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Bourgain Algebras of Ideals in H^∞ Generated by Inner Functions

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Abstract. In this paper we prove that Bourgain algebra of qX relative to L^∞ contains C if X is backward shift invariant analytic subalgebra and $q \in X$ is an inner function, i.e. $(qX, L^\infty)_b \supset C$. We also studied some Bourgain algebras of finitely generated ideals in A and H^∞.

1. Introduction

Let Y be a commutative Banach algebra with an identity and let X be a linear subspace of Y. J. Cima and R. Timoney [1] introduced the notion of the Bourgain algebra based on the ideas of J. Bourgain [2]. The Bourgain algebra X_b or $(X, Y)_b$ of X relative to L^∞ is the space of all functions f in Y such that if $f_n \to 0$ weakly in X, then $dist\,(f.f_n, X) \to 0$.

Distance $dist\,(f.f_n, X)$ between $f.f_n$ and X is the quotient norm of the coset $f.f_n + X$ in the space Y/X. The proof in [1] shows that $(X, Y)_b$ is a closed subalgebra of Y and contains the constant functions; if X is an algebra then $X \subset (X, Y)_b$.

Let D be the open unit disk and H^∞ denote the algebra of bounded analytic functions on D. Taking the boundary values of the functions on $T = \{z \in \mathbb{C} : |z| = 1\}$, we can consider $H^\infty = H^\infty(T)$ as an essentially supremum-norm closed subalgebra of $L^\infty = L^\infty(T)$. Let z belong to D and $\varphi_z(f) = f(z)$ for every $f \in H^\infty$. Then φ_z is a complex homomorphism "evaluation at the point z". Let $C = C(T)$ be the space of all continuous functions on the closed unit circle T.

A closed subalgebra B between H^∞ and L^∞ is called a Douglas algebra.
B coincides with the closed subalgebra generate by H^∞ and complex conjugate of interpolating Blaschke product [3]. The space $H^\infty + C = [H^\infty, \bar{z}]$ is a typical Douglas algebra [3]. In [4] J. Cima, Sv. Janson and K. Yale showed that H^∞_b relative L^∞ is H^∞+C. P. Gorkin, K. Izuchi and R. Mortini [5] present another proof. They also prove many properties of the Bourgain algebras $(X, L^\infty)_b$ where X is Douglas algebra. K Izuchi in [6] proved that the Bourgain algebra of a closed subalgebra E between disk algebra $A = H^\infty \cap C$ and H^∞ relative to L^∞ is always contained in H^∞+C and $E_{bb} = (E_b)_b = E_b$.

An inner function is a function $f \in H^\infty$ such that $\left|f(z)\right| = 1$ almost everywhere on T. A sequence $\{z_n\}_n$ is called interpolating if for every bounded sequence $\{a_n\}_n$ of complex numbers there is a function $f \in H^\infty$ such that $f(z_n) = a_n$ for all n. For a sequence in D with $\sum\limits_{n=1}^{\infty}(1 - |z_n|) < \infty$, the function:

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$$B(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, z \in D,$$

is called a Blaschke product with zeros $\{z_n\}_n$. If $\{z_n\}_n$ is an interpolating sequence, $B(z)$ then is also called interpolating. Some results for interpolating sequences and their applications in the description of Douglas algebras have been obtained in [3] and [7]. Interpolating sequences can be applied for obtaining new scalar solutions of nonlinear differential equations using results in [8], [9] and [10].

A norm closed subalgebra between A and H^∞ is called an analytic subalgebra. A subset X of H^∞ is called blackward shift invariant if

$$f^* = \frac{f(z) - f(0)}{z} \in X, \text{ for every } f \text{ in } X.$$

Let X be an analytic subalgebra. K. Nishizawa in [11] proved that the linear space $X + C$ is a closed subalgebra of L^∞ if and only if X is a backward shift invariant.

Some characterizations of backward shift invariant analytic subalgebras are given it the follow statement.

