



On Some Enriched Contractions in Banach Spaces and an Application

Pratikshan Mondal^a, Hiranmoy Garai^{b,c}, Lakshmi Kanta Dey^b

^aDepartment of Mathematics, Durgapur Government College, Durgapur, India.

^bDepartment of Mathematics, National Institute of Technology Durgapur, India.

^cDepartment of Science and Humanities, Siliguri Govt. Polytechnic, Siliguri, India.

Abstract. In this paper, we introduce two new types of enriched contractions, viz., enriched \mathcal{A} -contraction and enriched \mathcal{A}' -contraction. Then we obtain fixed points of mappings satisfying such contractions using the fixed point property of the average operator of the mappings. Further, we study the well-posedness and limit shadowing property of the fixed point problem involving the contractions, and give some examples to validate the results proved. Our work improves Berinde and Păcurar's recent results on different kinds enriched contractions and some well known classical fixed point results. As an application, we prove an existence and uniqueness theorem for an integro-differential equation.

1. Introduction and preliminaries

Fixed point theory is one of the most rapidly growing branch of mathematics due to its applicability in different areas of pure and applied mathematics. Several researchers are engaged in this field to study fixed point of a self-map by generalizing the underlying space as well as the contraction condition. One of the main focus of this study is to find a set of sufficient condition(s) to ensure the existence of a fixed point which is the self image of the map under consideration. To reach such conditions, several mathematicians have introduced a number of contractions and used these in several structures to obtain fixed point, common fixed point, coincidence point of a map. Some of such contractions are introduced recently and looks very interesting to study. One such contraction is enriched contraction, which is introduced very recently by Berinde and Păcurar, see [5]. We first recall the definition of enriched contraction.

Definition 1.1. ([5, p. 2, Definition 2.1]). Let $(X, \|\cdot\|)$ be a normed linear space. A self map T on X is called an enriched contraction if there exist $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that

$$\|b(u - v) + Tu - Tv\| \leq \theta\|u - v\|$$

for all $u, v \in X$. In this case the mapping T is called a (b, θ) -enriched contraction. The class of enriched contractions contains the Picard-Banach contractions as well as some non-expansive mappings, e.g., a Picard-Banach mapping with constant k is a $(0, k)$ -enriched contraction.

2020 Mathematics Subject Classification. Primary 47H05, 47H10, 54H25.

Keywords. Banach spaces; enriched \mathcal{A} -contraction; enriched \mathcal{A}' -contraction; well-posedness; limit shadowing property

Received: 14 December 2020; Accepted: 20 December 2021

Communicated by Pratulananda Das

Email addresses: real.analysis77@gmail.com (Pratikshan Mondal), hiran.garai24@gmail.com (Hiranmoy Garai), lakshmikdey@yahoo.co.in (Lakshmi Kanta Dey)

Berinde and Păcurar showed that an enriched contraction T defined on a Banach space possesses a unique fixed point. They obtained this result by using the fixed point property of the map T_λ , where $T_\lambda(u) = (1 - \lambda)u + \lambda Tu$, $0 < \lambda < 1$.

It is to be noted that the enriched contraction introduced by Berinde and Păcurar involves the displacement $\|u - v\|$ only, which indicates that this contraction is basically the enriched version of Banach contraction. We know that for two points u, v the other displacements are $\|u - Tu\|$, $\|v - Tv\|$, $\|u - Tv\|$, $\|v - Tu\|$ and there are a lot of interesting contractions involving these displacements, some of them are due to Bianchini [6], Chatterjea [7], Ćirić [8, 9], Kannan [10], Khan [11], and Reich [12] etc. (see [13] for more contractions of this type). So it will be attractive to apply enrichment technique to the contractions by the above mentioned mathematicians. Berinde and Păcurar did this task for Kannan and Chatterjea mappings, see [3, 4]. So it is open in the literature to apply enrichment technique to the contractions of Bianchini [6], Ćirić [8, 9], Khan [11], Reich [12] etc. If we want to apply this technique to the above-named contractions separately, then we have to prove a handful number of results. So if we can establish a very few results involving enriched contractions, from which all the enriched versions of the aforesaid contractions can be deduced, then it will be simple and concise to the literature. Motivated by this observation, in the present article, we introduce two new types of enriched contractions, viz., enriched \mathcal{A} -contraction and enriched \mathcal{A}' -contraction. Then we prove two fixed point results showing that a self mapping T satisfying anyone of these two contractions admits a fixed point if the underlying space is a Banach space. Further, we show that from our obtained results, the enriched versions of the contractions due to Kannan, Chatterjea, Reich, Bianchini, Khan and many other contractions can be deduced. Along with these, we show that the fixed point problems related to these two contractions are well-posed and possess limit shadowing property. We give few examples to support the validity of our results. Next, we study Ulam-Hyers stability of fixed point equation related to enriched \mathcal{A} -contraction. Finally, we apply one of our result to find a solvability condition to a certain kind of integro-differential equation.

Throughout the paper, \mathbb{R}_+ denote the set of all non-negative real numbers.

Before going to our main work, we first recollect a few result and definitions, which will be useful in our next sections. At first, we recall the following result due to Akram et al. [1].

