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Cline's Formula and Jacobson's Lemma for g-Drazin Inverse

Huanyin Chen[a], Marjan Sheibani Abdolyousefi[b]

[a]*School of Mathematics, Hangzhou Normal University, Hangzhou, China*
[b]*Farzanegan Campus, Semnan University, Semnan, Iran*

Abstract. We present new conditions under which Cline's formula and Jacobson's lemma for g-Drazin inverse hold. Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. We prove that a has g-Drazin inverse if and only if b^ka^k has g-Drazin inverse. In this case,

$$(b^ka^k)^d = b^k(a^d)^2a^k \text{ and } a^d = a^k[(b^ka^k)^d]^{k+1}.$$

Further, we study Jacobson's lemma for g-Drazin inverse in a Banach algebra under the preceding condition. The common spectral property of bounded linear operators on a Banach space is thereby obtained.

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. An element $a \in R$ has g-Drazin inverse if there exists $b \in R$ such that

$$b = bab, b \in comm^2(a), a - a^2b \in R^{qnil}.$$

The preceding b is unique if it exists, we denote it by a^d. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in R^{-1} \text{ for every } x \in comm(a)\}$. If we replace R^{qnil} by the set R^{nil} of all nilpotents in R, we call such b the Drazin inverse of a, and denote it by a^D. The set of all g-Drazin invertible elements in R will be denoted by R^d.

Let $a, b \in R$. Then $ab \in R^d$ if and only if $ba \in R^d$ and $(ba)^d = b[(ab)^d]^2a$. This was known as Cline's formula for g-Drazin inverse (see [7, Theorem 2.1]). Lian and Zeng extended Cline's formula for generalized Drazin inverse to the case when $aba = aca$ (see [6]). We refer the reader for further extensions of Cline's formula to [6, Theorem 2.3] and [14, Theorem 2.2].

For any $a, b \in R$, Jacobson's lemma for g-Drazin inverse states that $1 + ab \in R^d$ if and only if $1 + ba \in R^d$ (see [15, Theorem 2.3]). Corach et al. extended Jacobson's lemma to the case that $aba = aca$(see [2, Theorem 1]). Further extensions of Jacobson's lemma can be found in [12, Theorem 3.1] and [2, Theorem 1].

In [4, Theorem 2.20], Gupta and Kumar studied Cline's formula for Drazin inverse under the condition $a^kb^ka^k = a^{k+1}$. By the analysis technique, they investigated common spectral properties of linear operators

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Corresponding author: Marjan Sheibani Abdolyousefi
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Email addresses: huanyinchen@aliyun.com (Huanyin Chen), m.sheibani@semnan.ac.ir (Marjan Sheibani Abdolyousefi)

A and B satisfying $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$. The common spectral properties under the preceding operator equations are investigated by many authors. For instance, Schmoeger [9] considered the common spectral properties of bounded linear operators A and B under $ABA = A^2$ and $BAB = B^2$.

Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. The motivation of this paper is to investigate whether Cline's formula holds for g-Drazin inverse under the preceding equations. In Section 2, we proved that a has g-Drazin inverse if and only if b^ka^k has g-Drazin inverse. In this case,

$$(b^ka^k)^d = b^k(a^d)^2a^k \text{ and } a^d = a^k[(b^ka^k)^d]^{k+1}.$$

In Section 3, we further study Jacobson's lemma for g-Drazin inverse in a Banach algebra under the assumption that $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. Finally, in the last section, the common spectral property of bounded linear operators on a Banach space for g-Drazin inverse is thereby obtained. Let $A, B \in \mathcal{L}(X)$ such that $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$ for some $k \in \mathbb{N}$. We prove that $\sigma_d(A) = \sigma_d(A^kB^k) = \sigma_d(B^kA^k) = \sigma_d(B)$, where σ_d is the g-Drazin spectrum.

Throughout the paper, all rings are associative with an identity and all Banach algebras are complex. We use R^{-1} and R^{qnil} to denote the set of all units and the set of all quasinilpotents of the ring R, respectively. $\mathbb{C}$ stands for the field of all complex numbers.

2. Cline's Formula

In this section we propose to give Cline's formula for the g-Drazin inverse under the conditions $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. The following lemma is crucial in the sequel.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. If a has g-Drazin inverse, then b^ka^k has g-Drazin inverse. In this case,*

$$(b^ka^k)^d = b^k(a^d)^2a^k.$$

Proof. Let $y = b^k(a^d)^2a^k$.

