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Global Optimal Solutions of a System of Differential Equations via Measure of Noncompactness

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Abstract. We establish the existence of best proximity points (pairs) for a new class of cyclic (noncyclic) condensing operators by using the concept of measure of noncompactness. Our conclusions extend and improve the main results of [Indagationes Math. 29 (2018), 895-906]. By applying our results, we prove a coupled best proximity point theorem and investigate the existence of a solution for a system of differential equations.

1. Introduction

We study some best proximity point (pair) problems, strongly related to the existence/non-existence of fixed points for mappings. The main skill is the use of measures of noncompactness to get the key inequalities satisfied by the mappings. So, we depict a brief history to correctly understand our topic and its motivation. Indeed, we point out some cornerstones. From Schauder [14], we recall the classical fixed point problem in a Banach space X with some regularity assumptions.

Theorem 1.1. *Let $A \subseteq X$ be nonempty, compact and convex. If $T : A \to A$ is a continuous operator, then it admits at least a fixed point.*

Precisely, the Schauder's theorem can be considered as generalization of Brouwer fixed point theorem from $\mathbb{R}^n$ to infinite dimensional Banach spaces, via an approximation process (we refer to [2] for more information).

Let $K \subseteq H$ where H is a normed linear space. We recall that a mapping $T : K \to Y$ is said to be a *compact operator* if T is continuous and maps bounded sets into relatively compact sets (Y is also a normed linear space). Here, the interest for compact operators is motivated by the following result.

Theorem 1.2. *Let $K \subseteq H$ be nonempty, bounded, closed and convex. If $T : K \to K$ is a compact operator, then it admits a fixed point.*

To get significant extensions of the theory of compact operators, many researchers considered the concept of *measure of noncompactness*, which was firstly established by Kuratowski and further generalized by Hausdorff.

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Definition 1.3. *Let $\Sigma \subseteq X$ be the family of all nonempty and bounded subsets of a metric space (X, d). By $\alpha : \Sigma \to [0, \infty)$ given as*

$$\alpha(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \leq \varepsilon\},$$

for all $B \in \Sigma$, we mean the Kuratowski measure of noncompactness.
Also, by $\chi : \Sigma \to [0, \infty)$ given as

$$\chi(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\},$$

for all $B \in \Sigma$, we mean the Hausdorff measure of noncompactness.

For an exhaustive discussion on the Kuratowski-Hausdorff measures of noncompactness we refer to Akhmerov-Kamenskii-Patapov-Rodkina-Sadovskii [3]. Here, we recall the following definitions.

Definition 1.4. *Let $\Sigma \subseteq X$ be the family of all bounded subsets of a metric space (X, d). By $\mu : \Sigma \to [0, \infty)$ we mean a measure of noncompactness* (MNC) *if the following conditions hold:*
(*i*) $\mu(A) = 0$ *iff A is relatively compact,*
(*ii*) $\mu(A) = \mu(\overline{A})$ *for all $A \in \Sigma$,*
(*iii*) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ *for all $A, B \in \Sigma$.*

If μ is an MNC on Σ, then the following properties will be concluded immediately (see [2] for more information).
(1) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$,
(2) $\mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma$,
(3) If A is a finite set, then $\mu(A) = 0$,
(4) If $\{A_n\}$ is a decreasing sequence of nonempty, bounded and closed subsets of X such that $\lim_{n\to\infty} \mu(A_n) = 0$, then $A_\infty := \cap_{n\geq 1} A_n$ is nonempty and compact.
Also, if X is a Banach space, then
(5) $\mu(\overline{\text{con}}(A)) = \mu(A)$ for all $A \in \Sigma$,
(6) $\mu(tA) = |t|\mu(A)$, for any number t and $A \in \Sigma$,
(7) $\mu(A + B) \leq \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.

Next definition provides the notion of condensing operator.

Definition 1.5. *Let $A \subseteq X$ and μ be an* MNC *on X. If $T : A \to X$ is continuous and there exists $r \in [0, 1)$ such that $\mu(TK) \leq r\mu(K)$, for every bounded subset K of A, then T is a condensing operator.*

Contractions and compact operators are examples of condensing operators. On the basis of Schauder's theorem, Darbo (see [5]) proposed the following result

Theorem 1.6. *Let $A \subseteq X$ be nonempty, bounded, closed and convex. If $T : A \to A$ is a condensing operator, then T admits a fixed point.*

Various generalizations and extensions of such a theorem appeared in the literature, where the authors considered different classes of control functions and studied various applications of functional analysis, for example, in the theory of differential and integral equations. For instance, we recall the works of Aghajani-Sabzali [1] and Samadi-Ghaemi [13].

This paper is organized as follows: in Section 2, we recall some basic definitions and notions related to best proximity theory. In Section 3, we present some existence results of best proximity points (pairs) for new classes of cyclic (noncyclic) condensing operators. Also, we extend and improve the main conclusions of Gabeleh-Markin [11]. As an application of our results, we establish a new coupled best proximity point theorem in Section 4 and finally in Section 5, we study the existence of a solution for a system of differential equations under appropriate conditions.

2. Mathematical background

Here, for reader convenience, we collect some basic definitions and notations that are needed in the sequel of the paper.