Lemma 1.1. *[12] Let X be an analytic algebra. Then the following conditions are equivalent.*

1. *X is backward shift invariant.*
2. *For each a in D , $(f(z) - f(0))/(z - a) \in X$ for every $f \in X$.*
3. *For each a in D, $(z - a)X + A = X$.*
4. *$X + C$ is a closed subalgebra.*
5. *$X = H^\infty \cap [X + C]$.*

K. Izuchi has been shown that if X is a closed subalgebra, such that
$A \subset X \subset H^\infty$ and $X + C$ is an algebra then X_b relative L^∞ contains C [6]. In particular, this is valid for every backward shift invariant analytic subalgebra. There are many different subalgebras of H^∞ that do not have the property "backward shift invariant".

Example 1. Let X be an analytic subalgebra, $X_\alpha = \{f \in X : f(\alpha) = 0\}$ and $\alpha \in D$. Since X is a closed subalgebra between A and H^∞ then

$$z_\alpha = \frac{z - \alpha}{1 - \bar{\alpha} z} \in X_\alpha \text{ and } \frac{z_\alpha - z_\alpha(0)}{z - 0} = \frac{z_\alpha + \alpha}{z}.$$

1. If $\alpha \neq 0$ then $\left(\frac{z_\alpha + \alpha}{z}\right)(\alpha) = \frac{\alpha}{\alpha} = 1 \neq 0$
2. If $\alpha = 0$ then $\frac{z_0 + 0}{z}(0) = 1 \neq 0$.

Therefore, $\frac{z_\alpha - z_\alpha(0)}{z - 0} \notin X_\alpha$ and X_α is not backward shift invariant.

Example 2. Let X be an analytic subalgebra $\alpha \in D$, let q be an inner function in X and $q_\alpha = \frac{q - q(\alpha)}{1 - q(\alpha).q}$. We study the algebra $q_\alpha X$. Since $1 \in X$ then $q_\alpha \in q_\alpha X$. We assume that $\frac{q_\alpha - q_\alpha(0)}{z} = q_\alpha.p$ and $p \in X$. Then we get $q_\alpha = z.q_\alpha.p + q_\alpha(0)$ (1).

1. If α is such that $q(0) = q(\alpha)$ then $q_\alpha(0) = 0$ and we obtain that $z.p = 1$, i.e. z is invertible in$X$.
2. If α is such that $q(0) \neq q(\alpha)$ then $q_\alpha(0) \neq 0$. But for $z = \alpha$ in (1) we obtain that $q_\alpha(0) = 0$.

Therefore $\frac{q_\alpha - q_\alpha(0)}{z - 0} \notin q_\alpha X$ and $q_\alpha X$ is not backward shift invariant.

In particular, if $q(\alpha) = 0$ it follows that qX do not have the property "backward shift invariant". In this paper we prove that Bourgain algebra of qX relative to L^∞ contains C if X is backward shift invariant analytic subagbera and $q \in X$, i.e. $(qX, L^\infty)_b \supset C$. Other results related to Bourgain algebras of subspaces of H^∞ were obtained in [13], [14] and [15]. Bourgain algebras of some finitely generated ideals in A and H^∞ are also studied.

2. The main results

Theorem 2.1. *Let X be a backward shift invariant analytic subalgebra and let $q \in X$ be an inner function. The Bourgain algebra of qX relative L^∞ contains C, i.e. $(qX, L^\infty)_b \supset C$.*

Proof. Since qX is an algebra, the space $(qX, L^\infty)_b$ is a closed subalgebra of L^∞ and $qX \subset (qX, L^\infty)_b$. If $f \in X$ then $f.qg \in X$ for $g \in X$ and we obtain that $X \subset (qX, L^\infty)_b$.
First we will look at the case when $q(0) \neq 0$.

Let $f_n \to 0$ weakly in qX, i.e. $\varphi(f_n) \to 0$ for all $\varphi \in (X)^*$. For $\varphi = \varphi_0$ we have $\varphi_0(f_n) = f_n(0) \to 0$. If $f_n = qg_n$, then $f_n(0) = q(0).g_n(0)$, where $g_n \in X$ and we obtain $g_n(0) \to 0$.

