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On Block Diagonal Majorization and Basic Sequences

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Abstract. In this paper we generalize (finite) block diagonal matrices to infinite dimensions and then by using block diagonal row stochastic matrices (as a special case), we define the relation $\prec_{bdr}$ on $\mathfrak{c}_0$, which is said block diagonal majorization. We also obtain some important properties of $\mathcal{P}_{bdr}$, the set of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$, which preserve $\prec_{bdr}$. Further, it is obtained necessary conditions for a bounded linear operator T on $\mathfrak{c}_0$ to be a preserver of the block diagonal majorization $\prec_{bdr}$. Also, the notion of the basic sequences correspond to block diagonal row stochastic matrices with description of some relevant examples will be discussed.

1. Introduction

In 1992, Pierce obtained a survey of linear preserver problemmas [8]. The standard work on the theory of majorization and its applications is given by Marshall and Olkin in [7] and for relative papers, see [1–6].

We will make the following assumptions: $\mathfrak{c}_0$ is the Banach space of all real sequences converge to zero with the supremum norm. An elemmaent $x = (x_n) \in \mathfrak{c}_0$ can be represented by $\sum_{i\in\mathbb{N}} x_i \mathrm{e}_i$, where $\mathrm{e}_i : \mathbb{N} \to \mathbb{R}$ is defined by $\mathrm{e}_i(j) = \delta_{ij}$, the Kronecker delta. Also, M_n denotes the set of all $n \times n$ real matrices.

Recently, Armandnejad and Passandi [2] considered the notion of block diagonal majorization on $\mathfrak{c}_0$ and find the possible structure of the bounded linear operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ which preserve $\prec_{bdr}$, block diagonal majorization on $\mathfrak{c}_0$. We denote the set of such operators by $\mathcal{P}_{bdr}$.

In the next section, we introduce the notion of the basic sequence corresponds to a block diagonal row stochastic matrix. Also, we investigate some important properties of $\mathcal{P}_{bdr}$ and we show for any block diagonal row stochastic matrix, there corresponds uniquely a bounded linear operator with norm 1 on $\mathfrak{c}_0$ which is called block diagonal row stochastic operator. We obtain necessary conditions for a bounded linear operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ to be a preserve of block diagonal majorization. Moreover, some relevant examples are given.

2. Main results

For the convenience of the reader, we repeat the relevant material. We recall that a square matrix with nonnegative entries is called row stochastic if all its row sums equal 1. For $x, y \in \mathfrak{c}_0$, we say that x is row stochastic majorized by y, denoted by $x \prec_r y$ if there exists a row stochastic matrix R such that $x = Ry$.

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Definition 2.1. *[2] (i) Let $(n_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ and for any $i \in \mathbb{N}$ let $R_{n_i} \in M_{n_i}$ be a row stochastic matrix. Then $R = \oplus_{i=1}^{\infty} R_{n_i}$, that is*

$$R = \begin{bmatrix} R_{n_1} & O_{n_1\times n_2} & O_{n_1\times n_3} & O_{n_1\times n_4} & \cdots \\ O_{n_2\times n_1} & R_{n_2} & O_{n_2\times n_3} & O_{n_2\times n_4} & \cdots \\ O_{n_3\times n_1} & O_{n_3\times n_2} & R_{n_3} & O_{n_3\times n_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is called a block diagonal row stochastic matrix, where $O_{n_i\times n_j}$ is the zero $n_j \times n_j$ matrix.
(ii) For $x, y \in c_0$, we say that x is block diagonal majorized by y, denoted by $x \prec_{bdr} y$ if there exists a block diagonal row stochastic matrix R such that $x = Ry$. Also, x is said to be block diagonal equivalent to y, and denoted by $x \sim_{bdr} y$, whenever $x \prec_{bdr} y$ and $y \prec_{bdr} x$.

In what follows, we denote $\mathcal{M}_{bdr}$ for the set of all $\mathbb{N} \times \mathbb{N}$ block diagonal row stochastic matrix.

Example 2.2. *The following matrix*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is a block diagonal row stochastic matrix.

In this part, we introduce the basic sequence of any block diagonal row stochastic matrix and obtain that it is not unique. Also, the supersequence of any real sequence is defined. Some important properties of the basic sequences will be investigated.

Definition 2.3. *Let (x_n) and (y_n) be two real sequences. If there exists a sequence (k_n) in $\mathbb{N}$ such that*

$$\begin{aligned} y_1 &= x_1 + \cdots + x_{k_1}, \\ y_2 &= x_{k_1+1} + \cdots + x_{k_1+k_2}, \\ y_3 &= x_{k_1+k_2+1} + \cdots + x_{k_1+k_2+k_3}, \\ &\vdots \end{aligned}$$

Then we say that the real sequence (y_n) is a supersequence of (x_n) and denoted by $(x_n) \ll (y_n)$.

Remark 2.4. *Let $x = (x_n)$ and $y = (y_n)$ be two real sequences. If $(x_n) \ll (y_n)$, then we have the following assertions.*

(i) *If $x \geq 0$, then $y \geq 0$. Also, in this case, $\sum_{i=1}^{\infty} x_n$ converges if and only if $\sum_{i=1}^{\infty} y_n$ converges.*

(ii) $\|y\|_1 \leq \|x\|_1$.

