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Spectral Radius and Energy of Sombor Matrix of Graphs

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Abstract. Let G be a graph of order n. For $i = 1, 2, \ldots, n$, let d_i be the degree of the vertex v_i of G. The Sombor matrix $\mathcal{A}_{so}$ of G is defined so that its (i, j)-entry is equal to $\sqrt{d_i^2 + d_j^2}$ if the vertices v_i and v_j are adjacent, and 0 otherwise. The spectral radius η_1 and the energy E_{so} of $\mathcal{A}_{so}$ are examined. In particular, upper bounds on E_{so} are obtained, as well as Nordhaus–Gaddum–type results for η_1 and E_{so}.

1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let $|V| = n$ and $|E| = m$. The edge connecting the vertices u and v will be denoted by uv. Let d_i be the degree of vertex $v_i \in V(G)$ for $i = 1, 2, \ldots, n$. The maximum and minimum degree of G are denoted by Δ and δ, respectively.

The *adjacency matrix* $\mathbf{A} = \mathbf{A}(G) = (a_{ij})_{n\times n}$ of G is defined via

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathbf{A}(G)$, forming the *spectrum* of the graph G [5]. The maximum eigenvalue λ_1 is usually called as the *spectral radius* of G. The *energy* of G is defined as [13]

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

There are a number of graph energies, and the majority of them are based on some types of degree–based square, symmetric matrices; see [9, 10].

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Recently, a new vertex–degree–based graph invariant was introduced [8], defined as

$$\mathrm{SO} = \mathrm{SO}(G) = \sum_{ij \in E(G)} \sqrt{(d_i)^2 + (d_j)^2},$$

and named *Sombor index*. In fact, in [8], a novel approach to the interpretation of vertex–degree–based molecular structure descriptors was put forward. In [8], some basic properties of the Sombor index are established, including bounds for general graphs and trees. The relations between Sombor and other degree-based indices are given in [16]. Inspired by this new index, following the reasoning from [6], we define the *Sombor matrix* of the graph G, denoted by $\mathbf{A}_{so} = \mathbf{A}_{so}(G) = (a'_{ij})_{n\times n}$ via

$$a'_{ij} = \begin{cases} \sqrt{(d_i)^2 + (d_j)^2} & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\mathbf{A}_{so}(G) = \sqrt{2}\, r\, \mathbf{A}(G)$ if G is an r-regular graph.

We denote the eigenvalues of $\mathbf{A}_{so}(G)$ by $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$, which form the *Sombor spectrum* of G. Then, in analogy with the ordinary graph spectrum [5, 6], η_1 is the *Sombor spectral radius* and

$$\mathcal{E}_{so} = \mathcal{E}_{so}(G) = \sum_{i=1}^{n} |\eta_i|$$

is the *Sombor energy* of G.

Let $f(G)$ be a graph invariant and n a positive integer, The *Nordhaus–Gaddum problem* is to determine sharp bounds for $f(G) + f(\overline{G})$ and $f(G) \cdot f(\overline{G})$, as G ranges over the class of all graphs of order n, and to characterize the extremal graphs, i.e., graphs that achieve the bounds. Nordhaus–Gaddum type relations have received wide attention; see the recent survey [1] and the book chapter [14].

For other undefined notations and terminology from graph theory, refer to [2, 7]. The rest of the paper is structured as follows. In Section 2, we get some bounds for Sombor spectral radius and Sombor energy. In Section 3, we give Nordhaus–Gaddum type results for the spectral radius η_1. Analogous results for the Sombor energy are given in Section 4.

2. Bounds for Sombor spectral radius and Sombor energy

We state here some previously known results that are needed in the next two sections.

Lemma 2.1. (Rayleigh–Ritz) [17] *If* $\mathbf{M}$ *is a symmetric* $n \times n$ *matrix with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, *then for any* $\mathbf{x} \in R^n$, *such that* $\mathbf{x} \neq \mathbf{0}$,

$$\lambda_1\, \mathbf{x}^T \mathbf{x} \geq \mathbf{x}^T \mathbf{M}\, \mathbf{x}\,.$$

Equality holds if and only if $\mathbf{x}$ *is an eigenvector of* $\mathbf{M}$ *corresponding to the largest eigenvalue* λ_1.