Theorem 1.2. cf. ([1, p. 29, Theorem 5]). *Let a self map T on a complete metric space (X, d) satisfies the condition:*

$$d(Tu, Tv) \leq \alpha(d(u, v), d(u, Tu), d(v, Tv))$$

for all $u, v \in X$ and some $\alpha \in A$, where A is the collection of all functions $\alpha : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying

- (i) α is continuous on the set \mathbb{R}_+^3 (with respect to the Euclidean metric on \mathbb{R}^3).
- (ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b .

Then T is a Picard operator.

Next, we recall the well-posedness and limit shadowing property of a mapping.

Definition 1.3. *Let T be a self map defined on a metric space (X, d) . Then the fixed point problem concerning T is known as well-posed if the followings hold:*

- (i) T has a unique fixed point $p \in X$;
- (ii) for any sequence $\{u_n\}$ in X with $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, we have

$$\lim_{n \rightarrow \infty} d(u_n, p) = 0.$$

Definition 1.4. *Let T be a self-map defined on a metric space (X, d) . Then the fixed point problem involving T is said to possess limit shadowing property in X if for any sequence $\{u_n\}$ in X such that $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, we have $p \in X$ with $\lim_{n \rightarrow \infty} d(T^n p, u_n) = 0$.*

Let us now recall the definition of Ulam–Hyers stability for the fixed point equation.

Definition 1.5. Let (X, d) be a metric space and let $T : X \rightarrow X$ be a mapping. The fixed point equation $x = Tx$ is Ulam-Hyers stable if there is a constant $K > 0$ such that for each $\varepsilon > 0$ and each $v^* \in X$ with $d(v^*, Tv^*) \leq \varepsilon$, there exists an $u^* \in X$ with $u^* = Tu^*$ such that $d(v^*, u^*) \leq K\varepsilon$.

2. Enriched \mathcal{A} -contraction and enriched \mathcal{A}' -contraction

In this section we give the formal definitions of enriched \mathcal{A} -contraction and enriched \mathcal{A}' -contraction mappings. To do this, we need two family of mappings which satisfy certain properties. Let \mathcal{A} be the collection of all mappings $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (\mathcal{A}_1) f is continuous;
- (\mathcal{A}_2) there exists $k \in [0, 1)$ such that if $r \leq f(s, r, s)$ or $r \leq f(r, s, s)$, then $r \leq ks$ for all $r, s \in \mathbb{R}_+$;
- (\mathcal{A}_3) for $\lambda > 0$ and for all $r, s, t \in \mathbb{R}_+$, $\lambda f(r, s, t) \leq f(\lambda r, \lambda s, \lambda t)$.

Let \mathcal{A}' be the collection of all mappings $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- (\mathcal{A}'_1) f is continuous;
- (\mathcal{A}'_2) there exists $k \in [0, 1)$ such that if $r \leq f(r, s, s)$ or $r \leq f(s, s, r)$ or $r \leq f(s, 0, r+s)$, then $r \leq ks$ for all $r, s \in \mathbb{R}_+$;
- (\mathcal{A}'_3) for $\lambda > 0$ and for all $r, s, t \in \mathbb{R}_+$, $\lambda f(r, s, t) \leq f(\lambda r, \lambda s, \lambda t)$;
- (\mathcal{A}'_4) if $t \leq t_1$, then $f(r, s, t) \leq f(r, s, t_1)$ for all $r, s, t, t_1 \in \mathbb{R}_+$;
- (\mathcal{A}'_5) if $r \leq f(r, r, r)$, then $r = 0$.

Below we present few example of mappings f belonging to the class \mathcal{A} :

- (i) $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$;
- (ii) $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$;
- (iii) $f(r, s, t) = \alpha \max\{r, s, t\}$, where $0 \leq \alpha < 1$;
- (iv) $f(r, s, t) = \alpha_1 r + \alpha_2 s + \alpha_3 t$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

The following mappings f belong to the class \mathcal{A}' :

- (i) $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$;
- (ii) $f(r, s, t) = \alpha(r + s + t)$, where $0 \leq \alpha < \frac{1}{3}$;
- (iii) $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$.

Next, we define enriched \mathcal{A} -contraction and enriched \mathcal{A}' -contraction in the following ways:

Definition 2.1. Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \rightarrow X$ be a mapping such that there exists an $f \in \mathcal{A}$ with

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|) \tag{1}$$

for all $u, v \in X$ and $b \in [0, \infty)$. Then T is said to be an enriched \mathcal{A} -contraction.

Definition 2.2. Let $(X, \|\cdot\|)$ be a normed linear space. Let $T : X \rightarrow X$ be a mapping such that there exists an $f \in \mathcal{A}'$ with

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|(b + 1)(u - v) + v - Tv\|, \|(b + 1)(v - u) + u - Tu\|) \tag{2}$$

for all $u, v \in X$ and $b \in [0, \infty)$. Then T is said to be an enriched \mathcal{A}' -contraction.

Next, we give some examples of the above two types of contractions.

Example 2.3. Let $X = \mathbb{R}$ and take the usual norm on X . Consider the mappings $T_1, T_2 : X \rightarrow X$ defined by

$$T_1u = -2u$$

for all $u \in X$ and

$$T_2u = \begin{cases} u + 16, & \text{if } u \in [1, 2] \\ 16, & \text{if } u \notin [1, 2]. \end{cases}$$

Then T_1 is an enriched \mathcal{A} -contraction for $b = \frac{5}{4}$ and $f(r, s, t) = \frac{s+t}{3}$, and T_2 is an enriched \mathcal{A}' -contraction for $b = \frac{1}{3}$ and $f(r, s, t) = \frac{1}{3}r + \frac{1}{4}s + \frac{1}{4}t$.