(iii) *Let $(n_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ and $\mathrm{e} = (1, 1, 1, \ldots)$ be the constant sequence. Then $\mathrm{e} \ll (n_k)$.*

(iv) $(x_n) \ll (x_n)$; *(which implies by considering $(k_n) = \mathrm{e}$).*

Remark 2.5. *The relation $\ll$ is a partial ordering on the set of all sequences in $\mathbb{N}$.*

Proof. Reflexivity follows from (iv), Remark 2.4. Suppose that (m_i) and (n_i) are both two sequences in $\mathbb{N}$ with $(m_i) \ll (n_i)$ and $(n_i) \ll (m_i)$. Then there are natural numbers k_1 and s_1 such that $m_1 = n_1 + \cdots + n_{k_1}$ and $n_1 = m_1 + \cdots + m_{s_1}$. So, we have $m_1 \le n_1 \le m_1$ and therefore $m_1 = n_1$. A similar argument show that $m_k = n_k$. This implies $(m_i) = (n_i)$, i.e., the relation $\ll$ is antisymmetric. To show transitivity of $\ll$, let $(m_i) \ll (n_i) \ll (p_i)$. Then each p_i is of the form $\sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} n_j$ and each n_j is of the form $\sum_{t=l_1+\cdots+l_{j-1}+1}^{l_1+\cdots+l_j} m_t$. Thus we have $p_i = \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} n_j = \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} \sum_{t=l_1+\cdots+l_{j-1}+1}^{l_1+\cdots+l_j} m_t$, which shows $(m_i) \ll (p_i)$. □

Definition 2.6. *Let $R \in \mathcal{M}_{bdr}$. Then we say $(n_i)_{i\in\mathbb{N}}$ is a basic sequence of R if there is a sequence of matrices (R_{n_i}) such that $R = \oplus_{i=1}^{\infty} R_{n_i}$ and each $R_{n_i} \in M_{n_i}$ is a row stochastic matrix. If $(n_i)_{i\in\mathbb{N}}$ is a constant sequence, we say R is of constant basic sequence.*

Obviously, the basic sequence of any matrix $R \in \mathcal{M}_{bdr}$ is not unique.
For example, the constant sequence $e = (1, 1, 1, \ldots)$ is a basic sequence of the identity matrix I. Also, $(2, 2, 2, \ldots)$, $(3, 3, 3, \ldots)$ and $(1, 2, 3, \ldots)$ are all basic sequences of I.

Theorem 2.7. *Suppose that (m_i) is a basic sequence of the matrix $R \in \mathcal{M}_{bdr}$ and $(m_i) \ll (n_i)$. Then (n_i) is a basic sequence of R, but not vice versa.*

Proof. By using the assumptions it follows that there is a sequence of natural numbers (k_i) such that

$$\begin{aligned} n_1 &= m_1 + \cdots + m_{k_1}, \\ n_2 &= m_{k_1+1} + \cdots + m_{k_2}, \\ &\vdots \end{aligned}$$

Since (m_i) is a basic sequence of R, so

$$R = \begin{bmatrix} R_1 & O & \cdots \\ O & R_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

where any R_i is an $m_i \times m_i$ row stochastic matrix. Put

$$S_1 = \begin{bmatrix} R_1 & O & O \\ O & \ddots & O \\ O & O & R_{k_1} \end{bmatrix}, \quad S_2 = \begin{bmatrix} R_{k_1+1} & O & O \\ O & \ddots & O \\ O & O & R_{k_2} \end{bmatrix}, \ldots$$

Clearly, we have

$$R = \begin{bmatrix} S_1 & O & \cdots \\ O & S_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

and so (n_i) is a basic sequence of R.
The converse is not true, for example, the matrix

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

is a block diagonal row stochastic matrix with the basic sequence $(2, 1, 1, \ldots)$ and $(1, 1, 1, \ldots) \lll (2, 1, 1, \ldots)$, but $(1, 1, 1, \ldots)$ is not a basic sequence of R. □

Theorem 2.8. *Let (n_i) be a basic sequence of the matrix $R \in \mathcal{M}_{bdr}$ such that the sequence (n_i) from somewhere on, is constant. Then R has the constant basic sequence.*

Proof. Let (n_i) be a basic sequence of the matrix $R \in \mathcal{M}_{bdr}$ and for $k \in \mathbb{N}$, we have

$$m = n_{k+1} = n_{k+2} = \cdots .$$

Put $N = \operatorname{lcm}(m, n_1 + \cdots + n_k)$, the least common multiple. It is easy to show that the constant sequence (N) is a constant basic sequence of R. □

Remark 2.9. *Let $R, S \in \mathcal{M}_{bdr}$ have a common basic sequence (k_i). Then RS, SR $\in \mathcal{M}_{bdr}$. Also, $R^n \in \mathcal{M}_{bdr}$, for any $n \in \mathbb{N}$. If $R = \oplus_{i=1}^{\infty} R_{k_i}$ and $S = \oplus_{i=1}^{\infty} S_{k_i}$, then $RS = \oplus_{i=1}^{\infty} R_{k_i} S_{k_i}$.*

Theorem 2.10. *Let (n_i) be a sequence in $\mathbb{N}$. Then there exists $R \in \mathcal{M}_{bdr}$ with basic sequence (n_i). Also, the set of all matrix with the basic sequence (n_i) is a convex and closed (with respect to the pointwise convergence) and closed (with respect to the composition) subset of $\mathcal{M}_{bdr}$.*