Lemma 2.2. [11] *Let* $\mathbf{A} = (a_{ij})$ *and* $\mathbf{B} = (b_{ij})$ *be symmetric, non-negative matrices of order* n. *If* $A \geq B$, *i.e.* $a_{ij} \geq b_{ij}$ *for all* i, j, *then* $\lambda_1(\mathbf{A}) \geq \lambda_1(\mathbf{B})$, *where* λ_1 *is the largest eigenvalue.*

Lemma 2.3. [15] *Let* $\mathbf{M}$ *be a symmetric matrix of order* n, *and let* $\mathbf{M}_k$ *be its leading* $k \times k$ *submatrix. Then, for* $i = 1, 2, \ldots, k$,

$$\lambda_{n-i+1}(\mathbf{M}) \leq \lambda_{k-i+1}(\mathbf{M}_k) \leq \lambda_{k-i+1}(\mathbf{M})\,,$$

where $\lambda_i(\mathbf{M})$ *is the i-th largest eigenvalue of* $\mathbf{M}$.

Lemma 2.4. [3] *Let G be a graph of order n with m edges, minimum degree $\delta \geq 1$, and maximum degree Δ. Then*

$$\lambda_1 \leq \sqrt{2m - \delta(n-1) + (\delta - 1)\Delta},$$

with equality holding if and only if G is regular, a star plus copies of K_2, or a complete graph plus a regular graph whose degree is smaller.

The following result is immediate.

Corollary 2.5. *Let G be a graph of order n with m edges, minimum degree $\delta \geq 1$, and maximum degree Δ. Then*

$$\eta_1 \leq \sqrt{2}\,\Delta(G)\,\sqrt{2m - \delta(n-1) + (\delta - 1)\Delta}.$$

Moreover, the bound is sharp.

Proof. From Lemma 2.2, we have

$$\eta_1 \leq \sqrt{2}\,\Delta(G)\,\lambda_1 \leq \sqrt{2}\,\Delta(G)\,\sqrt{2m - \delta(n-1) + (\delta - 1)\Delta}.$$

If G is a regular graph of degree $\Delta = \delta$, then $\eta_1 = \sqrt{2}\,\Delta\,\lambda_1$, which means that the bound is sharp. □

Koolen and Moulton [12] obtained the following three results.

Lemma 2.6. [12] *If $2m \geq n$ and G is a graph on n vertices with m edges, then the inequality*

$$\mathcal{E}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}$$

holds. Moreover, equality holds if and only if G is either $\frac{n}{2}K_2$, K_n, or a non-complete connected strongly regular graph with two non-trivial eigenvalues both with absolute value

$$\sqrt{\frac{1}{n-1}\left[2m - \left(\frac{2m}{n}\right)^2\right]}.$$

Lemma 2.7. [12] *If $2m \leq n$ and G is a graph on n vertices with m edges, then the inequality*

$$\mathcal{E}(G) \leq 2m$$

holds. Moreover, equality holds if and only if G is disjoint union of edges and isolated vertices.

Lemma 2.8. [12] *Let G be a graph on n vertices. Then*

$$\mathcal{E}(G) \leq \frac{n}{2}(1 + \sqrt{n}),$$

holds, with equality holding if and only if G is a strongly regular graph with parameters $\left(n, (n+\sqrt{n})/2, (n+2\sqrt{n})/4, (n+2\sqrt{n})/4\right)$.

Following the idea from [12], we obtain the following results.

Theorem 2.9. *Let G be a graph on n vertices with m edges having no isolated vertices. If $2m \geq n$, then*

$$\mathcal{E}_{so}(G) \leq t + \sqrt{(n-1)\left[4m\Delta^2 - t^2\right]}, \tag{1}$$

where $t = \max\left\{\frac{2\sqrt{2}m\delta}{n}, \sqrt{\frac{4m\Delta^2}{n}}\right\}$. Moreover, the upper bound is sharp.

Proof. Suppose that $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ are the eigenvalues of $\mathcal{A}_{so}(G)$ (which are real as the Sombor matrix of is real and symmetric). Then, setting $\mathbf{x} = (1, 1, \ldots, 1)^T$ in Lemma 2.1, we get

$$\eta_1 \geq \frac{2\mathrm{SO}(G)}{n} \geq \frac{2\sqrt{2}\,m\delta}{n}.$$

Combining this with the Cauchy–Schwartz inequality, applied to the vectors $(|\eta_2|, |\eta_3|, \ldots, |\eta_n|)^T$ and $(1, 1, \ldots, 1)^T$ with $n-1$ entries, we obtain the inequality

$$\sum_{i=2}^{n} |\eta_i| \leq \sqrt{(n-1)\left(\sum_{i=1}^{n} \eta_i^2 - \eta_1^2\right)}.$$