Step 1. $yb^ka^ky = y$. We see that

$$\begin{array}{rcl} yb^ka^ky & = & b^k(a^d)^2a^k(b^ka^k)b^k(a^d)^2a^k \\ & = & b^k(a^d)^2(a^kb^ka^k)b^ka^k(a^d)^2 \\ & = & b^k(a^d)^2a^{k+1}b^ka^k(a^d)^2 \\ & = & b^k(a^d)^2a(a^kb^ka^k)(a^d)^2 \\ & = & b^k(a^d)^2aa^{k+1}(a^d)^2 \\ & = & b^k(a^d)^2a^k \\ & = & y. \end{array}$$

Step 2. We claim by induction that

$$[b^ka^k - (b^ka^k)y(b^ka^k)]^{n+1} \quad = \quad b^k(1 - aa^d)a^{k+n}.$$

Let $n = 1$

$$\begin{array}{rcl} [b^ka^k - (b^ka^k)y(b^ka^k)]^2 & = & b^k(1 - aa^d)a^kb^k(1 - aa^d)a^k \\ & = & b^k(1 - aa^d)a^{k+1}. \end{array}$$

And for n we have,

$$\begin{array}{rl} & [b^ka^k - (b^ka^k)y(b^ka^k)]^{n+1} \\ = & [b^ka^k - (b^ka^k)y(b^ka^k)]^n[b^ka^k - (b^ka^k)y(b^ka^k)] \\ = & b^k(1 - aa^d)a^{k+n-1}b^k(1 - aa^d)a^k \\ = & b^k(1 - aa^d)a^{k+n}. \end{array}$$

As a has g-Drazin inverse then, $a - a^2a^d \in \mathcal{A}^{qnil}$. Let $\alpha = a - a^2a^d$ and $\beta = a^k$. It is obvious that $\alpha^2\beta = \alpha\beta\alpha$ and $\beta^2\alpha = \beta\alpha\beta$. Then we deduce by [16, Lemma 2.11] that $\alpha\beta = a^k(a - a^2a^d) \in R^{qnil}$. Hence $a^kb^ka^k(1 - aa^d) = a^{k+1}(1 - aa^d) = a^k(a - a^2a^d) \in \mathcal{A}^{qnil}$. By using Cline's formula,

$$[b^ka^k - (b^ka^k)y(b^ka^k)]^{k+1} = b^k(1 - aa^d)a^{2k} = b^ka^k(1 - aa^d)a^k \in \mathcal{A}^{qnil},$$

and so

$$b^ka^k - (b^ka^k)y(b^ka^k) \in \mathcal{A}^{qnil}.$$

Step 3. $y \in comm^2(b^ka^k)$. Let $x \in comm(b^ka^k)$. Then $xb^ka^k = b^ka^kx$, and so $a^k(xb^ka^k) = a^k(b^ka^kx)$. Hence $(a^kxb^ka^k)b^ka^k = (a^kb^ka^kx)b^ka^k$. That is, $a^kxb^ka^{k+1} = a^{k+1}xb^ka^k$. Thus $(a^kxb^ka^k)a = a(a^kxb^ka^k)$, and then $a^{k+1}x = a^kxb^ka^k \in comm(a)$. This shows that $(a^{k+1}x)a^d = a^d(a^{k+1}x)$. We directly compute that

$$\begin{aligned} yx &= b^ka^k(a^d)^2x \\ &= b^k(a^d)^3(a^{k+1}x) \\ &= b^k(a^{k+1}x)(a^d)^3 \\ &= b^ka^{k+1}x(a^d)^3 \\ &= (b^ka^k)(b^ka^k)x(a^d)^3 \\ &= x(b^ka^k)(b^ka^k)(a^d)^3 \\ &= xb^ka^{k+1}(a^d)^3 = xb^ka^k(a^d)^2 \\ &= xy. \end{aligned}$$

Therefore $y \in comm^2(b^ka^k)$, and so $y = (b^ka^k)^d$, as required. □

We are ready to generalize [4, Theorem 2.10] to g-Drazin inverse as follow.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. Then a has g-Drazin inverse if and only if b^ka^k has g-Drazin inverse. In this case,*

$$(b^ka^k)^d = b^k(a^d)^2a^k \text{ and } a^d = a^k[(b^ka^k)^d]^{k+1}.$$

Proof. $\Longrightarrow$ This is proved in Lemma 2.1.