Proof. Let R_i be an $n_i \times n_i$ matrix as the following

$$R_i = \begin{bmatrix} \frac{1}{n_i} & \cdots & \frac{1}{n_i} \\ \vdots & \ddots & \vdots \\ \frac{1}{n_i} & \cdots & \frac{1}{n_i} \end{bmatrix}.$$

Now we put

$$R = \begin{bmatrix} R_1 & O & \cdots \\ O & R_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

thus (n_i) is a basic sequence of $R \in \mathcal{M}_{bdr}$.
Let the matrices $R, S \in \mathcal{M}_{bdr}$ be as the following

$$R = \begin{bmatrix} R_1 & O & \cdots \\ O & R_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & O & \cdots \\ O & S_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

both have (n_i) as a common basic sequence. For $0 \le \lambda \le 1$, we have

$$\lambda R + (1 - \lambda) S = \begin{bmatrix} \lambda R_1 + (1 - \lambda) S_1 & O & \cdots \\ O & \lambda R_2 + (1 - \lambda) S_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

and so $\lambda R + (1 - \lambda) S \in \mathcal{M}_{bdr}$ has (n_i) as a basic sequence.
To prove closedness with respect to the pointwise convergence, let (R_n) be a sequence in $\mathcal{M}_{bdr}$ such that

$$R_n = \begin{bmatrix} R_{1,n} & O & \cdots \\ O & R_{2,n} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = [r_{ij,n}],$$

and (n_i) is a basic sequence of R_n, and so each $R_{i,n}$ is the $n_i \times n_i$ row stochastic matrix. Assume that the sequence (R_n) is pointwise convergent to $R = [r_{ij}]$.
Let $(i, j) \notin \{1, \dots, n_1\}^2 \cup \{n_1 + 1, \dots, n_2\}^2 \cup \cdots$. Clearly, for all $n \in \mathbb{N}$, we have $r_{ij,n} = 0$, and so

$$\lim_{n\to\infty} r_{ij,n} = 0 = r_{ij}.$$

Thus R is a block diagonal matrix with blocks $(n_1, n_2, \dots)$.
On the other hand, for all $n \in \mathbb{N}$ and $i \in \mathbb{N}$, we have

$$\sum_{j=1}^{\infty} r_{ij,n} = 1. \tag{1}$$

In the above summation, there are finitely nonzero elemmaents. So, it follows from letting $n \to \infty$ in (1) that

$$1 = \lim_{n\to\infty} \sum_{j=1}^{\infty} r_{ij,n} = \sum_{j=1}^{\infty} \lim_{n\to\infty} r_{ij,n} = \sum_{j=1}^{\infty} r_{ij}. \tag{2}$$

Therefore (2) implies that $R \in \mathcal{M}_{bdr}$ with the basic sequence (n_i).
Remark 2.9 follows that the set of all matrices in $\mathcal{M}_{bdr}$ with the basic sequence (n_i) is closed under the composition. □

Example 2.11. *Let $N \in \mathbb{N}$. The matrix*

$$\begin{bmatrix} A & O & O & \dots \\ O & A & O & \dots \\ O & O & A & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3}$$

is block diagonal row stochastic, where O is the $N \times N$ zero matrix and A is the $N \times N$ matrix

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

The operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ corresponds to the matrix (3) *is*

$$Tx = \Big(\underbrace{x_1, \dots, x_1}_{N\text{-times}}, \underbrace{x_{N+1}, \dots, x_{N+1}}_{N\text{-times}}, \underbrace{x_{2N+1}, \dots, x_{2N+1}}_{N\text{-times}}, \dots\Big),$$

for $x = (x_n) \in \mathfrak{c}_0$.

In general, for any block diagonal row stochastic matrix, there corresponds a unique bounded linear operator on $\mathfrak{c}_0$ which its norm is one, as in the next theorem.

Theorem 2.12. *Let $[d_{mn}]_{m,n\in\mathbb{N}}$ be a block diagonal row stochastic matrix. Then there is a unique bounded linear operator $R : \mathfrak{c}_0 \to \mathfrak{c}_0$ such that*

$$\langle R\mathrm{e}_n, \mathrm{e}_m \rangle = (R\mathrm{e}_n)(m) = d_{mn},$$

where $\langle (a_n), (b_n) \rangle$ denotes the dual pairing of (a_n) and (b_n) which is defined by $\langle (a_n), (b_n) \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i$. Moreover, $\|R\| = 1$.

Proof. By assumption, for all $m \in \mathbb{N}$ we have $\sum_{n\in\mathbb{N}} d_{mn} = 1$ and in any row and column there are at most finitely many nonzero entries. Also, for $f \in \mathfrak{c}_0$ and $m \in \mathbb{N}$ the series $\sum_{n\in\mathbb{N}} d_{mn} f_n$ is absolutely convergent and we have

$$\Big|\sum_{n\in\mathbb{N}} d_{mn} f_n\Big| \leq \sum_{n\in\mathbb{N}} d_{mn}|f_n| \leq \|f\| \sum_{n\in\mathbb{N}} d_{mn} = \|f\|. \tag{4}$$

Since $f \in \mathfrak{c}_0$, for given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n| \leq \frac{\varepsilon}{2}$. As the sequences $(d_{m1})_{m\in\mathbb{N}}, (d_{m2})_{m\in\mathbb{N}}, \ldots, (d_{mN})_{m\in\mathbb{N}}$ tend to zero, there is $M \in \mathbb{N}$ such that

$$0 \leq d_{mj} < \frac{\varepsilon}{2N(\|f\|+1)}, \qquad \text{for all } m \geq M, \ \ j = 1, \ldots, N.$$