Now, the function $F(x) := x + \sqrt{(n-1)\,[4m\Delta^2 - x^2]}$ attains a maximum value when $x = \sqrt{\frac{4m\Delta^2}{n}}$. Thus $F(\eta_1) \leq F\left(\sqrt{\frac{4m\Delta^2}{n}}\right)$ holds. If $\frac{2\sqrt{2}\,m\delta}{n} \geq \sqrt{\frac{4m\Delta^2}{n}}$, then $F(\eta_1) \leq F(\frac{2\sqrt{2}\,m\delta}{n})$. Thus, we must have

$$\mathcal{E}_{so}(G) \leq x + \sqrt{(n-1)\left(\sum_{i=1}^{n} \eta_i^2 - x^2\right)} \leq t + \sqrt{(n-1)\,\{4m\Delta^2 - t^2\}}.$$

One can easily check that $\frac{n}{2}K_2$ and K_n attain this upper bound. □

Corollary 2.10. *Let G be a graph on n vertices with m edges having no isolated vertices. If* $2m \geq n$, *then*

$$\mathcal{E}_{so}(G) \leq \Delta\sqrt{4nm}.$$

Theorem 2.11. *Let G be a graph on n vertices with m edges. If* $2m \leq n$, *then*

$$\mathcal{E}_{so}(G) \leq 2\sqrt{2}\,m\Delta(G).$$

Moreover, the bound is sharp.

Proof. Since $2m \leq n$, it follows that G has at least $n - 2m$ isolated vertices. Consider the graph G' obtained from G by removing all isolated vertices. Then G' has at most $2m$ vertices and m edges. Thus we may apply Theorem 2.9 to see that $\mathcal{E}_{so}(G) = \mathcal{E}_{so}(G') \leq 2\sqrt{2}\,m\Delta(G') = 2\sqrt{2}\,m\Delta(G)$ holds. If G' is a disjoint union of edges, then G attains the upper bound. □

Theorem 2.12. *Let G be a graph on n vertices having no isolated vertices. Then*

$$\mathcal{E}_{so}(G) \leq \frac{(n + n\sqrt{n})\Delta^2}{\sqrt{2}\,\delta}. \tag{2}$$

Proof. Suppose that G is a graph with n vertices and m edges. If $2m \geq n$ and

$$\frac{2\sqrt{2}\,m\delta}{n} \geq \sqrt{\frac{4m\Delta^2}{n}},$$

then using routine calculus, it is seen that the right–hand side of inequality (1) – considered as a function of m – is maximized for

$$m = \frac{(n^2 + n\sqrt{n})\Delta^2}{4\,\delta^2}.$$

Substituting this value of m into (1), we get the bound (2).

If

$$\sqrt{\frac{4m\Delta^2}{n}} \geq \frac{2\sqrt{2}\,m\delta}{n},$$

then

$$\mathcal{E}_{so}(G) \leq \sqrt{4nm\Delta^2} \leq \frac{\sqrt{2}\,n\Delta^2}{\delta}.$$

If $2m \leq n$, then by Theorem 2.11, $\mathcal{E}_{so}(G) \leq \sqrt{2}\,n\Delta(G)$. It is clear that (2) follows. □

3. Nordhaus–Gaddum type results for Sombor spectral radius

In this section we present lower and upper bounds on $\eta_1 + \bar{\eta}_1$.

Theorem 3.1. *Let G be a graph of order n with m edges, minimum degree δ and maximum degree Δ. Then*

$$\eta_1 + \bar{\eta}_1 \geq \frac{2}{n}\left[\sqrt{2}\, m\delta(G) + \left[\binom{n}{2} - m\right]\left[n-1-\Delta(G)\right]\sqrt{2}\right]. \tag{3}$$

Moreover, the bound is sharp.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be any vector in R^n. We have

$$\begin{aligned}\mathbf{x}^T(\mathbf{A}_{so} + \bar{\mathbf{A}}_{so})\mathbf{x} &= \mathbf{x}^T\mathbf{A}_{so}\mathbf{x} + \mathbf{x}^T\bar{\mathbf{A}}_{so}\mathbf{x} \\ &= \sum_{v_iv_j\in E(G)} \left(2\sqrt{d_i^2 + d_j^2}\right) x_i x_j + \sum_{v_iv_j\in E(\bar{G})} \left(2\sqrt{\bar{d}_i^2 + \bar{d}_j^2}\right) x_i x_j\end{aligned}$$