$\Longleftarrow$ Let $p = (b^ka^k)^d$ and $z = a^kp^{k+1}$.

Step 1. $z \in comm(a)$. We see that

$$\begin{aligned} za &= a^kp^{k+1}a \\ &= a^kp^{k+2}b^ka^{k+1} \\ &= a^kp^{k+2}(b^ka^k)b^ka^k \\ &= a^kp^{k+1}(b^ka^k) \\ &= a^k(b^ka^k)p^{k+1} \\ &= a^{k+1}p^{k+1} \\ &= az, \end{aligned}$$

as desired.

Step 2. $zaz = z$. We compute that

$$\begin{aligned} zaz &= a^kp^{k+1}a^{k+1}p^{k+1} = a^kp^{k+1}a^k(b^ka^k)p^{k+1} = (a^kp^{k+1})a^kp^k = a^{2k}p^{2k+1} \\ &= a^{k-1}a^{k+1}p^{2k+1} = a^{k-1}a^kp^{2k} = a^{k-2}a^kp^{2k-1} = \cdots = a^kp^{k+1} = z, \end{aligned}$$

as required.

Step 3. $a - a^2z \in \mathcal{A}^{qnil}$. We check that

$$\begin{array}{rcl} (a-a^2z)^{k+1} & = & a^{k+1}(1-a^{k+1}p^{k+1}) \\ & = & a^kb^ka^k - (a^{k+1})^2p^{k+1} \\ & = & a^kb^ka^k - a^k(b^ka^{2k})b^ka^kp^{k+1} \\ & = & a^kb^ka^k - a^k(b^ka^{k+1})a^{k-1}b^ka^kp^{k+1} \\ & = & a^kb^ka^k - a^k(b^ka^k)^2a^{k-1}b^ka^kp^{k+1} \\ & = & a^kb^ka^k - a^k(b^ka^k)^3a^{k-2}b^ka^kp^{k+1} \\ & \vdots & \\ & = & a^kb^ka^k - a^k(b^ka^k)^{k+1}b^ka^kp^{k+1} \\ & = & a^k\Big[b^ka^k - (b^ka^k)^2p\Big]. \end{array}$$

We easily see that

$$\begin{array}{rcl} [b^ka^k - (b^ka^k)^2p]a^k & = & b^ka^{2k} - (b^ka^k)p(b^ka^{2k}) \\ & = & (b^ka^k)^2a^{k-1} - (b^ka^k)p(b^ka^k)^2a^{k-1} \\ & = & (b^ka^k)^3a^{k-2} - (b^ka^k)p(b^ka^k)^3a^{k-2} \\ & \vdots & \\ & = & (b^ka^k)^{k+1} - (b^ka^k)p(b^ka^k)^{k+1} \\ & = & (b^ka^k)\Big[b^ka^k - (b^ka^k)^2p\Big]. \end{array}$$

Since b^ka^k has g-Drazin inverse, $b^ka^k - (b^ka^k)^2p \in \mathcal{A}^{qnil}$ and so $(b^ka^k)(b^ka^k - (b^ka^k)^2p) \in \mathcal{A}^{qnil}$. Hence, $[b^ka^k - (b^ka^k)^2p]a^k \in \mathcal{A}^{qnil}$. By using Cline's formula (see [7, Theorem 2.1]), we have $(a - a^2z)^{k+1} \in \mathcal{A}^{qnil}$, and so $a - a^2z \in \mathcal{A}^{qnil}$. Therefore $a^d = z = a^kp^{k+1}$, as asserted. □

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ and $b^ka^kb^k = b^{k+1}$ for some $k \in \mathbb{N}$. Then a has g-Drazin inverse if and only if b has g-Drazin inverse. In this case,*

$$b^d = b^k[a^k(b^k(a^d)^2a^k)^2b^k]^{k+1}.$$

Proof. In view of Theorem 2.2, a has g-Drazin inverse if and only if b^ka^k has g-Drazin inverse, b has g-Drazin inverse if and only if a^kb^k has g-Drazin inverse. In light of Cline's formula, a^kb^k has g-Drazin inverse if and only if b^ka^k has g-Drazin inverse. Therefore $a \in \mathcal{A}^d$ if and only if $b \in \mathcal{A}^d$. Moreover we have,

$$\begin{array}{rcl} b^d & = & b^k[(a^kb^k)^d]^{k+1}, \\ (a^kb^k)^d & = & a^k[(b^ka^k)^d]^2b^k, \\ (b^ka^k)^d & = & b^k(a^d)^2a^k. \end{array}$$