Put $t_n(z) = g_n(z) - g(0)$. Since $\bar{z}t_n = [g_n(z) - g_n(0)].\bar{z} = [g_n(z) - g_n(0)]/z$ for $z \in T$ and X is a backward shift invariant then $\bar{z}t_n \in X$. Hence:

$$\begin{gathered}\operatorname{dist}(\bar{z}f_n, qX) = dist(\bar{z}qg_n, qX) \leq \operatorname{dist}(\bar{z}g_n, X) = \\ \operatorname{dist}(\bar{z}t_n + g_n(0).\bar{z}, X) = \operatorname{dist}(\bar{z}g_n(0), X) = \\ \inf\{\|h\|_\infty : h \in [\bar{z}g_n(0)]\} \leq \left\|\bar{z}g_n(0)\right\| = \left|g_n(0)\right| \to 0\end{gathered}$$

and we obtain $\bar{z} \in (qX, L^\infty)_b$.

Now let $q(0) = 0$.

We can assume that, $q = z^k.p$, $k \in \mathbb{N}$ and p is an inner function, such that $p(0) \neq 0$. But $p(z) = \frac{q(z)}{z^k} = \frac{1}{z^{k-1}}.\frac{q(z)}{z} = \frac{1}{z^{k-1}}.\frac{q(z)-q(0)}{z-0} = \frac{1}{z^{k-1}}.q_1(z)$, where $q_1(z) = \frac{q(z)-q(0)}{z-0}$ belong to the backward shift invariant algebra X. We get to the $k-1$ step:

$$p(z) = \frac{1}{z}.q_{k-1}(z) = \frac{q_{k-1}(z) - q_{k-1}(0)}{z-0} \in X,$$

that is $p \in X$ is an inner function and $p(0) \neq 0$. If $f_n \to 0$ weakly in qX, then $f_n \to 0$ in pX because $qX = z^kp.X \subset pX$. If $f_n = qg_n$, where $g_n \in X$ we have:

$$\operatorname{dist}(\bar{z}.f_n, qX) = dist\left(\bar{z}.f_n, z^k.pX\right) = \operatorname{dist}\left(\bar{z}^{k+1}.f_n, pX\right) \to 0$$

because of the first case $\bar{z}$ belongs to the algebra $(pX, L^\infty)_b$. It follows that, $\bar{z} \in (qX, L^\infty)_b$.

Since $z \in X \subset (qX, L^\infty)_b$ and $(qX, L^\infty)_b$ contains the constant functions, by the Weierstrase theorem we have $C \subset (qX, L^\infty)_b$. The theorem is proved. □

We need two lemmas.

Lemma 2.2. *[3] If $\{z_n\}_n \subset D$ is interpolating sequence, then there exist functions $\{f_n\}_n \subset H^\infty(D)$ and positive number M such that $f_n(z_n) = 1$ for all n, $f_n(z_k) = 0$ for $n \neq k$ and $\sum_{n=1}^{\infty} |f_n(z)| \leq M$ for $z \in D$.*

Lemma 2.3. *[4] Suppose that $\{f_n\}_n$ is a sequence in H^∞ such that $\sum_{n=1}^{\infty} |f_n(z)| \leq M$ for all $z \in D$. Then $f_n \to 0$ weakly in H^∞.*

Corollary 2.4. *Let I be a finitely generated rotation ideal in algebra H^∞, then $(I, L^\infty)_b = H^\infty + C$.*

Proof. By [16] $I = z^m.H^\infty$ for some positive integer m and we can applies Theorem 2.1 by $X = H^\infty$ and $q = z^m$. Therefore $(I, L^\infty)_b \supset H^\infty + C$.

By the Chang-Marshall theorem, every Douglas algebra B such that
$H^\infty + C \subsetneq B$ is generated by H^∞ and complex conjugate of infinity interpolating Blaschke product. Then for $(I, L^\infty)_b = H^\infty + C$ it is sufficient to prove that $(I, L^\infty)_b$ does not contain any complex conjugate of infinity interpolating Blaschke product.