Definition 2.1. *Let X be a Banach space. We say that*
(i) X is uniformly convex if there exists a strictly increasing function $\delta : (0,2] \to [0,1]$ such that the following implication holds for all $x, y, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\begin{cases} \|x-p\| \le R, \\ \|y-p\| \le R, \\ \|x-y\| \ge r \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| \le (1 - \delta(\frac{r}{R}))R;$$

(ii) X is strictly convex if the following implication holds for all $x, y, p \in X$ and $R > 0$:

$$\begin{cases} \|x-p\| \le R, \\ \|y-p\| \le R, \\ x \ne y \end{cases} \Rightarrow \|\frac{x+y}{2} - p\| < R.$$

Let A and B be two nonempty subsets of a normed linear space Y. The pair (A, B) satisfies a property if both A and B satisfy that property. So, we say that (A, B) is closed if and only if both A and B are closed; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,\ B \subseteq D$. From now on, $\mathcal{B}(x; r)$ will mean the closed ball in the Banach space X centered at $x \in X$ with radius $r > 0$. We shall also adopt the following notations

$$\begin{aligned} \delta_x(A) &= \sup\{d(x, y) \colon y \in A\} \text{ for all } x \in X, \\ \delta(A, B) &= \sup\{d(x, y) \colon x \in A,\ y \in B\}, \\ \mathrm{diam}(A) &= \delta(A, A). \end{aligned}$$

The closed and convex hull of a set A will be denoted by $\overline{\mathrm{con}}(A)$, which is the smallest closed and convex set that contains A. We mention that if A is a nonempty and compact subset of a Banach space X, then $\overline{\mathrm{con}}(A)$ is compact (see Dunford-Schwartz [6]).

In addition, we set

$$\mathrm{dist}(A, B) := \inf\{\|x - y\| : (x, y) \in A \times B\},$$

$$A_0 := \{x \in A : \exists\, y' \in B\ :\ \|x - y'\| = \mathrm{dist}(A, B)\ (y' \text{ is called a proximal point of } x)\},$$

$$B_0 := \{y \in B : \exists\, x' \in A\ :\ \|x' - y\| = \mathrm{dist}(A, B)\ (x' \text{ is called a proximal point of } y)\}.$$

Definition 2.2. *A nonempty pair (A, B) in a normed linear space Y is said to be proximinal if $A = A_0$ and $B = B_0$.*

It is remarkable to note that if (A, B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X, then (A_0, B_0) is also nonempty, closed and convex.

A mapping $T : A \cup B \to A \cup B$ is said to be
(i) relatively nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for any $(x, y) \in A \times B$,
(ii) cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$,
(iii) noncyclic if $T(A) \subseteq A$ and $T(B) \subseteq B$,
(iv) compact if the pair $(\overline{T(A)}, \overline{T(B)})$ is compact (see [11]).

Definition 2.3. *Let (A, B) be a nonempty pair in a Banach space X and $T : A \cup B \to A \cup B$ be a mapping. If T is cyclic, then a point $p \in A \cup B$ is said to be a best proximity point for T provided that*

$$\|p - Tp\| = \mathrm{dist}(A, B).$$

Also, if T is noncyclic, then the pair $(p, q) \in A \times B$ is called a best proximity pair for T provided that

$$p = Tp, \quad q = Tq, \quad \|p - q\| = \mathrm{dist}(A, B).$$

Existence of best proximity points (pairs) for cyclic (noncyclic) relatively nonexpansive mappings was first studied by Eldred-Kirk-Veeramani [7], under a geometric concept of *proximal normal structure*.

Definition 2.4. ([7]) *A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure (PNS) if for any bounded, closed, convex and proximinal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which* $\mathrm{dist}(H_1, H_2) = \mathrm{dist}(K_1, K_2)$ *and* $\delta(H_1, H_2) > \mathrm{dist}(H_1, H_2)$*, there exists $(x_1, x_2) \in H_1 \times H_2$ such that*

$$\max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} < \delta(H_1, H_2).$$

Notice that the pair (K, K) has PNS if and only if K has normal structure in the sense of Brodski-Milman [4].

It was announced in [7] that every nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the PNS. Also, every nonempty, compact and convex pair in a Banach space X has the PNS (Theorem 3.5 of Gabeleh [9]). We also mention that a characterization of proximal normal structure by *proximal diametral sequences* is given in [9].

By [7], we recall the following results.

Theorem 2.5. (Theorem 2.1 of [7]) *Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X and suppose (A, B) has PNS. If $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping, then it admits a best proximity point.*

Theorem 2.6. (Theorem 2.2 of [7]) *Let (A, B) be a nonempty, weakly compact and convex pair in a strictly convex Banach space X and suppose (A, B) has PNS. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping, then it admits a best proximity pair.*

The next corollaries play a crucial role to establish the results of this paper.

Corollary 2.7. *Let (A, B) be a nonempty, compact and convex pair in a Banach space X. If $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping, then it admits a best proximity point.*

Corollary 2.8. *Let (A, B) be a nonempty, compact and convex pair in a strictly convex Banach space X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping, then it admits a best proximity pair.*

Very recently, Corollaries 2.7, 2.8 were extended as follows.

Theorem 2.9. (Theorem 3.2 of [11]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If T is compact, then it admits a best proximity point.*

Theorem 2.10. (Theorem 4.1 of [11]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X. Assume that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. If T is compact, then it admits a best proximity pair.*

The cyclic (noncyclic) version of condensing mappings was introduced in [11] in order to study the existence of best proximity points (pairs) and to generalize Theorems 2.9 and 2.10 above.

Definition 2.11. *Let (A, B) be a nonempty and convex pair in a Banach space X and μ an* MNC *on X. A cyclic (noncyclic) mapping $T : A \cup B \to A \cup B$ is said to be a condensing operator if there exists $r \in (0, 1)$ such that for any nonempty, bounded, closed, convex, proximinal and T-invariant pair $(H_1, H_2) \subseteq (A, B)$ such that* $\mathrm{dist}(H_1, H_2) = \mathrm{dist}(A, B)$ *we have*

$$\mu(T(H_1) \cup T(H_2)) \le r\mu(H_1 \cup H_2).$$

Next results are real extensions of Theorem 1.6 due to Darbo.

Theorem 2.12. (Theorem 3.4 of [11]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X and μ an* MNC *on X. If $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping which is condensing in the sense of Definition 2.11, then it admits a best proximity point.*

Theorem 2.13. (Theorem 3.4 of [11]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X and μ an* MNC *on X. If $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping which is condensing in the sense of Definition 2.11, then it admits a best proximity pair.*

We also refer to Gabeleh-Vetro [12] for the generalizations of Theorems 2.12 and 2.13, by considering a class of cyclic (noncyclic) Meir-Keeler condensing operators.