Next, we present an example which is not an enriched \mathcal{A} -contraction.

Example 2.4. Let $X = \mathbb{R}$ be the Banach space equipped with usual norm. Define

$$Tu = \begin{cases} 1 + u & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

for all $u \in X$.

We claim that T is not an enriched \mathcal{A} -contraction on X . If possible, let T be an enriched \mathcal{A} -contraction on X . Then there exists $b \in [0, \infty)$ such that

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

for all $u, v \in X$.

Let us take $u, v \in X$ be such that $u, v \geq 0$. Then

$$\begin{aligned} \|b(u - v) + Tu - Tv\| &\leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|) \\ \implies \|(b + 1)(u - v)\| &\leq f((b + 1)\|u - v\|, 1, 1). \end{aligned}$$

Then there exists $k \in [0, \infty)$ such that

$$\begin{aligned} (b + 1)\|u - v\| &\leq k \cdot 1 \\ \implies \|u - v\| &\leq \frac{k}{b + 1} \end{aligned}$$

for all $u, v \geq 0$ with $u \neq v$, which is a contradiction.

Hence T is not an enriched \mathcal{A} -contraction on X . Note that T admits no fixed point in X .

3. Main Results

In this section, we present the following two results that ensure the existence of fixed point of the two contractions, mentioned here, if the domain of the mapping is a Banach space.

Theorem 3.1. Let X be Banach space and T be an enriched \mathcal{A} -contraction. Then T has a unique fixed point in X and there exists $\lambda \in (0, 1]$ such that the sequence $\{u_n\}$ defined by $u_{n+1} = (1 - \lambda)u_n + \lambda Tu_n$, $n \geq 0$ converges to that fixed point, for any $u_0 \in X$.

Proof. Let $b > 0$. Set $\lambda = \frac{1}{b+1} > 0$. Then from (1), we get

$$\begin{aligned} \|T_\lambda u - T_\lambda v\| &\leq \lambda f\left(\frac{1}{\lambda}\|u - v\|, \|u - Tu\|, \|v - Tv\|\right) \\ &\leq f(\|u - v\|, \|u - T_\lambda u\|, \|v - T_\lambda v\|) \end{aligned} \tag{3}$$

for all $u, v \in X$ with $u \neq v$.

Let $u_0 \in X$ be arbitrary. Define $u_n = T_\lambda^n u_0$ for all $n \geq 1$. Let us put $u = u_n$ and $v = u_{n-1}$ in (3). Then we have

$$\|u_{n+1} - u_n\| \leq f(\|u_n - u_{n-1}\|, \|u_n - u_{n+1}\|, \|u_{n-1} - u_n\|).$$

Therefore, by (\mathcal{A}_2) , there exists $k \in [0, 1)$ such that

$$\|u_{n+1} - u_n\| \leq k\|u_n - u_{n-1}\|.$$

Continuing in such way, we get

$$\|u_{n+1} - u_n\| \leq k^n \|u_1 - u_0\|.$$

Now for all $m, n \geq 1$, we have

$$\begin{aligned} \|u_{n+m} - u_n\| &\leq \|u_{n+m} - u_{n+m-1}\| + \|u_{n+m-1} - u_{n+m-2}\| + \dots + \|u_{n+1} - u_n\| \\ &\leq (k^{n+m-1} + k^{n+m-2} + \dots + k^n)\|u_1 - u_0\| \\ &= k^n \frac{1 - k^m}{1 - k} \|u_1 - u_0\| \end{aligned}$$

which implies that $\|u_{n+m} - u_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\{u_n\}$ is a Cauchy sequence in X and hence there exists an element $p \in X$ such that $u_n \rightarrow p$ as $n \rightarrow \infty$.

Now

$$\|T_\lambda p - u_{n+1}\| \leq f(\|p - u_n\|, \|p - T_\lambda p\|, \|u_n - u_{n+1}\|).$$

Taking limit as $n \rightarrow \infty$, we get

$$\|T_\lambda p - p\| \leq f(\|p - p\|, \|p - T_\lambda p\|, \|p - p\|)$$

which implies that

$$\|T_\lambda p - p\| \leq k \cdot \|p - p\| = 0.$$

This implies that $T_\lambda p = p$ and consequently $Tp = p$. Therefore, p is a fixed point of T .

Let $q \in X$ be another fixed point of T and consequently a fixed point of T_λ . Then

$$\begin{aligned} \|p - q\| &= \|T_\lambda p - T_\lambda q\| \\ &\leq f(\|p - q\|, \|p - T_\lambda p\|, \|q - T_\lambda q\|) \\ &= f(\|p - q\|, \|p - p\|, \|q - q\|) \\ &= f(\|p - q\|, 0, 0). \end{aligned}$$

This implies that $\|p - q\| \leq k \cdot 0 = 0$. Hence $p = q$.

If $b = 0$, then (1) reduces to the form

$$\|Tu - Tv\| \leq f(\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

for all $u, v \in X$ with $u \neq v$. Hence the result follows from Theorem 1.2. \square

Theorem 3.2. Let X be Banach space and T be an enriched \mathcal{A}' -contraction. Then T has a unique fixed point in X and there exists $\lambda \in (0, 1]$ such that the sequence $\{u_n\}$ defined by $u_{n+1} = (1 - \lambda)u_n + \lambda Tu_n$, $n \geq 0$ converges to that fixed point, for any $u_0 \in X$.