Therefore,

$$\begin{array}{rcl} b^d & = & b^k[a^k((b^ka^k)^d)^2b^k]^{k+1} \\ & = & b^k[a^k(b^k(a^d)^2a^k)^2b^k]^{k+1}, \end{array}$$

as desired. □

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}$ satisfying $a^kb^ka^k = a^{k+1}$ and $b^ka^kb^k = b^{k+1}$ for some $k \in \mathbb{N}$. Then $a \in \mathcal{A}^{qnil}$ if and only if $b \in \mathcal{A}^{qnil}$.*

Proof. Since $a \in \mathcal{A}^{qnil}$, we have $a^d = 0$. Then $b^d = 0$ by Corollary 2.3, which implies that $b \in \mathcal{A}^{qnil}$. The other direction is proved in the similar way. □

Example 2.5.

Let $R = M_3(\mathbb{C})$. Choose

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in R.$$

Then $A^2B^2A^2 = A^3, B^2A^2B^2 = B^3$. In view of Theorem 2.2, $(B^2A^2)^d = B^2(A^d)^2A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly we have,

$$A^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Jacobson's lemma

In this section we study Jacobson's lemma in the case that $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. We begin with the following theorem which is a part of our main results in this paper.

Theorem 3.1. *Let R be a ring, and let $a, b \in R$ satisfying $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. Then $1 - a \in R^{-1}$ if and only if $1 - b^ka^k \in R^{-1}$. In this case,*

$$(1 - b^ka^k)^{-1} = 1 + b^k(1 - a)^{-1}a^k,$$
$$(1 - a)^{-1} = 1 + a + \cdots + a^{k-1} + a^k(1 - b^ka^k)^{-1}.$$

Proof. $\Longrightarrow$ Set $s = (1 - a)^{-1}$. Then

$$\begin{array}{rcl} (1 - b^ka^k)(1 + b^ksa^k) & = & 1 + b^ksa^k - b^ka^k - b^k(a^kb^ka^k)s \\ & = & 1 + b^ksa^k - b^ka^k - b^ksa^{k+1} \\ & = & 1 + b^k(s - 1 - sa)a^k \\ & = & 1 + b^k[s(1 - a) - 1]a^k \\ & = & 1. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} (1 + b^ksa^k)(1 - b^ka^k) & = & 1 - b^ka^k + b^ksa^k - b^ksa^{k+1} \\ & = & 1 - b^k(1 - s)a^k - b^ksa^{k+1} \\ & = & 1 - b^ks(s^{-1} - 1)a^k - b^ksa^{k+1} \\ & = & 1. \end{array}$$

Therefore $(1 - b^ka^k)^{-1} = 1 + b^k(1 - a)^{-1}a^k$.

$\Longleftarrow$ One easily checks that

$$\begin{array}{rl} & (1 - a)\Big[1 + a + \cdots + a^{k-1} + a^k(1 - b^ka^k)^{-1}\Big] \\ = & 1 - a^k + (1 - a)a^k(1 - b^ka^k)^{-1} \\ = & 1 - a^k + (a^k - a^kb^ka^k)(1 - b^ka^k)^{-1} \\ = & 1. \end{array}$$

In view of [12, Corollary 2.5], $1 - a^kb^k \in R^{-1}$. Clearly, $(1 - a^kb^k)a^k = a^k(1 - b^ka^k)$; hence,

$$a^k(1 - b^ka^k)^{-1} = (1 - a^kb^k)^{-1}a^k.$$

By direct computation, we have

$$\begin{array}{rl} & \Big[1+a+\cdots+a^{k-1}+a^k(1-b^ka^k)^{-1}\Big](1-a)\\ = & 1-a^k+a^k(1-b^ka^k)^{-1}(1-a)\\ = & 1-a^k(1-b^ka^k)^{-1}\Big[1-b^ka^k-(1-a)\Big]\\ = & 1-a^k(1-b^ka^k)^{-1}(a-b^ka^k)\\ = & 1-(1-a^kb^k)^{-1}a^k(a-b^ka^k)\\ = & 1-(1-a^kb^k)^{-1}(a^{k+1}-a^kb^ka^k)\\ = & 1. \end{array}$$