3. Best proximity point (pair) results

Motivated by the fixed point results of Samadi-Ghaemi [13], we denote by Ψ the set of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = 0$ if and only if $t = 0$, and denote by Γ the family of all functions $\gamma : [0, \infty) \to [0, 1)$ for which $\limsup_{t\to r^+} \gamma(t) < 1$ for all $r \geq 0$.

Definition 3.1. *Let (A, B) be a nonempty and convex pair in a Banach space X and μ an* MNC *on X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic (noncyclic) $\psi - \gamma$ condensing operator if for any nonempty, bounded, closed, convex, proximinal and T-invariant pair $(K_1, K_2) \subseteq (A, B)$ with $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$ we have*

$$\psi\Big(\mu((T(K_1) \cup T(K_2)))\Big) \leq \gamma\Big(\psi(\mu(K_1 \cup K_2))\Big)\psi(\mu(K_1 \cup K_2)),$$

where $\psi \in \Psi$ and $\gamma \in \Gamma$.

Obviously, if ψ is the identity function on $[0, \infty)$ and $\gamma = r$ with $r \in [0, 1)$, then we retrieve the class of condensing mappings in Definition 2.11.

Let (A, B) be a nonempty pair in a normed linear space X and $T : A \cup B \to A \cup B$ be a cyclic (noncyclic) mapping. By $\mathcal{M}_T(A, B)$ we denote the set of all nonempty, bounded, closed, convex, proximinal and T-invariant pair $(U, V) \subseteq (A, B)$ with $\operatorname{dist}(U, V) = \operatorname{dist}(A, B)$. Notice that $\mathcal{M}_T(A, B)$ may be empty, but in particular if (A, B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X and T is cyclic (noncyclic) relatively nonexpansive, then $(A_0, B_0) \in \mathcal{M}_T(A, B)$ (see Gabeleh [8] for more details).

Here, we state the first result of this section.

Theorem 3.2. *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X and μ an* MNC *on X. Let $T : A \cup B \to A \cup B$ be a cyclic $\psi - \gamma$ condensing operator for some $\psi \in \Psi$ and $\gamma \in \Gamma$. If T is a relatively nonexpansive mapping, then it admits a best proximity point.*

Proof. It is worth noticing that $(A_0, B_0) \in \mathcal{M}_T(A, B) \neq \emptyset$. Set

$$A_n := \overline{\text{con}}(T(A_{n-1})), \quad B_n := \overline{\text{con}}(T(B_{n-1})).$$

By the fact that $A_1 = \overline{\text{con}}(T(A_0)) \subseteq B_0$, we get

$$T(A_1) \subseteq T(B_0) \subseteq \overline{\text{con}}(T(B_0)) = B_1.$$

Equivalently, $T(B_1) \subseteq A_1$, which implies that T is cyclic on $A_1 \cup B_1$. Continuing this process and using inductive reasoning, we deduce that T is cyclic on $A_n \cup B_n$ for any $n \in \mathbb{N}$. Besides, we have

$$A_{n+1} = \overline{\text{con}}(T(A_n)) \subseteq B_n = \overline{\text{con}}(T(B_{n-1})) \subseteq A_{n-1}, \quad \forall n \in \mathbb{N}, \tag{1}$$

$$B_{n+1} = \overline{\text{con}}(T(B_n)) \subseteq A_n = \overline{\text{con}}(T(A_{n-1})) \subseteq B_{n-1}, \quad \forall n \in \mathbb{N}, \tag{2}$$

which ensure that the sequence $\{(A_{2n}, B_{2n})\}_{n\in\mathbb{N}\cup\{0\}}$ is decreasing. On the other hand, for $(x_0, y_0) \in A_0 \times B_0$ with $\|x_0 - y_0\| = \text{dist}(A, B)$ we have

$$\text{dist}(A_{2n}, B_{2n}) \le \|T^{2n}x_0 - T^{2n}y_0\| \le \|x_0 - y_0\| = \text{dist}(A, B), \quad \forall n \in \mathbb{N},$$

and so, $\text{dist}(A_{2n}, B_{2n}) = \text{dist}(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, let $u \in \text{con}(T(A_0))$. Then $u = \sum_{j=1}^{m} \lambda_j T(u_j)$ for some $m \in \mathbb{N}$, where $u_j \in A_0$ for all $1 \le j \le m$. Because of the fact that (A_0, B_0) is proximinal, for all $1 \le j \le m$ there exists an element $v_j \in B_k$ for which $\|u_j - v_j\| = \text{dist}(A, B_0)(= \text{dist}(A, B))$. Suppose $v := \sum_{j=1}^{m} \lambda_j T(v_j)$. Then $v \in \text{con}(T(B_0))$. Since T is a relatively nonexpansive mapping,

$$\begin{aligned}\|u - v\| &= \|\sum_{j=1}^{m} \lambda_j T(u_j) - \sum_{j=1}^{m} \lambda_j T(v_j)\| \le \sum_{j=1}^{m} \lambda_j \|T(u_j) - T(v_j)\| \\ &\le \sum_{j=1}^{m} \lambda_j \|u_j - v_j\| = \text{dist}(A, B).\end{aligned}$$

Therefore, each point of $\text{con}(T(A_0))$ has a proximal point in $\text{con}(T(B_0))$. By a similar argument, we can see that each point of $\text{con}(T(B_0))$ has a proximal point in $\text{con}(T(A_0))$ too, that is, the pair $(\text{con}(T(A_0)), \text{con}(T(B_0)))$ is proximinal. Moreover, if $q \in \overline{\text{con}}(T(A_0))$, then there exists a sequence $\{y_n\}$ in $\text{con}(T(A_0))$ such that $y_n \to q$. Since $(\text{con}(T(A_0)), \text{con}(T(B_0)))$ is proximinal, for any $n \in \mathbb{N}$ there exists a point $x_n \in \text{con}(T(B_k))$ such that

$$\|x_n - y_n\| = \text{dist}(\text{con}(T(A_0)), \text{con}(T(B_0))) = \text{dist}(A, B).$$