Proof. Let $b > 0$. Set $\lambda = \frac{1}{b+1}$ so that $0 < \lambda < 1$. Now note that

$$u - T_\lambda v = (u - v) + \lambda(v - Tv) \text{ and } v - T_\lambda u = (v - u) + \lambda(u - Tu).$$

Then from (2), we get

$$\begin{aligned} \|T_\lambda u - T_\lambda v\| &\leq \lambda f\left(\frac{1}{\lambda}\|u - v\|, \left\|\frac{1}{\lambda}(u - v) + v - Tv\right\|, \left\|\frac{1}{\lambda}(v - u) + u - Tu\right\|\right) \\ &\leq f(\|u - v\|, \|(u - v) + \lambda(v - Tv)\|, \|(v - u) + \lambda(u - Tu)\|) \\ &= f(\|u - v\|, \|u - T_\lambda v\|, \|v - T_\lambda u\|) \end{aligned} \tag{4}$$

for all $u, v \in X$ with $u \neq v$.

Let $u_0 \in X$ be arbitrary. Define $u_n = T_\lambda^n u_0$ for all $n \geq 1$. Let us put $u = u_n$ and $v = u_{n-1}$ in (4). Then we have

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq f(\|u_n - u_{n-1}\|, \|u_n - u_n\|, \|u_{n-1} - u_{n+1}\|) \\ &\leq f(\|u_n - u_{n-1}\|, \|u_n - u_n\|, \|u_{n-1} - u_n\| + \|u_n - u_{n+1}\|) \end{aligned}$$

and hence by (\mathcal{A}'_2) , there exists $k \in [0, 1)$ such that

$$\|u_{n+1} - u_n\| \leq k\|u_n - u_{n-1}\|.$$

Proceeding in such way, we get

$$\|u_{n+1} - u_n\| \leq k^n \|u_1 - u_0\|.$$

Now for all $m, n \geq 1$, we have

$$\begin{aligned} \|u_{n+m} - u_n\| &\leq \|u_{n+m} - u_{n+m-1}\| + \|u_{n+m-1} - u_{n+m-2}\| + \dots + \|u_{n+1} - u_n\| \\ &\leq (k^{n+m-1} + k^{n+m-2} + \dots + k^n)\|u_1 - u_0\| \\ &= k^n \frac{1 - k^m}{1 - k} \|u_1 - u_0\| \end{aligned}$$

which implies that $\|u_{n+m} - u_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Hence $\{u_n\}$ is a Cauchy sequence in X and hence there exists an element $p \in X$ such that $u_n \rightarrow p$ as $n \rightarrow \infty$.

Now

$$\|T_\lambda p - u_{n+1}\| \leq f(\|p - u_n\|, \|p - u_{n+1}\|, \|u_n - T_\lambda p\|).$$

Taking limit as $n \rightarrow \infty$, we get

$$\|T_\lambda p - p\| \leq f(\|p - p\|, \|p - p\|, \|p - T_\lambda p\|)$$

which implies that

$$\|T_\lambda p - p\| \leq k \cdot \|p - p\| = 0.$$

Hence $T_\lambda p = p$ and consequently, $Tp = p$. Thus p is a fixed point of T .

Let $q \in X$ be another fixed point of T and hence a fixed point of T_λ . Then

$$\begin{aligned} \|p - q\| &= \|T_\lambda p - T_\lambda q\| \\ &\leq f(\|p - q\|, \|p - T_\lambda q\|, \|q - T_\lambda p\|) \\ &= f(\|p - q\|, \|p - q\|, \|q - p\|) \end{aligned}$$

which implies, by (\mathcal{A}'_5) , that $\|p - q\| = 0$. Hence $p = q$.

If $b = 0$, then (2) reduces to the form

$$\|Tu - Tv\| \leq f(\|u - v\|, \|u - Tv\|, \|v - Tu\|)$$

for all $u, v \in X$ with $u \neq v$.

Let $u_0 \in X$. Define the sequence $\{u_n\}$ by $u_n = T^n u_0$ for all $n \geq 1$. Then just replacing T_λ by T in the above proof, the result follows. \square

We now note an important observation about the above two results, which is presented in the following remark.

Remark 3.3. In most of the contractions for Banach space valued mappings, we see that if the domain of the mapping is a closed subset (not necessarily a subspace) of a Banach space, then the mapping possesses a fixed point. But this fact is not true in case of enriched contractions. More precisely, in the above two theorems, if the domain of the mapping T is a closed subset of a Banach space, then T may not acquire a fixed point. To support this, we consider the following example.

Example 3.4. Choose $X = \mathbb{R}$ with usual norm and take $C = (-\infty, -1] \cup [1, \infty)$. Define $T : C \rightarrow C$ by $Tu = -2u$ for all $u \in C$. Then T is an enriched \mathcal{A} -contraction for $b = \frac{5}{4}$ and $f(r, s, t) = \frac{s+t}{3}$ but T is fixed point free.

