcal{M})$), then the curvature tensor is as below

$$R_{ijk}{}^{l} = k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}), \tag{3.16}$$

In this case, if $k(p) = const$, then the Riemannian manifold $(\mathcal{M}, g)$ is a constant curvature manifold.

Theorem 3.5. (Schur's Theorem) *Suppose that $(\mathcal{M}^n, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic mutual connection $\overset{sm}{\nabla}$. Then the Riemannian manifold $(\mathcal{M}^n, g, \overset{sm}{\nabla})$ is a constant curvature manifold.*

Proof. Substituting (3.16) into the second Bianchi identity of the curvature tensor of the mutual connection $\overset{sm}{\nabla}$ of the symmetric-type quarter-symmetric non-metric connection $\overset{s}{\nabla}$, we have

$$\overset{sm}{\nabla}_h R_{ijk}{}^{l} + \overset{sm}{\nabla}_i R_{jhk}{}^{l} + \overset{sm}{\nabla}_j R_{hik}{}^{l} = \overset{sm}{T}{}^{p}_{hi} R_{jpk}{}^{l} + \overset{sm}{T}{}^{p}_{ij} R_{hpk}{}^{l} + \overset{sm}{T}{}^{p}_{jh} R_{ipk}{}^{l},$$

By a direct computation, we obtain

$$\begin{aligned}
&\overset{sm}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{sm}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{sm}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\
&\quad + k(p)[(\delta^l_i \overset{sm}{\nabla}_h g_{jk} - \delta^l_j \overset{sm}{\nabla}_h g_{ik}) + (\delta^l_j \overset{sm}{\nabla}_i g_{hk} - \delta^l_h \overset{sm}{\nabla}_i g_{jk}) + (\delta^l_h \overset{sm}{\nabla}_j g_{ik} - \delta^l_i \overset{sm}{\nabla}_j g_{hk})] \\
&= k(p)[(\pi_h U^p_i - \pi_i U^p_h)(\delta^l_j g_{pk} - \delta^l_p g_{jk}) + (\pi_i U^p_j - \pi_j U^p_i)(\delta^l_h g_{pk} - \delta^l_p g_{hk}) + (\pi_j U^p_h - \pi_h U^p_j)(\delta^l_i g_{pk} - \delta^l_p g_{ik})].
\end{aligned}$$

Using (3.10), then we obtain

$$\begin{aligned}
&\overset{sm}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{sm}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{sm}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\
&\quad - 2k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})] \\
&= k(p)[(\delta^l_j \pi_h U_{ik} - \delta^l_j \pi_i U_{hk} - \pi_h U^l_i g_{jk} + \pi_i U^l_h g_{jk}) + (\delta^l_h \pi_i U_{jk} - \delta^l_h \pi_j U_{ik} - \pi_i U^l_j g_{hk} + \pi_j U^l_i g_{hk}) \\
&\quad + (\delta^l_i \pi_j U_{hk} - \delta^l_i \pi_h U_{jk} - \pi_j U^l_h g_{ik} + \pi_h U^l_j g_{ik})].
\end{aligned}$$

Contracting the indices i, l, we have

$$\begin{aligned}
&(n-2)[g_{jk} \overset{sm}{\nabla}_h k(p) - g_{hk} \overset{sm}{\nabla}_j k(p) - 2k(p)(n-2)(\pi_h U_{jk} - \pi_j U_{hk})] \\
&= k(p)[(n-3)(\pi_j U_{hk} - \pi_h U_{jk}) - g_{hk}(U^i_h \pi_i - U^i_i \pi_j) + g_{jk}(U^i_h \pi_i - U^i_i \pi_h)].
\end{aligned}$$

Multiplying both sides of this expression again by g^{jk}, then we arrive at

$$(n-1)(n-2)\overset{sm}{\nabla}_h k(p) = 0.$$

According to $dim\mathcal{M} \geq 3$, we have $k = const$. The connected condition implies that Theorem 3.5 is tenable. □

4. Ackonowedement

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