Weakly compactness of $\overline{\text{con}}(T(B_0))$ ensures the existence of a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ for which $x_{n_j} \rightharpoonup p \in \overline{\text{con}}(T(B_0))$, where " $\rightharpoonup$ " stands for weakly convergence in the Banach space X. Weakly lower semi-continuity of the norm implies that

$$\|p - q\| \le \liminf_{j\to\infty} \|x_{n_j} - y_{n_j}\| = \text{dist}(A, B).$$

Thereby, (A_1, B_1) is proximinal. Again by using mathematical induction we conclude that each pair $(A_n, B_n)\}_{n\in\mathbb{N}\cup\{0\}}$ is proximinal. Therefore, $\{(A_{2n}, B_{2n})\}_{n\in\mathbb{N}\cup\{0\}}$ is a descending sequence in $\mathcal{M}_T(A, B)$. Notice that if for some $n \in \mathbb{N}$ we have $\mu(A_{2n} \cup B_{2n}) = 0$, then the pair (A_{2n}, B_{2n}) is compact and convex and T is cyclic on $A_{2n} \cup B_{2n}$. Hence, the result follows from Corollary 2.7. So, we may assume that $\mu(A_{2n} \cup B_{2n}) > 0$ for all $n \in \mathbb{N}$. By the fact that T is a cyclic $\psi - \gamma$ condensing operator for some $\psi \in \Psi$ and $\gamma \in \Gamma$ and using (1), (2) we obtain

$$\begin{aligned}\psi\Big(\mu(A_{2n+2} \cup B_{2n+2})\Big) &= \psi\Big(\max\{\mu(A_{2n+2}), \mu(B_{2n+2})\}\Big) \\ &\le \psi\Big(\max\{\mu(B_{2n+1}), \mu(A_{2n+1})\}\Big) \\ &= \psi\Big(\max\{\mu(\overline{\text{con}}(T(B_{2n}))), \mu(\overline{\text{con}}(T(A_{2n})))\}\Big) \\ &= \psi\Big(\max\{\mu((T(B_{2n}))), \mu((T(A_{2n})))\}\Big) \\ &= \psi\Big(\max\{\mu(\overline{\text{con}}(T(B_{2n}))), \mu(\overline{\text{con}}(T(A_{2n})))\}\Big) \\ &= \psi\Big(\max\{\mu((T(B_{2n}))), \mu((T(A_{2n})))\}\Big) \\ &= \psi\Big(\mu(T(A_{2n}) \cup T(B_{2n}))\Big) \\ &\le \gamma\Big(\psi(\mu(A_{2n} \cup B_{2n}))\Big)\psi(\mu(A_{2n} \cup B_{2n})) \qquad (3) \\ &< \psi(\mu(A_{2n} \cup B_{2n})).\end{aligned}$$

This ensures that the sequence $\{\psi(\mu(A_{2n} \cup B_{2n}))\}$ is strict decreasing and so we have

$$r := \sup_{n\in\mathbb{N}} \gamma\Big(\psi(\mu(A_{2n} \cup B_{2n}))\Big) < 1.$$

Also, from (3) we have

$$\begin{aligned}\psi\Big(\mu(A_{2n+2}\cup B_{2n+2})\Big) &\leq \gamma\Big(\psi(\mu(A_{2n}\cup B_{2n}))\Big)\psi(\mu(A_{2n}\cup B_{2n}))\\ &\leq r\psi(\mu(A_{2n}\cup B_{2n}))\\ &\leq r^2\psi(\mu(A_{2n-2}\cup B_{2n-2}))\\ &\leq \cdots \leq r^{2n}\psi(\mu(A_0\cup B_0)).\end{aligned}$$

Hence, $\psi(\mu(A_{2n}\cup B_{2n}))\to 0$ as $n\to+\infty$. Moreover, as $\{\mu(A_{2n}\cup B_{2n})\}_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence of nonnegative real numbers, we deduce that $\mu(A_{2n}\cup B_{2n})\to t$ for some $t\geq 0$ as $n\to+\infty$. Since ψ is an increasing function, $\psi(t)\leq\psi(\mu(A_{2n}\cup B_{2n}))$ for all $n\in\mathbb{N}$, which implies that $\psi(t)=0$ and by the property of ψ we must have $t=0$, that is, $\mu(A_{2n}\cup B_{2n})\to 0$ as $n\to+\infty$. Now if we set

$$A_\infty := \bigcap_{n=0}^{\infty} A_{2n},\quad B_\infty := \bigcap_{n=0}^{\infty} B_{2n},$$

then (A_∞, B_∞) is nonempty, compact, convex and T-invariant with $\operatorname{dist}(A_\infty, B_\infty)=\operatorname{dist}(A,B)$. Thus the result follows from Corollary 2.7.

□

Remark 3.3. *It is remarkable to note that the reflexivity condition in Theorem 3.2 is essential in order to establish nonemptiness of the proximal pair* (A_0, B_0).

The following best proximity pair theorem is an extension of Theorem 2.13.