In view of Remark 3.3., we pose the following open question:

Open Question 3.5. Does there exist a self map defined on a subspace (not necessarily closed) of a Banach space satisfying enriched \mathcal{A} -contraction (resp. enriched \mathcal{A}' -contraction) having a fixed point?

By choosing particular f in above two theorems, we have the following remarks.

Remark 3.6. If we choose $f(r, s, t) = \alpha r$, where $0 \leq \alpha < 1$; $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$; $f(r, s, t) = \alpha_1 r + \alpha_2 s + \alpha_3 t$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$; $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$; $f(r, s, t) = \alpha \sqrt{st}$, where $0 \leq \alpha < 1$ in Theorem 3.1, then we get the enriched versions of the contractions of Banach [2], Kannan [10], Reich [12], Bianchini [6] and Khan [11] respectively.

Remark 3.7. If we choose $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$ in Theorem 3.2, then we get the enriched version of the Chatterjea contraction [7]. Also, for $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$; $f(r, s, t) = \alpha_1 r + \alpha_2 s + \alpha_3 t$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$ in Theorem 3.2, the enriched versions of the following two contractions respectively, can be acquired:

- $\|Tu - Tv\| \leq \alpha \max\{\|u - Tv\|, \|v - Tu\|\}, 0 \leq \alpha < 1$;
- $\|Tu - Tv\| \leq \alpha_1 \|u - v\| + \alpha_2 \|u - Tv\| + \alpha_3 \|v - Tu\|, 0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

Remark 3.8. One can perceive from the above remarks that Berinde and Păcurar's results regarding existence of fixed point of different types of enriched contractions [3–5] are particular cases of our results.

For $b = 0$ and some particular f in Theorem 3.1 and Theorem 3.2, we obtain some classical fixed point results, which is presented in the next remarks.

Remark 3.9. In particular for $b = 0$ and for $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$; $f(r, s, t) = \alpha_1 r + \alpha_2 s + \alpha_3 t$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$; $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$; $f(r, s, t) = \alpha \sqrt{st}$, where $0 \leq \alpha < 1$ in Theorem 3.1, we can obtain the classical fixed point results of Kannan [10], Reich [12], Bianchini [6] and Khan [11] respectively as corollaries.

Remark 3.10. For $b = 0$ and $f(r, s, t) = \alpha(s + t)$, where $0 \leq \alpha < \frac{1}{2}$ in Theorem 3.2, the classical fixed point theorem of Chatterjea [7] can be deduced as a corollary. Also for $b = 0$ and $f(r, s, t) = \alpha \max\{s, t\}$, where $0 \leq \alpha < 1$; $f(r, s, t) = \alpha_1 r + \alpha_2 s + \alpha_3 t$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$ in Theorem 3.2, two fixed point results concerning the following contractions respectively, can be deduced:

- $\|Tu - Tv\| \leq \alpha \max\{\|u - Tv\|, \|v - Tu\|\}, 0 \leq \alpha < 1$;
- $\|Tu - Tv\| \leq \alpha_1 \|u - v\| + \alpha_2 \|u - Tv\| + \alpha_3 \|v - Tu\|, 0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

We now take a couple of examples in support of Theorem 3.1.

Example 3.11. Let us consider the Banach space $(X, \|\cdot\|)$, where $X = C[0, 1]$ and $\|\cdot\|$ is the sup norm. Next, we define a mapping $T : X \rightarrow X$ by $(Tu)(t) = -2u(t)$ for all $u \in X$ and $t \in [0, 1]$. We choose $f \in \mathcal{A}$, defined by $f(r, s, t) = \frac{1}{3}(s + t)$ and $b = \frac{5}{4}$. Then for any $u, v \in X$, we have

$$\begin{aligned} f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|) &= \frac{1}{3}(\|u - Tu\| + \|v - Tv\|) \\ &= \|u\| + \|v\|, \end{aligned}$$

and

$$\begin{aligned} \|b(u - v) + Tu - Tv\| &= \|b(u - v) - 2u + 2v\| \\ &= |b - 2|\|u - v\| = \frac{3}{4}\|u - v\|. \end{aligned}$$

Therefore,

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

holds for all $u, v \in X$. So T is an enriched \mathcal{A} -contraction. So by Theorem 3.1, T has a unique fixed point. Note that $u \in X$ defined by $u(t) = 0$ for all $t \in [0, 1]$, is the unique fixed point of T .

Example 3.12. Let $X = \mathbb{R}$ be the Banach space equipped with the usual norm. Define $T : X \rightarrow X$ by $Tu = 6 - u$ for all $u \in X$.

Let us consider the mapping $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ defined by $f(r, s, t) = \frac{1}{6} \max\{s, t\}$ for $r, s, t \in \mathbb{R}_+$. Then $f \in \mathcal{A}$. Let $b = 1$. Then we have

$$\|b(u - v) + Tu - Tv\| = 0$$

and

$$f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|) = \frac{1}{6} \max\{\|2u - 6\|, \|2v - 6\|\},$$

which shows that

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

for all $u, v \in X$. Therefore, T is an enriched \mathcal{A} -contraction on X . It is to be noticed that 3 is the unique fixed point of T .

Next, we present another couple of examples, which validate Theorem 3.2.