Thus for $m \geq M$, we have

$$\begin{aligned}
\Big|\sum_{n=1}^{\infty} d_{mn} f_n\Big| &\leq \sum_{n=1}^{\infty} d_{mn}|f_n| \\
&\leq \sum_{n=1}^{N} d_{mn}|f_n| + \sum_{n=N+1}^{\infty} d_{mn}|f_n| \\
&\leq \frac{\varepsilon}{2N(\|f\|+1)} \sum_{n=1}^{N} |f_n| + \frac{\varepsilon}{2} \sum_{n=N+1}^{\infty} d_{mn} \\
&\leq \frac{\varepsilon N \|f\|}{2N(\|f\|+1)} + \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

The above relations show that $\lim_{m\to\infty} \sum_{n=1}^{\infty} d_{mn} f_n = 0$. So, the operator $R : \mathfrak{c}_0 \to \mathfrak{c}_0$ which is defined by

$$Rf = \sum_{m\in\mathbb{N}} \Big(\sum_{n\in\mathbb{N}} d_{mn} f_n\Big) \mathrm{e}_m,$$

is clearly linear. Also, (4) implies that R is bounded, $\|Rf\| \leq \|f\|$, and then

$$\|R\| \leq 1. \tag{5}$$

For $k \in \mathbb{N}$, it follows that

$$R\Big(\sum_{i=1}^{k} \mathrm{e}_i\Big) = \sum_{m\in\mathbb{N}} \Big(\sum_{n=1}^{k} d_{mn}\Big) \mathrm{e}_m, \qquad \Big\|\sum_{i=1}^{k} \mathrm{e}_i\Big\| = 1,$$

and so

$$\|R\| \geq \Big\|R\Big(\sum_{i=1}^{k} \mathrm{e}_i\Big)\Big\| = \Big\|\sum_{m\in\mathbb{N}} \Big(\sum_{n=1}^{k} d_{mn}\Big) \mathrm{e}_m\Big\| \geq \Big|\sum_{n=1}^{k} d_{1n}\Big| = \sum_{n=1}^{k} d_{1n}.$$

That is $\|R\| \geq \sum_{n=1}^{k} d_{1n}$. As k tends to infinity, we get $\|R\| \geq 1$. Together (5) it implies that $\|R\| = 1$. The definition of R follows that

$$R\mathrm{e}_n = \sum_{m\in\mathbb{N}} d_{mn} \mathrm{e}_m, \qquad \text{for } n \in \mathbb{N},$$

and for $m \in \mathbb{N}$, we get $\langle R\mathrm{e}_n, \mathrm{e}_m\rangle = (R\mathrm{e}_n)(m) = d_{mn}$.

Now we show that R is unique, suppose that $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ be a bounded linear operator such that

$$\langle T\mathrm{e}_n, \mathrm{e}_m\rangle = (T\mathrm{e}_n)(m) = d_{mn}, \qquad \text{for all } m, n \in \mathbb{N}.$$

Thus for $f \in \mathfrak{c}_0$ and $k \in \mathbb{N}$, since T is continuous and linear, we have

$$\begin{aligned}(Tf)(k) &= \Big(T\Big(\sum_{n\in\mathbb{N}} f_n \mathrm{e}_n\Big)\Big)(k)\\ &= \Big(\sum_{n\in\mathbb{N}} f_n T\mathrm{e}_n\Big)(k)\\ &= \sum_{n\in\mathbb{N}} f_n\,(T\mathrm{e}_n)\,(k)\\ &= \sum_{n\in\mathbb{N}} f_n d_{kn}.\end{aligned}$$

Therefore

$$Tf = \sum_{m\in\mathbb{N}} (Tf)(m)\mathrm{e}_m = \sum_{m\in\mathbb{N}} \Big(\sum_{n\in\mathbb{N}} f_n d_{mn}\Big)\mathrm{e}_m = Rf.$$

It follows the uniqueness of R. □

In the next example, we consider two elemmaents in $\mathfrak{c}_0$, which are block diagonal equivalent.

Example 2.13. *Let $x, y \in \mathfrak{c}_0$ be as follows:*

$$x = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{9} \\ \frac{1}{10} \\ \frac{1}{13} \\ \vdots \end{bmatrix}, \qquad y = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{7} \\ \frac{1}{8} \\ \frac{1}{11} \\ \frac{1}{12} \\ \vdots \end{bmatrix}.$$

We show that $x \sim_{bdr} y$. Put

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{8}{15} & \frac{7}{15} & 0 & 0 & \cdots \\ 0 & 0 & \frac{2}{9} & \frac{7}{9} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{16}{27} & \frac{11}{27} & \cdots \\ 0 & 0 & 0 & 0 & \frac{4}{15} & \frac{11}{15} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

then $x = Ry$ and so $x \prec_{bdr} y$. Also, let

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{15} & \frac{4}{15} & \frac{10}{15} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{4}{24} & \frac{20}{24} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{12}{21} & \frac{9}{21} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{3}{12} & \frac{9}{12} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{20}{33} & \frac{13}{33} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{18} & \frac{13}{18} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

then we have $y = Sx$ and so $y \prec_{bdr} x$. Therefore $x \sim_{bdr} y$.