which for $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right)^T$ becomes

$$\begin{aligned} &= \frac{2}{n}\sum_{v_iv_j\in E(G)} \sqrt{d_i^2 + d_j^2} + \frac{2}{n}\sum_{v_iv_j\in E(\bar{G})} \sqrt{\bar{d}_i^2 + \bar{d}_j^2} \\ &= \frac{2}{n}[\mathrm{SO}(G) + \mathrm{SO}(\overline{G})] \\ &\geq \frac{2}{n}\left[\sqrt{2}\, m\delta(G) + \left[\binom{n}{2} - m\right]\left[n-1-\Delta(G))\right]\sqrt{2}\right]. \end{aligned} \tag{4}$$

From Lemma 2.1, we get $\eta_1 \geq \mathbf{x}^T\mathbf{A}_{so}\,\mathbf{x}$ and $\bar{\eta}_1 \geq \mathbf{x}^T\,\bar{\mathbf{A}}_{so}\mathbf{x}$. Combining this with inequality (4), we obtain ((3)).

Suppose now that equality holds in (3). Therefore G is regular. Conversely, it is not difficult to see that the equality (3) holds for regular graphs. □

Corollary 3.2. *If G is regular, then*

$$\eta_1 + \bar{\eta}_1 \geq \mathbf{x}^T(\mathbf{A}_{so} + \bar{\mathbf{A}}_{so})\mathbf{x} \geq \sqrt{2}\,\Delta^2 + \sqrt{2}(n-1-\Delta)^2\,.$$

We are now in the position to give Nordhaus–Gaddum type results for η_1.

Theorem 3.3. *Let G be a connected graph of order n with m edges. Then*

$$\begin{aligned}\eta_1 + \bar{\eta}_1 \leq\ & \sqrt{2}\,\Delta(G)\sqrt{2m - \delta(n-1) + (\delta-1)\Delta} \\ &+ \sqrt{2}(n-1-\delta)\sqrt{2\binom{n}{2} - 2m - (\delta+1)(n-1) + \delta(\Delta+1)}\,.\end{aligned}$$

Moreover, the bound is sharp.

Proof. From Corollary 2.5, we have

$$\eta_1 \leq \sqrt{2}\,\Delta\,\sqrt{2m - \delta(n-1) + (\delta - 1)\Delta}$$

and

$$\begin{aligned}\overline{\eta}_1 &\leq \sqrt{2}\,\overline{\Delta}\,\sqrt{2\binom{n}{2} - 2m - \overline{\delta}(n-1) + (\overline{\delta} - 1)\overline{\Delta}}\\ &= \sqrt{2}(n-1-\delta)\,\sqrt{2\binom{n}{2} - 2m - (\delta+1)(n-1) + \delta(\Delta+1)}\,,\end{aligned}$$

and hence

$$\begin{aligned}\eta_1 + \overline{\eta}_1 &\leq \sqrt{2}\,\Delta\,\sqrt{2m - \delta(n-1) + (\delta - 1)\Delta}\\ &+ \sqrt{2}(n-1-\delta)\,\sqrt{2\binom{n}{2} - 2m - (\delta+1)(n-1) + \delta(\Delta+1)}\,.\end{aligned}$$

□

In order to show the sharpness of the upper bound in Theorem 3.3, we consider the following examples.

Example 1. Let G be a regular graph. Then $\overline{G}$ is also regular. From Corollary 2.5, we have

$$\begin{aligned}\eta_1 + \overline{\eta}_1 &= \sqrt{2}\,\Delta\,\sqrt{2m - \delta(n-1) + (\delta - 1)\Delta}\\ &+ \sqrt{2}(n-1-\delta)\,\sqrt{2\binom{n}{2} - 2m - (\delta+1)(n-1) + \delta(\Delta+1)}\,,\end{aligned}$$

which implies that the upper bound of Theorem 3.3 is sharp.

4. Nordhaus–Gaddum type results for Sombor energy

We first give a lower bound for $\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G})$.

Theorem 4.1. *Let G be a connected graph of order n with m edges, and let $C_1, C_2, \ldots, C_r$ be the connected components of $\overline{G}$. If $\eta(C_1) \geq \eta(C_2) \geq \ldots \geq \eta(C_r)$, then*

$$\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) \geq \frac{4\sqrt{2}\,m\delta}{n} + \sum_{i=1}^{r} \frac{4\sqrt{2}\,m(C_i)\delta(C_i)}{n}.$$

Moreover, the bound is sharp.