Example 3.13. Let us take the Banach space $X = C[0, \frac{1}{4}]$, equipped with sup norm. We define a mapping $T : X \rightarrow X$ by $(Tu)(t) = tu(t)$ for all $u \in X$ and for all $t \in [0, \frac{1}{4}]$. We choose $b = \frac{1}{4}$ and $f \in \mathcal{A}$, defined by $f(r, s, t) = \frac{9}{20} \max\{r, s, t\}$. Then for any $u, v \in X$, we have

$$\begin{aligned} &\|b(u - v) + Tu - Tv\| \\ &= \sup_{t \in [0, \frac{1}{4}]} |bu(t) - bv(t) + tu(t) - tv(t)| \\ &\leq \sup_{t \in [0, \frac{1}{4}]} |b + t| \sup_{t \in [0, \frac{1}{4}]} |u(t) - v(t)| \\ &= \frac{1}{2}\|u - v\| \\ &\leq \frac{9}{20}(b + 1)\|u - v\| \\ &\leq f((b + 1)\|u - v\|, \|(b + 1)(u - v) + v - Tv\|, \|(b + 1)(v - u) + u - Tu\|). \end{aligned}$$

Thus T is an enriched \mathcal{A} '-contraction. So by Theorem 3.2, it follows that T possesses a unique fixed point in X and note the fixed point is u' , where $u'(t) = 0$ for all $t \in [0, \frac{1}{4}]$.

Example 3.14. Let $X = \mathbb{R}$ be the Banach space equipped with usual norm. Define $T : X \rightarrow X$ by $Tu = 2 - u$ for all $u \in X$.

Let us consider the mapping $f : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ defined by $f(r, s, t) = \frac{s+t}{6}$ for $r, s, t \in \mathbb{R}_+$. Then $f \in \mathcal{A}'$. Let us choose $b = 1$. Then we have

$$\|b(u - v) + Tu - Tv\| = 0$$

and

$$f((b + 1)\|u - v\|, \|(b + 1)(u - v) + v - Tv\|, \|(b + 1)(v - u) + u - Tu\|) = \frac{|u - 1| + |v - 1|}{3}.$$

So

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|(b + 1)(u - v) + v - Tv\|, \|(b + 1)(v - u) + u - Tu\|)$$

for all $u, v \in X$. Therefore, T is an enriched \mathcal{A}' -contraction on X . It is to be noticed that 1 is the unique fixed point of T .

Finally, we study well-posedness and limit shadowing property of fixed point problem for both types of contractions defined in the present paper.

Theorem 3.15. Let X be Banach space and T be an enriched \mathcal{A} -contraction. Then the fixed point problem involving T is well-posed.

Proof. Theorem 3.1 ensures us that T possesses a unique fixed point p , say.

It has been shown in Theorem 3.1 that if $b \in (0, \infty)$, then the contraction condition

$$\|b(u - v) + Tu - Tv\| \leq f((b + 1)\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

for all $u, v \in X$ with $u \neq v$, can be reduced to the form

$$\|T_\lambda u - T_\lambda v\| \leq f(\|u - v\|, \|u - T_\lambda u\|, \|v - T_\lambda v\|)$$

for some $\lambda > 0$. In this case it is to be noted that $\|u - T_\lambda u\| = \lambda\|u - Tu\|$ for all $u \in X$.

Therefore, $\lim_{n \rightarrow \infty} \|u_n - Tu_n\| = 0 \iff \lim_{n \rightarrow \infty} \|u_n - T_\lambda u_n\| = 0$.

Now let $\{u_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} \|u_n - T_\lambda u_n\| = 0$. Then

$$\begin{aligned} \|u_n - p\| &\leq \|u_n - T_\lambda u_n\| + \|T_\lambda u_n - p\| \\ &= \|u_n - T_\lambda u_n\| + \|T_\lambda u_n - T_\lambda p\| \\ &\leq \|u_n - T_\lambda u_n\| + f(\|u_n - p\|, \|u_n - T_\lambda u_n\|, \|p - T_\lambda p\|). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|u_n - p\| \leq f(\lim_{n \rightarrow \infty} \|u_n - p\|, 0, 0).$$

Therefore there exist $k \in [0, 1)$ such that $\lim_{n \rightarrow \infty} \|u_n - p\| \leq k \cdot 0$ which implies that $\lim_{n \rightarrow \infty} \|u_n - p\| = 0$.

Now, if $b = 0$, then the contraction condition reduces to

$$\|Tu - Tv\| \leq f(\|u - v\|, \|u - Tu\|, \|v - Tv\|)$$

for all $u, v \in X$ with $u \neq v$.

In a similar way, it can be shown that $\lim_{n \rightarrow \infty} \|u_n - p\| = 0$. Hence the result follows. \square

Theorem 3.16. Let X be Banach space and T be an enriched \mathcal{A}' -contraction. Then the fixed point problem involving T is well-posed.

Proof. Theorem 3.2 ensures us that T possesses a unique fixed point p , say.

It has been shown in Theorem 3.2 that if $b \in (0, \infty)$, then the contraction condition

$$\|b(u - v) + Tu - Tv\| \leq f(\|u - v\|, \|(b + 1)(u - v) + v - Tv\|, \|(b + 1)(v - u) + u - Tu\|)$$

for all $u, v \in X$ with $u \neq v$, can be reduced to the form

$$\|T_\lambda u - T_\lambda v\| \leq f(\|u - v\|, \|u - T_\lambda v\|, \|v - T_\lambda u\|)$$

for some $\lambda > 0$. In this case it is to be noted that

$$\|u - T_\lambda v\| = \|(u - v) + \lambda(v - Tv)\|$$

and

$$\|v - T_\lambda u\| = \|(v - u) + \lambda(u - Tu)\|$$

for all $u, v \in X$.