Theorem 3.4. *Let* (A,B) *be a nonempty, bounded, closed and convex pair in a reflexive Banach space* X *and* μ *an* MNC *on* X. *Let* $T: A\cup B\to A\cup B$ *be a noncyclic* $\psi-\gamma$ *condensing operator for some* $\psi\in\Psi$ *and* $\gamma\in\Gamma$. *If* T *is a relatively nonexpansive mapping, then it admits a best proximity pair.*

Proof. As in the proof of Theorem 3.2, let $A_n=\overline{\operatorname{con}}(T(A_{n-1}))$ and $B_n=\overline{\operatorname{con}}(T(B_{n-1}))$ for all $n\in\mathbb{N}$. Since T is noncyclic, $A_1=\overline{\operatorname{con}}(T(A_0))\subseteq A_0$, and so

$$T(A_1)\subseteq T(A_0)\subseteq\overline{\operatorname{con}}(T(A_0))=A_1.$$

Similarly, $T(B_1)\subseteq B_1$, that is, T is noncyclic on $A_1\cup B_1$. Continuing this process we obtain that T is noncyclic on $A_n\cup B_n$ for all $n\in\mathbb{N}$. Also, for all $n\in\mathbb{N}$ we have

$$A_{n+1}=\overline{\operatorname{con}}(T(A_n))\subseteq A_n,\quad B_{n+1}=\overline{\operatorname{con}}(T(B_n))\subseteq B_n.$$

By a similar argument as in the proof of Theorem 3.2, we conclude that $\{(A_n,B_n)\}$ is a descending sequence of nonempty, weakly compact, convex, T-invariant and proximinal pairs in $\mathcal{M}_T(A,B)$ such that $\operatorname{dist}(A_n,B_n)=\operatorname{dist}(A,B)$ for all $n\in\mathbb{N}$. Now, if we define

$$(A_\infty,B_\infty)=\left(\bigcap_{n=0}^{\infty}A_n,\bigcap_{n=1}^{\infty}B_n\right),$$

then (A_∞,B_∞) is nonempty, compact, convex and T-invariant pair in a strictly convex Banach space X, and by Corollary 2.8, T has a best proximity pair.

□

4. Coupled best proximity points

Let (A, B) be a nonempty pair in a normed linear space X and $F : (A \times A) \cup (B \times B) \to A \cup B$ be a cyclic mapping, that is, $F(A \times A) \subseteq B$ and $F(B \times B) \subseteq A$. A point $(u, v) \in (A \times A) \cup (B \times B)$ is called a *coupled best proximity point* for the mapping F provided that

$$\|u - F(u, v)\| = \|v - F(v, u)\| = \mathrm{dist}(A, B).$$

This notion was first introduced in the Ph.D Thesis of the first author ([10]).

In this section we establish a coupled best proximity point which is based on Theorem 3.2 above. To this end, we need the following lemmas.

Lemma 4.1. ([3]) *Suppose that $\mu_1, \mu_2, \cdots, \mu_n$ are measures of noncompactness in the Banach spaces $X_1, X_2, \cdots, X_n$, respectively. Moreover, assume that the function $\Theta : [0, \infty)^n \to [0, \infty)$ is convex and $\Theta(x_1, x_2, \cdots, x_n) = 0$ if and only if $x_j = 0$ for all $j = 1, 2, \cdots, n$. Then*

$$\mu(E) = \Theta\Big(\mu_1(E_1), \mu_2(E_2), \cdots, \mu_n(E_n)\Big),$$

defines a measure of noncompactness in $X_1 \times X_2 \times \cdots \times X_n$, where E_j denotes the natural projection of E into E_j for $j = 1, 2, \cdots, n$.

Lemma 4.2. *Let (A, B) be a nonempty, bounded, closed and convex pair in a Banach space X. Consider the Banach space $X \times X$ with the norm*

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X.$$

Then the pair (A, B) is proximinal in X if and only if $(A \times A, B \times B)$ is proximinal in $X \times X$.

Proof. At first we note that $\mathrm{dist}(A \times A, B \times B) = \mathrm{dist}(A, B)$. Indeed, for any $(a, a') \in A \times A, (b, b') \in B \times B$ we have

$$\begin{aligned} \mathrm{dist}(A \times A, B \times B) &= \inf_{\big((a,a'),(b,b')\big) \in (A\times A)\times(B\times B)} \|(a, a') - (b, b')\| \\ &= \inf_{\big((a,a'),(b,b')\big) \in (A\times A)\times(B\times B)} \max\{\|a - b\|, \|a' - b'\|\} \\ &= \mathrm{dist}(A, B). \end{aligned}$$

If (A, B) is proximinal and $(a, a') \in A \times A$, then we can find $b, b' \in B$ such that $\|a - b\| = \|a' - b'\| = \mathrm{dist}(A, B)$. So, for $(b, b') \in B \times B$ we have $\|(a, a') - (b, b')\| = \mathrm{dist}(A, B)\Big(= \mathrm{dist}(A \times A, B \times B)\Big)$, that is, $(A \times A)_0 = A \times A$. By a similar argument we can see that $(B \times B)_0 = B \times B$ and so the pair $\Big((A \times A), (B \times B)\Big)$ is proximinal in $X \times X$. Besides, if $\Big((A \times A), (B \times B)\Big)$ is proximinal and $a \in A$, then $(a, a) \in A \times A$ and so there exists a point $(b, b') \in B \times B$ for which $\|(a, a) - (b, b')\| = \mathrm{dist}(A, B)$, and hence $\|a - b\| = \|a - b'\| = \mathrm{dist}(A, B)$, that is, $A_0 = A$. Equivalently, $B_0 = B$ which implies that (A, B) is proximinal and hence the conclusion of lemma holds true. □

In what follows we assume that Ψ' denotes a subclass of the set Ψ with the additional condition

$$\psi(t + s) \leq \psi(t) + \psi(s), \quad \forall s, t \in [0, \infty),$$

for any $\psi \in \Psi'$.

Theorem 4.3. *Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X and μ an* MNC *on X. Let $F : (A \times A) \cup (B \times B) \to A \cup B$ be a cyclic mapping such that for all nonempty, bounded, closed, convex, proximinal and T-invariant pairs $(K_1, K_2) \subseteq (A, B)$ and $(K_1', K_2') \subseteq (A, B)$ with* $\mathrm{dist}(K_1, K_2) = \mathrm{dist}(A, B) = \mathrm{dist}(K_1', K_2')$ *we have*

$$\psi\Big(\mu((F(K_1 \times K_1') \cup F(K_2 \times K_2')))\Big) \leq \frac{1}{2}\gamma\Big(\psi\Big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\Big)\Big) \\ \times \psi\Big(\max\{\mu(K_1 \cup K_1'), \mu(K_2 \cup K_2')\}\Big), \tag{4}$$

where $\psi \in \Psi'$ and $\gamma \in \Gamma$. If

$$\|F(x, x') - F(y, y')\| \leq \max\{\|x - y\|, \|x' - y'\|\}, \\ \forall (x, x') \in A \times A,\ \forall (y, y') \in B \times B, \tag{5}$$

then F admits a coupled best proximity point.