Let $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ be a bounded linear operator with matrix representation $[t_{ij}]_{i,j\in\mathbb{N}}$ where $t_{ij} = (T\mathrm{e}_j)(i)$. We will incorporate T to its matrix form. Therefore,

$$(Tx)(i) = \sum_{j\in\mathbb{N}} t_{ij}x(j), \qquad \text{for } x \in \mathfrak{c}_0 \text{ and } i \in \mathbb{N}.$$

Definition 2.14. *A bounded linear operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ is called block diagonal row stochastic operator on $\mathfrak{c}_0$ if the matrix representation of T, i.e. $[t_{ij}]_{i,j\in\mathbb{N}}$ belongs to $\mathcal{M}_{bdr}$.*

Definition 2.15. *A linear operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ is called a preserver of $\prec_{bdr}$ if for $x, y \in \mathfrak{c}_0$ the relation $x \prec_{bdr} y$ implies $Tx \prec_{bdr} Ty$. We denote by $\mathcal{P}_{bdr}$ the set of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ which preserve $\prec_{bdr}$.*

Now we investigate some important properties of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ which preserves $\prec_{bdr}$. It is clear that for such an operator and for each $k \in \mathbb{N}$, because $\mathrm{e}_k \sim_{bdr} \mathrm{e}_1$, it follows that $T\mathrm{e}_k \sim_{bdr} T\mathrm{e}_1$. Therefore, we can consider $a = \sup T\mathrm{e}_k = \sup T\mathrm{e}_1$ and $b = \inf T\mathrm{e}_k = \inf T\mathrm{e}_1$.

Definition 2.16. *[2] Let $T \in \mathcal{P}_{bdr}$. For any $k \in \mathbb{N}$, let $a \geq 0$, and $b \leq 0$. We define*

$$I_k = \{i \in \mathbb{N};\ t_{ik} = a\}, \qquad J_k = \{j \in \mathbb{N};\ t_{jk} = b\},$$

where $t_{ij} = (T\mathrm{e}_j)(i)$.

Lemma 2.17. *Let $T \in \mathcal{P}_{bdr}$. If $k \in \mathbb{N}$ and $a > 0$, then $a = \max T\mathrm{e}_k$, and for any $n_0 \neq k$,*

$$\sup\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\} = a = \max\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\}.$$

Proof. As $T\mathrm{e}_k$ is a sequence in $\mathfrak{c}_0$, with $\sup T\mathrm{e}_k > 0$, then it has some positive elemmaents and so, $T\mathrm{e}_k$ attains the maximum. Thus $a = \max T\mathrm{e}_k$, that is $I_k \neq \emptyset$.

Now let $i_0 \in \mathbb{N}$ be such that $t_{i_0k} = a$. As the sequence $(t_{mk})_{m\in\mathbb{N}}$ converges to zero, for any $0 < \varepsilon < \frac{a}{2}$, there is $M \in \mathbb{N}$ such that for all $m \geq M$,

$$|t_{mk}| < \varepsilon.$$

On the other hand, since the sequences $(t_{1,n})_{n\in\mathbb{N}}, (t_{2,n})_{n\in\mathbb{N}}, \ldots, (t_{M-1,n})_{n\in\mathbb{N}}$ belong to $\mathfrak{c}_0^* = \ell^1$, all of them converge to zero and so one can choose $N \in \mathbb{N}$ such that for all $1 \leq i \leq M-1$, and $n \geq N$, we have

$$|t_{in}| < \varepsilon.$$

Since $1 \leq i_0 \leq M-1$ and $t_{i_0k} = a > \varepsilon$ it follows that $1 \leq k \leq N-1$.
Let $n_0 \neq k$. Since $\mathrm{e}_k + \mathrm{e}_{n_0} \sim_{bdr} \mathrm{e}_k + \mathrm{e}_N$, it follows that

$$T\mathrm{e}_k + T\mathrm{e}_{n_0} \sim_{bdr} T\mathrm{e}_k + T\mathrm{e}_N.$$

Thus

$$\begin{aligned}
&\sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\}\\
&\quad = \sup\{T\mathrm{e}_k + T\mathrm{e}_N\}\\
&\quad = \max\left\{\sup\{t_{mk} + t_{mN};\ 1 \le m \le M-1\},\ \sup\{t_{mk} + t_{mN};\ m \ge M\}\right\}\\
&\quad \le \max\left\{a+\varepsilon, \sup\{\varepsilon + t_{mN};\ m \ge M\}\right\}\\
&\quad = \max\{a+\varepsilon, a+\varepsilon\} = a+\varepsilon.
\end{aligned}$$

Hence for $0 < \varepsilon < \frac{a}{2}$, we see that

$$\sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\} \le a + \varepsilon,$$

and so

$$\sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\} \le a. \tag{6}$$

On the other hand, for all $n \in \mathbb{N}$, we have

$$a + t_{i_0 n} = t_{i_0 k} + t_{i_0 n} \le \sup\{T\mathrm{e}_k + T\mathrm{e}_n\} = \sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\},$$

and as $(t_{i_0 n})_{n\in\mathbb{N}}$ is a sequence in ℓ^1, it tends to zero. Now, in the above inequality when $n \to \infty$ we obtain that

$$a \le \sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\}. \tag{7}$$

The inequalities (6) and (7) imply that $\sup\{T\mathrm{e}_k + T\mathrm{e}_{n_0}\} = a$. In the same manner, we can prove that $\sup\{T\mathrm{e}_k - T\mathrm{e}_{n_0}\} = a$.