This completes the proof. □

Corollary 3.2. *Let R be a ring, and let $a,b\in R$ satisfying $a^kb^ka^k=a^{k+1}$ and $b^ka^kb^k=b^{k+1}$ for some $k\in\mathbb{N}$. Then $1-a\in R^{-1}$ if and only if $1-b\in R^{-1}$. In this case,*

$$(1-b)^{-1}=\sum_{i=0}^{k+1}b^i+b^{k+1}(1-a)^{-1}a^kb^k.$$

Proof. In view of Theorem 3.1, $1-a\in R^{-1}$ if and only if $1-b^ka^k\in R^{-1}$. Furthermore, we have

$$(1-b)^{-1}=1+b+\cdots b^{k-1}+b^k(1-a^kb^k)^{-1},$$

$$(1-b^ka^k)^{-1}=1+b^k(1-a)^{-1}a^k.$$

In light of Jacobson's lemma, we get

$$(1-a^kb^k)^{-1}=1+a^k(1-b^ka^k)^{-1}b^k.$$

Therefore we have

$$(1-b)^{-1}=\sum_{i=0}^{k-1}b^i+b^k[1+a^k(1+b^k(1-a)^{-1}a^k)b^k],$$

$$=\sum_{i=0}^{k+1}b^i+b^{k+1}(1-a)^{-1}a^kb^k.$$

as asserted. □

For a Banach algebra $\mathcal{A}$, it is well known that

$$a\in\mathcal{A}^{qnil}\Leftrightarrow\lim_{n\to\infty}\parallel a^n\parallel^{\frac{1}{n}}=0\Leftrightarrow\lambda-a\in\mathcal{A}^{-1}\text{ for any scarlar }\lambda\neq 0.$$

Many papers discussed Jacobson's lemma for g-Drazin inverse in the setting of matrices, operators and elements of Banach algebras.

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a,b\in\mathcal{A}$ satisfying $a^kb^ka^k=a^{k+1}$ for some $k\in\mathbb{N}$. If $1-a$ has g-Drazin inverse, then $1-b^ka^k$ has g-Drazin inverse. In this case,*

$$\begin{array}{rl} & (1-b^ka^k)^d\\ = & 1+b^k\Big[(1-a)^d-(1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big]a^k. \end{array}$$

Proof. $\Longrightarrow$ Let $y = 1 + b^k\Big[(1-a)^d - (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big]a^k$. We shall prove that $(1-b^ka^k)^d = y$.

Step 1. $y(1-b^ka^k)y = y$. We see that

$$\begin{aligned}
& y(1-b^ka^k) \\
&= 1 - b^ka^k + b^k\Big[(1-a)^d - (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big] \\
& (a^k - a^kb^ka^k) \\
&= 1 - b^ka^k + b^k\Big[(1-a)^d - (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big] \\
& (1-a)a^k \\
&= 1 - b^k\Big[(1-a)^\pi + (1-a)^\pi(1-a)(1-(1-a)^\pi(1-a))^{-1}\Big]a^k \\
&= 1 - b^k(1-(1-a)^\pi(1-a))^{-1}\Big[(1-a)^\pi(1-(1-a)^\pi(1-a)) \\
& +(1-a)^\pi(1-a)\Big]a^k \\
&= 1 - b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k.
\end{aligned}$$

Then we have

$$1 - y(1-b^ka^k) = b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k.$$

Therefore we check that

$$\begin{aligned}
& y - y(1-b^ka^k)y \\
&= \Big[1 - y(1-b^ka^k)\Big]y \\
&= b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k + b^k(1-(1-a)^\pi(1-a))^{-1} \\
& (1-a)^\pi a^kb^k\Big[(1-a)^d - (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big]a^k \\
&= b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k + b^k(1-(1-a)^\pi(1-a))^{-1} \\
& (1-a)^\pi)(a^kb^ka^k)\Big[(1-a)^d - (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big] \\
&= b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k - b^ka(1-(1-a)^\pi(1-a))^{-1} \\
& (1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}a^k \\
&= b^k(1-a)^\pi\Big[(1-(1-a)^\pi(1-a)) - a\Big](1-(1-a)^\pi(1-a))^{-2}a^k \\
&= b^k(1-a)^\pi\Big[(1-a) - (1-a)^\pi(1-a)\Big](1-(1-a)^\pi(1-a))^{-2}a^k \\
&= 0;
\end{aligned}$$

hence, $y = y(1-b^ka^k)y$, as required.