Proof. Set $\widetilde{\mu}(E) := \max\{\mu(E_1), \mu(E_2)\}$, where E_j denotes the natural projection of E into E_j for $j = 1, 2$. From Lemma 5.1 $\widetilde{\mu}$ is a MNC on $X \times X$. Define the mapping $T : (A \times A) \cup (B \times B) \to (A \times A) \cup (B \times B)$ by

$$T(u, v) = (F(u, v), F(v, u)), \quad \forall (u, v) \in (A \times A) \cup (B \times B).$$

Note that if $(u, v) \in A \times A$, then by the fact that F is cyclic, $(F(u, v), F(v, u)) \in B \times B$, that is, $T(A \times A) \subseteq B \times B$. Similarly, $T(B \times B) \subseteq A \times A$. Thus T is cyclic on $(A \times A) \cup (B \times B)$. For any $((a, a'), (b, b')) \in (A \times A) \times (B \times B)$ we have

$$\begin{aligned}
&\|T(a, a') - T(b, b')\| \\
&= \|\Big(F(a, a'), F(a', a)\Big) - \Big(F(b, b'), F(b', b)\Big)\| \\
&= \|\Big(F(a, a') - F(b, b'), F(a', a) - F(b', b)\Big)\| \\
&= \max\{\|F(a, a') - F(b, b')\|, \|F(a', a) - F(b', b)\|\} \\
&\leq \max\Big\{\max\{\|a - b\|, \|a' - b'\|\}, \max\{\|a' - b'\|, \|a - b\|\}\Big\} \quad \text{(by (5))} \\
&= \max\{\|a - b\|, \|a' - b'\|\} \\
&= \|(a, a') - (b, b')\|,
\end{aligned}$$

which implies that T is relatively nonexpansive. On the other hand (recall (4)), we have

$$\begin{aligned}
&\psi\Big(\widetilde{\mu}\Big[\Big(T(K_1 \times K_1')\Big) \cup \Big(T(K_2 \times K_2')\Big)\Big]\Big) \\
&= \psi\Big(\max\{\widetilde{\mu}\Big(T(K_1 \times K_1')\Big), \widetilde{\mu}\Big(T(K_2 \times K_2')\Big)\}\Big) \\
&= \psi\Big(\max\{\widetilde{\mu}\Big(F(K_1 \times K_1') \times F(K_1' \times K_1)\Big), \widetilde{\mu}\Big(F(K_2 \times K_2') \times F(K_2' \times K_2)\Big)\}\Big) \\
&= \psi\Big(\max\Big\{\max\{\mu\Big(F(K_1 \times K_1')\Big), \mu\Big(F(K_1' \times K_1)\Big)\}, \\
&\quad \max\{\mu\Big(F(K_2 \times K_2')\Big), \mu\Big(F(K_2' \times K_2)\Big)\}\Big\}\Big) \\
&= \psi\Big(\max\Big\{\max\{\mu\Big(F(K_1 \times K_1')\Big), \mu\Big(F(K_2 \times K_2')\Big)\}, \\
&\quad \max\{\mu\Big(F(K_1' \times K_1)\Big), \mu\Big(F(K_2' \times K_2)\Big)\}\Big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \psi\Big(\max\Big\{\mu\Big(F(K_1\times K_1')\cup F(K_2\times K_2')\Big), \mu\Big(F(K_1'\times K_1)\cup F(K_2'\times K_2)\Big)\Big\}\Big)\\
&\leq \psi\Big(\mu\Big(F(K_1\times K_1')\cup F(K_2\times K_2')\Big)\Big) + \psi\Big(\mu\Big(F(K_1'\times K_1)\cup F(K_2'\times K_2)\Big)\Big)\\
&\leq \frac{1}{2}\gamma\Big(\psi\Big(\max\{\max\{\mu(K_1),\mu(K_1')\},\max\{\mu(K_2),\mu(K_2')\}\}\Big)\Big)\\
&\quad\times \psi\Big(\max\{\{\max\{\mu(K_1),\mu(K_1')\},\max\{\mu(K_2),\mu(K_2')\}\}\Big)\\
&\quad+ \frac{1}{2}\gamma\Big(\psi\Big(\max\{\max\{\mu(K_1'),\mu(K_1)\},\max\{\mu(K_2'),\mu(K_2)\}\}\Big)\Big)\\
&\quad\times \psi\Big(\max\{\{\max\{\mu(K_1'),\mu(K_1)\},\max\{\mu(K_2'),\mu(K_2)\}\}\Big)\\
&= \gamma\Big(\psi\Big(\max\{\max\{\mu(K_1),\mu(K_1')\},\max\{\mu(K_2),\mu(K_2')\}\}\Big)\Big)\\
&\quad\times \psi\Big(\max\{\{\max\{\mu(K_1),\mu(K_1')\},\max\{\mu(K_2),\mu(K_2')\}\}\Big)\\
&= \gamma\Big(\psi\Big(\max\{\widetilde{\mu}(K_1\times K_1'),\widetilde{\mu}(K_2\times K_2')\}\Big)\Big)\psi\Big(\max\{\widetilde{\mu}(K_1\times K_1'),\widetilde{\mu}(K_2\times K_2')\}\Big)\\
&= \gamma\Big(\psi\Big(\widetilde{\mu}\Big[(K_1\times K_1')\cup(K_2\times K_2')\Big]\Big)\Big)\psi\Big(\widetilde{\mu}\Big[(K_1\times K_1')\cup(K_2\times K_2')\Big]\Big),
\end{aligned}
$$

which ensures that T is a cyclic $\psi-\gamma$ condensing operator. It now follows from Theorem 3.2 that T admits a best proximity point, that is, there exists a point $(p,q)\in(A\times A)\cup(B\times B)$ for which

$$
\begin{aligned}
\operatorname{dist}(A,B) &= \|(p,q)-T(p,q)\|\\
&= \|(p,q)-(F(p,q),F(q,p))\|\\
&= \max\{\|p-F(p,q)\|,\|q-F(q,p)\|\}.
\end{aligned}
$$

Therefore, (p,q) is a coupled best proximity point of F.
□

We conclude this section with the following result.