Because $T\mathrm{e}_k \pm T\mathrm{e}_{n_0}$ converges to zero with $\sup\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\} = a > 0$, it follows that some of the values of these sequences are positive. So, $\max\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\} = a$. □

Lemma 2.18. *Let $T \in \mathcal{P}_{bdr}$. Let $k \in \mathbb{N}$ and $b = \inf T\mathrm{e}_k < 0$. Then $b = \min T\mathrm{e}_k$, and for any $n_0 \neq k$, we have*

$$\inf\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\} = \min\{T\mathrm{e}_k \pm T\mathrm{e}_{n_0}\} = b.$$

Proof. Put $S = -T$ and apply Lemma 2.17 for the operator S. □

Corollary 2.19. *Let $T \in \mathcal{P}_{bdr}$ and $k \in \mathbb{N}$. Let $[t_{ij}]_{i,j\in\mathbb{N}}$ be the matrix representation of T. Then the following assertions hold.*

(i) *If $a > 0$, then I_k is a nonempty finite set and therefore, for all $i \in I_k$, and for any $n \neq k$, we have*

$$\langle T\mathrm{e}_n, \mathrm{e}_i\rangle = t_{in} = (T\mathrm{e}_n)(i) = 0.$$

(ii) *If $b < 0$, then J_k is a nonempty finite set and for all $j \in J_k$, and for any $n \neq k$, we have*

$$\langle T\mathrm{e}_n, \mathrm{e}_j\rangle = t_{jn} = (T\mathrm{e}_n)(j) = 0.$$

Proof. (i) Let $a > 0$. According to Lemma 2.17, we have $a = \max T\mathrm{e}_k$, and so $I_k \neq \emptyset$. On the other hand, since the sequence $(t_{mk})_{m\in\mathbb{N}}$ tends to zero, the set $I_k = \{i \in \mathbb{N};\ t_{ik} = a\}$ is a finite set. Now let $i \in I_k$. Let $n \neq k$ be such that $\langle T\mathrm{e}_n, \mathrm{e}_i\rangle = t_{in} \neq 0$, then we consider the following two cases:
Case I. If $t_{in} > 0$, according to Lemma 2.17, we get

$$a = \sup\{T\mathrm{e}_k + T\mathrm{e}_n\} \ge t_{ik} + t_{in} = a + t_{in} > a.$$

Case II. If $t_{in} < 0$, according to Lemma 2.17, we obtain

$$a = \sup\{T\mathrm{e}_k - T\mathrm{e}_n\} \ge t_{ik} - t_{in} = a - t_{in} > a.$$

In both cases, we get a contradiction. So $t_{in} = 0$.
(ii) One can apply part (i) for $-T$ instead of T. □

Lemma 2.20. *Let $f, g \in c_0$ be such that $f \prec_{bdr} g$. Then $\|f\| \leq \|g\|$.*

Proof. According to

$$\Big\{f(n);\ n \in \mathbb{N}\Big\} \subseteq \mathrm{co}\Big\{f(n);\ n \in \mathbb{N}\Big\} \subseteq \mathrm{co}\Big\{g(n);\ n \in \mathbb{N}\Big\} \subseteq [b, a],$$

where $b = \inf_{n\in\mathbb{N}} g(n)$, $a = \sup_{n\in\mathbb{N}} g(n)$, for all $n \in \mathbb{N}$, it follows that $b \leq f(n) \leq a$ and

$$-\max\{a, -b\} = \min\{b, -a\} \leq b \leq f(n) \leq a \leq \max\{a, -b\}.$$

Since $a \geq 0$, $b \leq 0$, we have $\max\{a, -b\} \geq 0$ and for all $n \in \mathbb{N}$, we have

$$|f(n)| \leq |\max\{a, -b\}| = \max\{a, -b\}.$$

Therefore $\|f\| \leq \max\{a, -b\} = \|g\|$. □

Lemma 2.21. *If $T \in \mathcal{P}_{bdr}$, then for all distinct $m, n \in \mathbb{N}$,*

$$\|Te_m - Te_n\| = \max\{a, -b\}.$$

Proof. Since for all distinct $m, n \in \mathbb{N}$, we have $Te_m - Te_n \sim_{bdr} Te_1 - Te_2$, Lemma 2.20 implies $\|Te_m - Te_n\| = \|Te_1 - Te_2\|$. Hence it remains to prove

$$\|Te_1 - Te_2\| = \max\{a, -b\}. \tag{8}$$

To this end, we consider the following two cases.
Case I. If $\max\{a, -b\} = 0$, then $a = b = 0$, and so for all $n \in \mathbb{N}$, we have $Te_n = 0$. Hence $T \equiv 0$, and (8) satisfies.
Case II. If $c = \max\{a, -b\} > 0$, then for any $0 < \varepsilon < c$, there exist $M, N \geq 2$ such that

$$|t_{m1}| < \varepsilon, \qquad \text{for all } m \geq M,$$

and

$$|t_{in}| < \varepsilon, \qquad \text{for all } i \in \{1, \ldots, M-1\} \text{ and } n \geq N.$$