Therefore, for any sequence $\{u_n\}$ in X , we have

$$\lim_{n \rightarrow \infty} \|u_n - Tu_n\| = 0 \iff \lim_{n \rightarrow \infty} \|u_n - T_\lambda u_n\| = 0.$$

Now let $\{u_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} \|u_n - T_\lambda u_n\| = 0$. Then

$$\|u_n - p\| \leq \|u_n - T_\lambda u_n\| + \|T_\lambda u_n - p\|.$$

Now,

$$\begin{aligned} \|T_\lambda u_n - p\| &= \|T_\lambda u_n - T_\lambda p\| \\ &\leq f(\|u_n - p\|, \|u_n - T_\lambda p\|, \|p - T_\lambda u_n\|) \\ &= f(\|u_n - p\|, \|u_n - p\|, \|p - T_\lambda u_n\|) \end{aligned}$$

Then we get a $k \in [0, 1)$ such that $\|T_\lambda u_n - p\| \leq k \cdot \|u_n - p\|$.

Therefore, we have

$$\|u_n - p\| \leq \|u_n - T_\lambda u_n\| + k \cdot \|u_n - p\|$$

for some $k \in [0, 1)$. This implies that

$$\|u_n - p\| \leq \frac{1}{1 - k} \|u_n - T_\lambda u_n\|$$

and therefore by taking limit as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \|u_n - p\| = 0$.

Now, if $b = 0$, then the contraction condition reduces to

$$\|Tu - Tv\| \leq f(\|u - v\|, \|u - Tv\|, \|v - Tu\|)$$

for all $u, v \in X$ with $u \neq v$.

In a similar way, it can be shown that $\lim_{n \rightarrow \infty} \|u_n - p\| = 0$. Hence the result follows. \square

Theorem 3.17. Let X be Banach space and T be an enriched \mathcal{A} -contraction (resp. an enriched \mathcal{A}' -contraction). Then the fixed point problem involving T possesses limit shadowing property in X .

Proof. We have already shown in Theorem 3.1 (resp. in Theorem 3.2) that $\lim_{n \rightarrow \infty} \|u_n - p\| = 0$, where p is the unique fixed point of T . Then for any $n \in \mathbb{N}$, $T^n p = p$ and therefore

$$\lim_{n \rightarrow \infty} \|u_n - T^n p\| = 0.$$

That is

$$\lim_{n \rightarrow \infty} \|T^n p - u_n\| = 0.$$

Hence the fixed point problems has limit shadowing property in both the cases. \square

We close this section with the following theorem concerning Ulam-Hyers stability of the fixed point equation of an enriched \mathcal{A} -contraction:

Theorem 3.18. *Let X be Banach space and let T be an enriched \mathcal{A} -contraction on X . Suppose that there exists an α with $0 < \alpha < 1$ such that $f(r, s, t) \leq \alpha r + t$ for all $r, s, t \in \mathbb{R}_+$. Then the fixed point equation $x = Tx$ is Ulam-Hyers stable.*

Proof. Theorem 3.1 ensures that T has a unique fixed point, say $u^* \in X$.

Let $\varepsilon > 0$ and let $v^* \in X$ be an ε -solution i.e.,

$$\|v^* - Tv^*\| \leq \varepsilon.$$

Since $u^* \in X$ and $\|u^* - Tu^*\| = 0 \leq \varepsilon$, it follows that u^* is also an ε -solution.

Now,

$$\begin{aligned} \|u^* - v^*\| &= \|Tu^* - v^*\| \\ &\leq \|Tu^* - Tv^*\| + \|Tv^* - v^*\| \\ &\leq f(\|u^* - v^*\|, \|u^* - Tu^*\|, \|v^* - Tv^*\|) + \varepsilon \\ &\leq \alpha \|u^* - v^*\| + \|v^* - Tv^*\| + \varepsilon \end{aligned}$$

which implies that

$$\|u^* - v^*\| \leq K\varepsilon$$

where $K = \frac{2}{1-\alpha}$. Hence the result follows. \square

4. Application to integro-differential equation

In this section, we study on solvability conditions of the following integro-differential equation:

$$\left. \begin{aligned} x^{(4)}(t) &= F_1(t, x(t)) + \int_0^t F_2(s, x(s))ds, \quad 0 < t < 1; \\ x(0) &= x'(0) = x''(1) = x'''(1) = 0, \end{aligned} \right\} \tag{1}$$

where $F_1, F_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Our main goal is to obtain sufficient conditions on K_1, K_2 in presence of which the equation (1) possesses a unique solution. Equation (1) is equivalent to the following integral equation

$$x(t) = \int_0^1 G(t, s)F_1(s, x(s))ds + \int_0^t \frac{(t-s)^4}{12}F_2(s, x(s))ds, \quad 0 < t < 1, \tag{2}$$

where $G : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is the Green's function, given by

$$G(t, s) = \begin{cases} \frac{1}{6}s^2(t-s), & \text{if } 0 \leq s \leq t \leq 1; \\ \frac{1}{6}t^2(3s-t), & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

We now have the following theorem:

Theorem 4.1. *Suppose that there exist two positive constants α_1, α_2 such that*

(i) $|F_1(t, a) - F_1(t, b)| \leq \alpha_1|a - b|;$

(ii) $|F_2(t, a) - F_2(t, b)| \leq \alpha_2|a - b|,$

for all $t \in [0, 1]$ and $a, b \in \mathbb{R}$. Then equation (1) has a unique continuous solution if $\frac{\alpha_1}{20} + \frac{\alpha_2}{60} \leq 1$.