Corollary 4.4. *Let (A,B) be a nonempty, bounded, closed and convex pair in a reflexive Banach space X and μ an* MNC *on X. Let $F:(A\times A)\cup(B\times B)\to A\cup B$ be a cyclic mapping such that for all nonempty, bounded, closed, convex, proximinal and T-invariant pairs $(K_1,K_2)\subseteq(A,B)$ and $(K_1',K_2')\subseteq(A,B)$ with* $\operatorname{dist}(K_1,K_2)=\operatorname{dist}(A,B)=\operatorname{dist}(K_1',K_2')$ *we have*

$$\mu((F(K_1\times K_1')\cup F(K_2\times K_2'))\leq\frac{r}{2}\max\{\mu(K_1\cup K_1'),\mu(K_2\cup K_2')\}$$

for some $r\in(0,1)$. If

$$\|F(x,x')-F(y,y')\|\leq\max\{\|x-y\|,\|x'-y'\|\},\ \forall(x,x')\in A\times A,\ \forall(y,y')\in B\times B,$$

then F admits a coupled best proximity point.

5. Application to a system of differential equations

In this section, we prove a theorem establishing the existence of an optimal solution of certain systems of differential equations with local initial conditions. We recall the following well-known version of the *Mean-Value Theorem*.

Lemma 5.1. *Let J be a real interval, X a Banach space and $f:J\to X$ a differentiable mapping. Let $a,b\in J$ with $a<b$. Then*

$$f(b)-f(a)\in(b-a)\overline{\operatorname{con}}(\{f'(t):t\in[a,b]\}).$$

Definition 5.2. *Let $a,b \in (0,\infty)$, $I := [t_0 - a, t_0 + a]$ and $V_1 = \mathcal{B}(x_0;b)$, $V_2 = \mathcal{B}(x_1;b)$ be closed balls in a Banach space X, where t_0 is a real number and $x_0, x_1 \in X$. Let $f : I \times V_1 \to X$ and $g : I \times V_2 \to X$ be two continuous mappings. We study the system of differential equations:*

$$\begin{cases} x'(t) = f(t,x(t)); & x(t_0) = x_0, \\ y'(t) = g(t,y(t)); & y(t_0) = x_1, \end{cases} \tag{6}$$

defined on a closed real interval $J = [t_0 - h, t_0 + h]$ for some $h \in (0,\infty)$, $x_0 \neq x_1$. Denote by $C(J,X)$ the set of all continuous mappings from J into X with the supremum norm and set

$$C(J,V_1) := \{x \in C(J,X) : x(t_0) = x_0\},$$

$$C(J,V_2) := \{y \in C(J,X) : y(t_0) = x_1\}.$$

So, for all $(x,y) \in C(J,V_1) \times C(J,V_2)$ we get

$$\|x - y\|_\infty = \sup_{t\in J} \|x(t) - y(t)\| \geq \|x_0 - x_1\|,$$

and hence $\operatorname{dist}(C(J,V_1), C(J,V_2)) = \|x_0 - x_1\|$. *Let define the operator*

$$T : C(J,V_1) \cup C(J,V_2) \to C(J,X),$$

given by

$$Tx(t) = x_1 + \int_{t_0}^{t} g(s,x(s))ds, \quad x \in C(J,V_1),$$

$$Ty(t) = x_0 + \int_{t_0}^{t} f(s,y(s))ds, \quad y \in C(J,V_2).$$

Now, $z \in C(J,V_1) \cup C(J,V_2)$ is an optimal solution for (6) *whenever* $\|z - Tz\|_\infty = \operatorname{dist}(C(J,V_1), C(J,V_2))$.

We establish the following result.

Theorem 5.3. *Let α be an MNC on $C(J,X)$. Under the assumptions of Definition 5.2, we assume that*

$$\alpha(f(I \times W_2) \cup g(I \times W_1)) \leq b\,\varphi(\alpha(W_1 \cup W_2)),$$

$$\|f(t,x) - g(t,y)\| \leq \frac{1}{h}(\|x(t) - y(t)\| - \|x_1 - x_0\|),$$

for all $(x,y) \in C(J,V_1) \times C(J,V_2)$, some upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$, and for any $(W_1,W_2) \subseteq (V_1,V_2)$ and $h \leq \min\{a, \frac{b}{M_1}, \frac{b}{M_2}, \frac{1}{2b}\}$, where

$$M_1 = \sup\{\|f(t,x)\| : (t,x) \in I \times V_1\}, \quad M_2 = \sup\{\|g(t,y)\| : (t,y) \in I \times V_2\}.$$

Therefore, the system (6) *admits an optimum solution.*

Proof. Notice that $(C(J,V_1), C(J,V_2))$ is a bounded, closed and convex pair in $C(J,X)$ and that T is cyclic on $C(J,V_1) \cup C(J,V_2)$. We show that $T(C(J,V_1))$ is a bounded and equicontinuous subset of $C(J,V_2)$. Set $t,t' \in J$ and $x \in C(J,V_1)$ so that we have

$$\begin{aligned} \|Tx(t)\| &= \|x_1 + \int_{t_0}^{t} g(s,x(s))ds\| \leq \|x_1\| + \int_{t_0}^{t} \|g(s,x(s))\|ds \\ &\leq \|x_1\| + M_2 h \leq \|x_1\| + b, \end{aligned}$$