So, for all $\varepsilon > 0$, we have

$$\begin{aligned}
\|Te_1 - Te_2\| \\
&= \|Te_1 - Te_N\| \\
&= \max\Big\{|t_{11} - t_{1N}|, \ldots, |t_{M-1,1} - t_{M-1,N}|, \sup_{m\geq M}\{|t_{m1} - t_{mN}|\}\Big\} \\
&\leq \max\Big\{|t_{11}| + |t_{1N}|, \ldots, |t_{M-1,1}| + |t_{M-1,N}|, \sup_{m\geq M}\{|t_{m1}| + |t_{mN}|\}\Big\} \\
&= c + \varepsilon.
\end{aligned}$$

It follows that

$$\|Te_1 - Te_2\| \leq c. \tag{9}$$

On the other hand, as $c > 0$ and the sequence Te_1 converges to zero, there is $i \in \mathbb{N}$ such that $|t_{i1}| = c$. Hence for any $n \geq 2$,

$$|t_{i1} - t_{in}| \leq \|Te_1 - Te_n\| = \|Te_1 - Te_2\|.$$

Since $\lim_{n\to\infty} t_{in} = 0$, the latter inequality implies that

$$c \leq \|Te_1 - Te_2\|. \tag{10}$$

Therefore (9) and (10) imply that $\|Te_1 - Te_2\| = \max\{a, -b\}$. □

Lemma 2.22. *Let $T \in \mathcal{P}_{bdr}$. Then* $\|T\| = \max\{a, -b\}$.

Proof. Suppose that $f \in \mathfrak{c}_0$ such that $\|f\| = \sup_{n\in\mathbb{N}} |f(n)| \leq 1$. Then

$$f \prec_{bdr} \mathrm{e}_1 - \mathrm{e}_2,$$

and so $Tf \prec_{bdr} T\mathrm{e}_1 - T\mathrm{e}_2$. Thus Lemmas 2.20 and 2.21 imply that

$$\|Tf\| \leq \|T\mathrm{e}_1 - T\mathrm{e}_2\| = \max\{a, -b\}.$$

It follows that $\|T\| \leq \max\{a, -b\}$. On the other hand, Definition 2.16, $\|T\mathrm{e}_1\| = \max\{a, -b\}$. This follows the assertion. □

Theorem 2.23. *Let $T \in \mathcal{P}_{bdr}$ and $[t_{ij}]_{i,j\in\mathbb{N}}$ be the matrix representation of T. If for any $n \in \mathbb{N}$, $\alpha_n \in [-1, 1]$, then for $m \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} \alpha_n t_{mn}$ converges and its absolute value is at most* $\max\{a, -b\}$.

Proof. Let $m \in \mathbb{N}$. Since

$$\begin{aligned}\sum_{n=1}^{\infty} |\alpha_n t_{mn}| &\leq \sum_{n=1}^{\infty} |t_{mn}| \\ &= \sum_{n=1}^{\infty} |\langle T\mathrm{e}_n, \mathrm{e}_m\rangle| = \sum_{n=1}^{\infty} |\langle \mathrm{e}_n, T^*\mathrm{e}_m\rangle| = \sum_{n=1}^{\infty} |\langle T^*\mathrm{e}_m, \mathrm{e}_n\rangle| \\ &= \sum_{n=1}^{\infty} |(T^*\mathrm{e}_m)(n)| = \|T^*\mathrm{e}_m\|_1 \\ &\leq \|T^*\| = \|T\| < \infty,\end{aligned}$$

thus this series is absolutely convergent and so converges. Therefore, Lemma 2.22 implies $\left|\sum_{n=1}^{\infty} \alpha_n t_{mn}\right| \leq \sum_{n=1}^{\infty} |\alpha_n t_{mn}| \leq \|T\| = \max\{a, -b\}$. □

Theorem 2.24. *If $\alpha, \beta \in \mathbb{R}$, then the operator*

$$T = \begin{bmatrix} \alpha & 0 & 0 & \cdots \\ \beta & 0 & 0 & \cdots \\ 0 & \alpha & 0 & \cdots \\ 0 & \beta & 0 & \cdots \\ 0 & 0 & \alpha & \cdots \\ 0 & 0 & \beta & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

on $\mathfrak{c}_0$ preserves $\prec_{bdr}$.

Proof. Suppose that $f, g \in \mathfrak{c}_0$ and $f \prec_{bdr} g$. So, there is a matrix $D = [d_{ij}]_{i,j\in\mathbb{N}} \in \mathcal{M}_{bdr}$ such that $f = Dg$. Let $\tilde{D}$ be the following matrix

$$\tilde{D} = \begin{bmatrix} d_{11} & 0 & d_{12} & 0 & d_{13} & 0 & d_{14} & \cdots \\ 0 & d_{11} & 0 & d_{12} & 0 & d_{13} & 0 & \cdots \\ d_{21} & 0 & d_{22} & 0 & d_{23} & 0 & d_{24} & \cdots \\ 0 & d_{21} & 0 & d_{22} & 0 & d_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Obviously, $\tilde{D} \in \mathcal{M}_{bdr}$ and $Tf = \tilde{D}Tg$, which follows that $Tf \prec_{bdr} Tg$. Therefore T preserves $\prec_{bdr}$. □

Theorem 2.25. *If $T \in \mathcal{P}_{bdr}$, then each columns of T attains the values $a = \sup T\mathrm{e}_1$ and $b = \inf T\mathrm{e}_1$; and so $a = \max T\mathrm{e}_1$ and $b = \min T\mathrm{e}_1$.*