Let c be an interpolating Blaschke product with zeros $\{z_n\}_n \subset D$. Notice that $|z_n| \to 1$. According to Lemma 2.2 there exist functions $\{f_n\}_n \subset H^\infty$ and positive number M such that $f_n(z_n) = 1$ for all n,

$f_k(z_n) = 0$ for $n \neq k$ and $\sum_{n=1}^{\infty} \left|f_n(z)\right| \leq M$ for $z \in D$. Then for the functions $g_n(z) = z^m f_n(z)$, $n \in \mathbb{N}$ we obtain $g_n \in z^m H^\infty, g_n(z_n) = z_n^m f_n(z_n) = z_n^m$ for all n, $g_n(z_k) = 0$ for $n \neq k$ and $\sum_{n=1}^{\infty} \left|g_n(z)\right| \leq M$ for $z \in D$. By Lemma 2.3 (with $z^m H^\infty$ instead H^∞) we have $g_n \to 0$ weakly in $z^m H^\infty$ but

$$\operatorname{dist}(\bar{c}g_n, z^m H^\infty) = dist\,(g_n, c.z^m.H^\infty) =$$

$$\inf\left(\sup_{z\in D} \left|g_n(z) - c(z).y(z)\right| : y \in z^m H^\infty\right) \geq$$

$$\inf\left(\left|g_n(z_n) - c(z_n).y(z_n)\right| : y \in z^m H^\infty\right) = \left|g_n(z_n)\right| = |z_n|^m \to 1.$$

Thus $\bar{c} \notin (I, L^\infty)_b$ and the corollary is proved. □

By [17], if I is a finitely generated prime ideal in H^∞ then there exist $z_0 \in D$ such that $I = \{f \in H^\infty : f(z_0) = 0\}$ i.e. $I = (z - z_0)H^\infty$. As in Corollary 2.4, the following statement is proved.

Corollary 2.5. *Let I be a finitely generated prime ideal of H^∞. Then it is fulfilled $(I, L^\infty)_b = H^\infty + C$.*

Theorem 2.6. *If B is a finite Blachke product, then $(BA, C)_b = C$, where $C = C(T)$.*

Proof. Since $zB \in C$ then $\bar{z}.\bar{B} = \overline{zB} \in C$. Let $f_n = Bg_n$, $g_n \in A$ and $f_n \to 0$ weakly in BA. Since BA contains in A therefore $f_n \to 0$ weakly in A. Then

$$\operatorname{dist}(\bar{z}.f_n, BA) = dist\,(\bar{z}.\bar{B}.f_n, A) = dist\left(\overline{zB}.f_n, A\right) \to 0,$$

because $(A, C)_b = C$, i.e. $\bar{z} \in (BA, C)_b$.

Since $zg_n \in A$ we obtain: $dist\,(zf_n, BA) = dist\,(Bzg_n, BA) = 0$, i.e.// $z \in (BA, C)_b$.

Since z and $\bar{z}$ belong to the closed algebra $(BA, C)_b$, then by the Weierstrase theorem we have that $C \in (BA, C)_b$. The inclusion is obvious. □

Let $f \in A$ and $Z(f) = \{z \in \bar{D} : f(z) = 0\}$. For an ideal $I \subset A$ let $Z(I) = \bigcap_{f\in I} Z(f)$ denote its zero set.

Corollary 2.7. *Let $I \neq \{0\}$ be an ideal in A that $Z(I) \cap T = \varnothing$. Then $(I, C)_b = C$.*

Proof. By [18] I is a principal ideal generated by a finite Blaschke product B, i.e. $I = BA$. By Theorem 2.6 we obtain $(I, C)_b = C$.

By [17] if I is a finitely generated prime ideal in A then $I = (z - z_0)A$ for some $z_0 \in D$. But $(z - z_0)\,A = \frac{z-z_0}{1-\bar{z}_0 z}A$, since $\frac{1}{1-\bar{z}_0 z} \in A$. By Theorem 2.6 we obtain the following statement. □

Corollary 2.8. *If I is a finitely generated prime ideal in A then $(I, C)_b = C$.*

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