which leads to boundedness of $T(C(J, V_1))$. Also,

$$\begin{aligned}\|Tx(t) - Tx(t')\| &= \|\int_{t_0}^{t} g(s, x(s))ds - \int_{t_0}^{t'} g(s, x(s))ds\| \\ &\leq \int_{t}^{t'} \|g(s, x(s))\|ds \leq M_2|t - t'|,\end{aligned}$$

that is, $T(C(J, V_1))$ is equicontinuous. In a similar fashion, we deduce that $T(C(J, V_2))$ is bounded and equicontinuous, too. Now, by Arzela-Ascoli's theorem we conclude that the pair $(C(J, V_1), C(J, V_2))$ is relatively compact. We show that the operator T is cyclic $\psi - \gamma$ condensing, for some $\psi \in \Psi$ and $\gamma \in \Gamma$. Indeed, assume that $(K_1, K_2) \subseteq (C(J, V_1), C(J, V_2))$ is a nonempty, closed, convex and proximinal pair which is T-invariant and that

$$\mathrm{dist}(K_1, K_2) = \mathrm{dist}(C(J, V_1), C(J, V_2)) \, (= \|x_0 - x_1\|).$$

By Theorem 2.11 of [2] we get

$$\begin{aligned}&\alpha(T(K_1), T(K_2)) \\ &= \max\{\alpha(T(K_1)), \alpha(T(K_2))\} \\ &= \max\{\sup_{t\in J}\{\alpha(\{Tx(t) : x \in K_1\})\}, \sup_{t\in J}\{\alpha(\{Ty(t) : y \in K_2\})\}\} \\ &= \max\{\sup_{t\in J}\{\alpha(\{x_1 + \int_{t_0}^{t} g(s, x(s))ds : x \in K_1\})\}, \\ &\qquad \sup_{t\in J}\{\alpha(\{x_0 + \int_{t_0}^{t} f(s, y(s))ds : y \in K_2\})\}\}.\end{aligned}$$

By Lemma 5.1 we deduce that

$$\begin{aligned}&x_1 + \int_{t_0}^{t} g(s, x(s))ds \in x_1 + (t - t_0)\overline{\mathrm{con}}(\{g(s, x(s)) : s \in [t_0, t]\}), \\ &x_0 + \int_{t_0}^{t} f(s, y(s))ds \in x_0 + (t - t_0)\overline{\mathrm{con}}(\{f(s, y(s)) : s \in [t_0, t]\}),\end{aligned}$$

and so

$$\begin{aligned}\alpha(T(K_1), T(K_2))) &\leq \max\{\sup_{t\in J}\{\alpha(\{x_1 + (t - t_0)\overline{\mathrm{con}}(\{g(s, x(s)) : s \in [t_0, t]\})\})\}, \\ &\qquad \sup_{t\in J}\{\alpha(\{x_0 + (t - t_0)\overline{\mathrm{con}}(\{f(s, y(s)) : s \in [t_0, t]\})\})\}\} \\ &\leq \max\{\sup_{0\leq\lambda\leq h}\{\alpha(\{x_1 + \lambda\overline{\mathrm{con}}(\{g(J \times K_1)\})\})\}, \\ &\qquad \sup_{0\leq\lambda\leq h}\{\alpha(\{x_0 + \lambda\overline{\mathrm{con}}(\{f(J \times K_2)\})\})\}\} \\ &= \max\{h\alpha(g(J \times K_1)), h\alpha(f(J \times K_2))\} \\ &= h\alpha(\{g(J \times K_1) \cup f(J \times K_2)\}) \\ &\leq \frac{1}{2b} b\,\varphi(\alpha(K_1 \cup K_2)) \\ &= \frac{1}{2}\varphi(\alpha(K_1 \cup K_2)).\end{aligned}$$

It follows that the operator T is $\psi - \gamma$ condensing, where $\gamma(t) := \frac{\varphi(t)}{2t} \in \Gamma$ and $\psi \in \Psi$ is the identity function. Next, we show that T is relatively nonexpansive. For any $(x, y) \in C(J, V_1) \times C(J, V_1)$ we get

$$\begin{aligned}\|Tx(t) - Ty(t)\| &= \|(x_1 + \int_{t_0}^{t} g(s, x(s))ds) - (x_0 + \int_{t_0}^{t} f(s, y(s))ds)\| \\ &\leq \|x_1 - x_0\| + \int_{t_0}^{t} \|g(s, x(s)) - f(s, y(s))\|ds \\ &\leq \|x_1 - x_0\| + \frac{1}{h}\int_{t_0}^{t} (\|x(s) - y(s)\| - \|x_1 - x_0\|)ds \\ &\leq \|x_1 - x_0\| + (\|x - y\|_\infty - \|x_1 - x_0\|) = \|x - y\|_\infty,\end{aligned}$$

and so $\|Tx - Ty\|_\infty \leq \|x - y\|_\infty$. The conclusion follows immediately by Theorem 3.2 (see Remark 5.4 below). □

Remark 5.4. *Note that in Theorem 4.3, as we know, the Banach space $C(J, X)$ is not reflexive and the reflexivity condition in Theorem 3.2 is essential (see Remark 3.3). Since in Theorem 4.3 $(x_0, x_1) \in \Big(\big(C(J, V_1)\big)_0, \big(C(J, V_2)\big)_0\Big)$, the proximal pair $\Big(\big(C(J, V_1)\big)_0, \big(C(J, V_2)\big)_0\Big)$ is nonempty, automatically and we do not need the reflexivity condition of the Banach space $C(J, X)$.*

6. Conclusions

It was proved by Gabeleh and Markin that every cyclic (noncyclic) relatively nonexpansive mapping which is condensing has a best proximity point (pair) (Theorems 2.12, 2.13). We have extended these theorems by considering some appropriate control functions (see Theorems 3.2, 3.4). Moreover, we have studied the existence of a coupled best proximity point for a class of condensing mappings by considering an appropriate measure of noncompactness (see Theorem 4.3).

In the last section, we have established the existence of a solution of a system of differential equations by using the existence of a best proximity point which was scurvied in Theorem 3.2.

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