Proof. Let $n \in \mathbb{N}$. We consider the n-th column of T. Since $T\mathrm{e}_n \sim_{bdr} T\mathrm{e}_1$, it follows that $a = \sup T\mathrm{e}_n$ and $b = \inf T\mathrm{e}_n$. We need only consider four cases:
Case I. Let $b < 0 < a$. Since $T\mathrm{e}_n \in \mathfrak{c}_0$, clearly

$$a = \max T\mathrm{e}_n, \qquad b = \min T\mathrm{e}_n.$$

Case II. Let $b = 0 < a$. Then $a = \max T\mathrm{e}_n$, and also $a = \max T\mathrm{e}_{n+1}$, and so there is $m \in \mathbb{N}$ such that $(T\mathrm{e}_{n+1})(m) = a$. Thus $m \in I_{n+1}$. Now part (i) of Corollary 2.19 implies that $(T\mathrm{e}_n)(m) = 0 = b$. Therefore $b = \min T\mathrm{e}_n$.
Case III. Let $b < 0 = a$. Then $b = \min T\mathrm{e}_n$ and also $b = \min T\mathrm{e}_{n+1}$, and so there is $m \in \mathbb{N}$ such that $(T\mathrm{e}_{n+1})(m) = b$. Thus $m \in J_{n+1}$. Now part (ii) of Corollary 2.19 implies that $(T\mathrm{e}_n)(m) = 0 = a$. Therefore $a = \max T\mathrm{e}_n$.
Case IV. Let $a = b = 0$. Then $T\mathrm{e}_n = 0$, and as T is continuous, we have $T \equiv 0$ and the assertion holds. □

Theorem 2.26. *Let $T \in \mathcal{P}_{bdr}$. Then exactly one of the following assertions hold.*

(i) *In all columns of T, there are finitely many nonzero entries.*

(ii) *In all columns of T, there are infinitely many nonzero entries.*

Proof. On the contrary, suppose that there are $m, n \in \mathbb{N}$ such that in the mth column of T, there are finitely many nonzero entries and in the nth column there are infinitely many nonzero entries. Therefore all entries of $T\mathrm{e}_m$ are zero except for finitely many, and so the relation $T\mathrm{e}_m \sim_{bdr} T\mathrm{e}_n$ can not be satisfied. □

In the following, we obtain some examples of bounded linear operators on $\mathfrak{c}_0$ which preserve $\prec_{bdr}$, and these operators need not to be block diagonal row stochastic operators (as Theorem 2.24).

Example 2.27. *The operator $D : \mathfrak{c}_0 \to \mathfrak{c}_0$ defined by the matrix form*

$$D = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is $(x_1, x_2, x_3, \dots) \longmapsto \Big(\sum_{n=1}^{\infty} \frac{1}{2^n} x_n, x_1, x_2, x_3, \dots \Big)$, *preserves* $\prec_{bdr}$.

Example 2.28. *The bounded linear operator $D : \mathfrak{c}_0 \to \mathfrak{c}_0$ with the matrix form*

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is $(x_1, x_2, x_3, \dots) \longmapsto (x_1, x_1, x_2, x_3, \dots)$, *preserves* $\prec_{bdr}$.

Remark 2.29. *If $D \in \mathcal{P}_{bdr}$, then*

(i) *One can add finitely many zero rows to the matrix form of D and it still preserves $\prec_{bdr}$.*

(ii) *One can repeat finitely many of any row of D, and it still preserves* $\prec_{bdr}$.

In Example 2.28, the first row of the identity operator is repeated.

Example 2.30. *The bounded linear operator* $D : \mathfrak{c}_0 \to \mathfrak{c}_0$ *with the matrix form*

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is $(x_1, x_2, x_3, \ldots) \longmapsto \left(x_1, \frac{1}{2}x_1 + \frac{1}{2}x_2, x_2, x_3, \ldots\right)$, *preserves* $\prec_{bdr}$.

Example 2.31. *The bounded linear operator* $D : \mathfrak{c}_0 \to \mathfrak{c}_0$ *with the matrix form*

$$D = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

that is $(x_1, x_2, x_3, \ldots) \longmapsto \left(\frac{1}{2}x_1, x_1, x_2, x_3, \ldots\right)$, *preserves* $\prec_{bdr}$, *by the following reason.*
Suppose that $x, y \in \mathfrak{c}_0$ *and* $x \prec_{bdr} y$. *Since* $x_n \to 0$ *and* $y_n \to 0$, *there are* $n_1, n_2 \in \mathbb{N}$ *such that*

$$\frac{1}{2}x_1 \in \mathrm{co}\{x_1, \ldots, x_{n_1}\} \quad \textit{and} \quad \frac{1}{2}y_1 \in \mathrm{co}\{y_1, \ldots, y_{n_2}\}.$$

One can choose the integer $n > \max\{n_1, n_2\}$ *such that*

$$\mathrm{co}\{x_1, \ldots, x_n\} \subseteq \mathrm{co}\{y_1, \ldots, y_n\}.$$

Remark 2.32. *Let* $f = (f_1, f_2, \ldots)$, $g = (g_1, g_2, \ldots) \in \mathfrak{c}_0$ *and* $f \prec_{bdr} g$. *Then there is a sequence of natural numbers* $(n_i)_{i \in \mathbb{N}}$ *such that*

$$F_{n_i} \prec_r G_{n_i},$$

where $F_{n_i} = (f_{n_{i-1}+1}, \ldots, f_{n_{i-1}+n_i})$ *and* $G_{n_i} = (g_{n_{i-1}+1}, \ldots, g_{n_{i-1}+n_i})$.

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