Proof. Let us consider the Banach space $C[0, 1]$ equipped with sup norm, and define a mapping $T : C[0, 1] \rightarrow C[0, 1]$ by

$$(Tx)(t) = \int_0^1 G(t, s)F_1(s, x(s))ds + \int_0^t \frac{(t-s)^4}{12} F_2(s, x(s))ds$$

for all $x \in C[0, 1]$ and $t \in [0, 1]$. Then an element $u \in C[0, 1]$ is a solution of (1) if and only if u is a fixed point of T .

Let $x, y \in C[0, 1]$ be arbitrary. Then for each $t \in [0, 1]$, we have

$$\begin{aligned} & |(Tx)(t) - (Ty)(t)| \\ &= \left| \int_0^1 G(t, s)F_1(s, x(s))ds + \int_0^t \frac{(t-s)^4}{12} F_2(s, x(s))ds - \int_0^1 G(t, s)F_1(s, y(s))ds \right. \\ &\quad \left. - \int_0^t \frac{(t-s)^4}{12} F_2(s, y(s))ds \right| \\ &\leq \left| \int_0^1 G(t, s)F_1(s, x(s))ds - \int_0^1 G(t, s)F_1(s, y(s))ds \right| \\ &\quad + \left| \int_0^t \frac{(t-s)^4}{12} F_2(s, x(s))ds - \int_0^t \frac{(t-s)^4}{12} F_2(s, y(s))ds \right| \\ &\leq \int_0^1 G(t, s)|F_1(s, x(s)) - F_1(s, y(s))|ds + \int_0^t \frac{(t-s)^4}{12} |F_2(s, x(s)) - F_2(s, y(s))|ds \\ &\leq \alpha_1 \int_0^1 G(t, s)|x(s) - y(s)|ds + \alpha_2 \int_0^t \frac{(t-s)^4}{12} |x(s) - y(s)|ds \\ &\leq \alpha_1 \|x - y\| \int_0^1 G(t, s)ds + \alpha_2 \|x - y\| \int_0^t \frac{(t-s)^4}{12} ds \\ &\leq \frac{\alpha_1}{20} \|x - y\| + \frac{\alpha_2}{60} \|x - y\| \leq \|x - y\|. \end{aligned}$$

Thus for any $b \in [0, \infty)$ and $t \in [0, 1]$, we have

$$\begin{aligned} |b(x - y)(t) + (Tx)(t) - (Ty)(t)| &\leq |b(x - y)(t)| + |(Tx)(t) - (Ty)(t)| \\ &\leq (b + 1)\|x - y\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|b(x - y) + Tx - Ty\| &\leq (b + 1)\|x - y\| \\ \implies \|b(x - y) + Tx - Ty\| &\leq f((b + 1)\|x - y\|, \|x - Tx\|, \|y - Ty\|), \end{aligned}$$

where $f(r, s, t) = r$ ($f \in \mathcal{A}$).

Thus T is an enriched \mathcal{A} -contraction. Therefore, by Theorem 3.1, T has a unique fixed point in $C[0, 1]$, and hence the equation (1) possesses a unique continuous solution. \square

Acknowledgement: The authors appreciate the anonymous referee and the section editor for their constructive comments towards improvement of the first draft of the article.

References

- [1] M. Akram, A.A. Zafar, and A.A. Siddiqui, A general class of contractions: A -contractions, Novi Sad J. Math. 38(1) (2008) 25–33.

- [2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* 3 (1922) 133–181. JFM 48.1204.02.
- [3] V. Berinde and M. Păcurar, Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative method in Banach spaces, (2019) 1–13. arXiv:1909.03494v1.
- [4] V. Berinde and M. Păcurar, Fixed point theorems for Kannan type mappings with applications to split feasibility and variational inequality problems, (2019) 1–13. arXiv:1909.02379v1.
- [5] V. Berinde and M. Păcurar, Approximating fixed points of enriched contractions in Banach spaces, *J. Fixed Point Theory Appl.* 22, 38 (2020).
- [6] R.M.T. Bianchini, Su un problema di S. Reich riguardante la teoria dei punti fissi, *Boll. Un. Mat. Ital.* 5 (1972) 103–108.
- [7] S.K. Chatterjea, Fixed point theorems, *C. R. Acad. Bulgare Sci.* 25 (1972) 727–730.
- [8] LB. Ćirić, Generalized contractions and fixed-point theorems, *Publ. Inst. Math.* 12(26) (1971) 19–26.
- [9] LB. Ćirić, A generalization of Banach’s contraction principle, *Proc. Amer. Math. Soc.* 45(2) (1974) 267–273.
- [10] R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.* 60 (1968) 71–76.
- [11] M.S. Khan, On fixed point theorems, *Math. Japonica* 23(2) (1978/79) 201–204.
- [12] S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.* 14(1) (1971) 121–124.
- [13] B.E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* 226 (1977) 257–290.