Step 2. $(1-b^ka^k) - (1-b^ka^k)y(1-b^ka^k) \in \mathcal{A}^{qnil}$. We verify that

$$\begin{aligned}
& (1-b^ka^k) - (1-b^ka^k)y(1-b^ka^k) \\
&= (1-b^ka^k)[1 - y(1-b^ka^k)] \\
&= (1-b^ka^k)b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k \\
&= (1-b^ka^k)b^ka^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi \\
&= b^ka^k(1-a)(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi \\
&= b^ka^k(1-a)^\pi(1-a)(1-(1-a)^\pi(1-a))^{-1}
\end{aligned}$$

By induction, we have

$$\begin{aligned}
& \Big[(1-b^ka^k) - (1-b^ka^k)y(1-b^ka^k)\Big]^n \\
= \ & b^ka^{k+n}\Big[(1-a)^\pi(1-a)\Big]^n[1-(1-a)^\pi(1-a)]^{-n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \| \Big[(1-b^ka^k) - (1-b^ka^k)y(1-b^ka^k)\Big]^n \|^{\frac{1}{n}} \le \| b^k \|^{\frac{1}{n}} \| a \|^{1+\frac{k}{n}} \\
& \| \Big[(1-a) - (1-a)^2(1-a)^d\Big]^n \|^{\frac{1}{n}} \| [1-(1-a)^\pi(1-a)]^{-1} \| .
\end{aligned}$$

Accordingly,

$$\lim_{n\to\infty} \| \Big[(1-b^ka^k) - (1-b^ka^k)y(1-b^ka^k)\Big]^n \|^{\frac{1}{n}} = 0,$$

and so $(1-b^ka^k)-(1-b^ka^k)y(1-b^ka^k)\in \mathcal{A}^{qnil}$.

Step 3. $y\in comm(1-b^ka^k)$. We directly compute that

$$\begin{aligned}
&(1-b^ka^k)y\\
&=(1-b^ka^k)\Big[1+b^k((1-a)^d-(1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big)a^k\Big]\\
&=1-b^ka^k+b^k(1-a)a^k\Big[(1-a)^d-(1-a)^\pi(1-(1-a)^\pi(1-a))^{-1}\Big]\\
&=1-b^k\Big[1-(1-a)((1-a)^d-(1-a)^\pi(1-(1-a)^\pi(1-a))^{-1})a^k\Big]\\
&=1-b^k\Big[(1-a)^\pi+(1-a)^\pi(1-a)(1-(1-a)^\pi(1-a))^{-1})a^k\Big]\\
&=1-b^k(1-(1-a)^\pi(1-a))^{-1}(1-a)^\pi a^k.
\end{aligned}$$

Therefore $(1-b^ka^k)y=y(1-b^ka^k)$, as desired. □

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra, let $\lambda\in\mathbb{C}$ and let $a,b\in\mathcal{A}$ satisfying $a^kb^ka^k=a^{k+1}$ for some $k\in\mathbb{N}$. If $\lambda-a$ has g-Drazin inverse, then $\lambda-b^ka^k$ has g-Drazin inverse. In this case,*

$$\begin{aligned}
&(\lambda-b^ka^k)^d\\
&=\begin{cases}-b^k(a^d)^2a^k & \lambda=0\\ \frac{1}{\lambda}+\frac{1}{\lambda}b^k[(\lambda-a)^d-(\lambda-a)^\pi(\lambda-(\lambda-a)^\pi(\lambda-a)^{-1}]a^k & \lambda\neq 0\end{cases}
\end{aligned}$$

Proof. Case 1. $\lambda=0$. The result follows by Theorem 2.2,

Case 2. $\lambda\neq 0$. Set $c=\frac{a}{\lambda}$ and $d=\lambda^{1-\frac{1}{k}}b$. Then $c^kd^kc^k=c^{k+1}$. By virtue of Theorem 3.3, $1-d^kc^k\in\mathcal{A}^d$ if and only if $1-c\in\mathcal{A}^d$. We see that

$$\begin{aligned}
1-d^kc^k &= \tfrac{1}{\lambda}(\lambda-b^ka^k),\\
1-c &= \tfrac{1}{\lambda}(\lambda-a).
\end{aligned}$$