Proof. Note that

$$\begin{aligned}\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) &= \sum_{i=1}^{n} |\eta_i| + \sum_{i=1}^{n} |\bar{\eta}_i| \geq 2\eta_1 + 2\sum_{i=1}^{r} \eta(C_i)\\ &\geq \frac{4SO(G)}{n} + \sum_{i=1}^{r} \frac{4SO(C_i)}{n} \geq \frac{4\sqrt{2}\,m\delta}{n} + \sum_{i=1}^{r} \frac{4\sqrt{2}\,m(C_i)\delta(C_i)}{n}\,.\end{aligned}$$

One can check that the regular complete bipartite graph attains this lower bound. □

The following corollary is immediate.

Corollary 4.2. *If G is a regular graph of order n, then*

$$\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) \geq 2\sqrt{2}\,\Delta^2 + 2\sqrt{2}(n-1-\Delta)^2.$$

Next, we give an upper bound for $\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G})$.

Theorem 4.3. *Let G be a graph of order n having no isolated vertices. If $\overline{G}$ has no isolated vertices, then*

$$\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) \leq \left(\frac{\Delta^2}{\delta} + \frac{(n-1-\delta)^2}{n-1-\Delta}\right)\frac{(n+n\sqrt{n})}{\sqrt{2}}\,. \tag{5}$$

Proof. From Theorem 2.12, we have

$$\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) \leq \frac{(n+n\sqrt{n})\Delta^2}{\sqrt{2}\,\delta} + \frac{(n+n\sqrt{n})\bar{\Delta}^2}{\sqrt{2}\,\bar{\delta}}$$

which straightforwardly implies (5). □

The above lower bound can be improved as follows.

Theorem 4.4. *Let G be a connected graph of order n with m edges. If $\eta_1 \geq \bar{\eta}_1$, then*

$$\mathcal{E}_{so}(G) + \mathcal{E}_{so}(\overline{G}) \leq \sqrt{8mn\Delta^2 + 8\left[\binom{n}{2} - m\right]n(n-1-\delta)^2}\,.$$

Proof. Since $\mathbf{A}^2\,\mathbf{x} = \lambda^2\,\mathbf{x}$, it follows that

$$\sum_{i=1}^{n}\eta_i^2 = \sum_{i=1}^{n}\mathbf{A}_{so}^2(ii) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a'_{ij})^2 = 2\sum_{v_iv_j\in E(G)}(a'_{ij})^2\,.$$

From the Cauchy–Schwarz inequality,

$$(\mathcal{E}_{so} - \eta_1)^2 \leq \left(2\sum_{v_iv_j\in E(G)}(a'_{ij})^2 - \eta_1^2\right)(n-1)\,,$$

and thus

$$\mathcal{E}_{so} \leq \eta_1 + \sqrt{n-1}\sqrt{2\sum_{v_iv_j\in E(G)}(a'_{ij})^2 - \eta_1^2}$$

implying

$$\begin{aligned}
\mathcal{E}_{so}(G)+\mathcal{E}_{so}(\overline{G}) &\leq \eta_1+\sqrt{n-1}\sqrt{2\sum_{v_iv_j\in E(G)}(a'_{ij})^2-\eta_1^2} \\
&+ \bar{\eta}_1+\sqrt{n-1}\sqrt{2\sum_{v_iv_j\in E(\bar{G})}(\bar{a}'_{ij})^2-\bar{\eta}_1^2} \\
&\leq \eta_1+\bar{\eta}_1+\sqrt{2(n-1)}\sqrt{2\sum_{v_iv_j\in E(G)}(a'_{ij})^2+2\sum_{v_iv_j\in E(\bar{G})}(\bar{a}'_{ij})^2-\eta_1^2-\bar{\eta}_1^2} \\
&\leq \eta_1+\bar{\eta}_1+\sqrt{n-1}\sqrt{4\sum_{v_iv_j\in E(G)}(a'_{ij})^2+4\sum_{v_iv_j\in E(\bar{G})}(\bar{a}'_{ij})^2-(\eta_1+\bar{\eta}_1)^2} \\
&\leq \eta_1+\bar{\eta}_1+\sqrt{n-1}\sqrt{8m\Delta^2+8\left[\binom{n}{2}-m\right](n-1-\delta)^2-(\eta_1+\bar{\eta}_1)^2} \\
&\leq \sqrt{8mn\Delta^2+8\left[\binom{n}{2}-m\right]n(n-1-\delta)^2}\,.
\end{aligned}$$

□

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