Therefore $\lambda-a\in\mathcal{A}^d$ if and only if $\lambda-b^ka^k\in\mathcal{A}^d$. Moreover, we have

$$\begin{aligned}
&(\lambda-b^ka^k)^d=\tfrac{1}{\lambda}(1-d^kc^k)^d\\
&=\tfrac{1}{\lambda}+\tfrac{1}{\lambda}d^k[(1-c)^d-(1-c)^\pi(1-(1-c)^\pi(1-c)^{-1})]c^k\\
&=\tfrac{1}{\lambda}+\tfrac{1}{\lambda}\lambda^{k-1}b^k[\lambda(\lambda-a)^d-(\lambda-a)^\pi(1-\tfrac{1}{\lambda}(\lambda-a)^\pi(\lambda-a)^{-1})\tfrac{1}{\lambda^k}a^k]\\
&=\tfrac{1}{\lambda}+\tfrac{1}{\lambda}b^k[(\lambda-a)^d-(\lambda-a)^\pi(\lambda-(\lambda-a)^\pi(\lambda-a))^{-1}a^k],
\end{aligned}$$

as asserted. □

4. Common spectral property

Let X be a Banach space, and let $\mathcal{L}(X)$ denote the set of all bounded linear operators from Banach space to itself, and let $A\in\mathcal{L}(X)$. The g-Drazin spectrum $\sigma_d(A)$ are defined by

$$\sigma_d(A)=\{\lambda\in\mathbb{C}\mid \lambda I-A\notin\mathcal{L}(X)^d\}.$$

The aim of this section is to concern on common spectrum property of $\mathcal{L}(X)$. We now ready to prove the following.

Theorem 4.1. *Let $A,B\in\mathcal{L}(X)$ such that $A^kB^kA^k=A^{k+1}$ for some $k\in\mathbb{N}$., then*

$$\sigma_d(A)=\sigma_d(B^kA^k).$$

Proof. Case 1. In view of Theorem 2.2, $B^kA^k\notin\mathcal{L}(X)^d$ if and only if $A\notin\mathcal{L}(X)^d$. Hence, $0\in\sigma_d(B^kA^k)$ if and only if $0\in\sigma_d(A)$.

Case 2. Let $\lambda\neq 0$. If $\lambda\in\sigma_d(B^kA^k)$, then $\lambda\in acc\sigma(B^kA^k)$. Hence,

$$\lambda=\lim_{n\to\infty}\{\lambda_n\mid \lambda_nI-B^kA^k\notin\mathcal{L}(X)^{-1}\}.$$

If $\lambda \in \sigma_d(A)$, anagously, we have

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - A \notin \mathcal{L}(X)^{-1}\}.$$

For any $\lambda_n \neq 0$, we have

$$(\lambda_n^{-1}A)^k(\lambda_n^{\frac{k-1}{k}}B)^k(\lambda_n^{-1}A)^k = (\lambda_n^{-1}A)^{k+1}.$$

In light of Theorem 3.1,

$$I - (\lambda_n^{\frac{k-1}{k}}B)^k(\lambda_n^{-1}A)^k \in \mathcal{L}(X)^{-1}$$

if and only if

$$I - \lambda_n^{-1}A \in \mathcal{L}(X)^{-1}.$$

That is, $\lambda_n I - B^kA^k \in \mathcal{L}(X)^{-1}$ if and only if $\lambda_n I - A \in \mathcal{L}(X)^{-1}$. Accordingly, $\sigma_d(B^kA^k) = \sigma_d(A)$. □

Corollary 4.2. *Let $A, B \in \mathcal{L}(X)$ such that $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$ for some $k \in \mathbb{N}$, then*

$$\sigma_d(A) = \sigma_d(B).$$

Proof. By Theorem 4.1, $\sigma_d(A) = \sigma_d(B^kA^k)$ and $\sigma_d(B) = \sigma_d(A^kB^k)$. It is clear that $\sigma_d(A^kB^k) = \sigma_d(B^kA^k)$ which implies that $\sigma_d(A) = \sigma_d(B)$. □

Example 4.3.

Let $R = M_2(\mathbb{C})$. Choose

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $ABA = A^2, BAB = B^2$. In view of Theorem 4.1, $\sigma_d(A) = \sigma_d(B)$. In this case, $A^2 \neq B^2$.

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