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Certain Results of Conformal and ∗-Conformal Ricci Soliton on Para-Cosymplectic and Para-Kenmotsu Manifolds

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Abstract. The goal of the paper is to deliberate conformal Ricci soliton and ∗-conformal Ricci soliton within the framework of paracontact geometry. Here we prove that if an η-Einstein para-Kenmotsu manifold admits conformal Ricci soliton and ∗-conformal Ricci soliton, then it is Einstein. Further we have shown that 3-dimensional para-cosymplectic manifold is Ricci flat if the manifold satisfies conformal Ricci soliton where the soliton vector field is conformal. We have also constructed some examples of para-Kenmotsu manifold that admits conformal and ∗-conformal Ricci soliton and verify our results.

1. Introduction

The notion of almost paracontact manifold was first introduced by Sato [23]. Later Kaneyuki and Williams [15] associated pseudo-Riemannian metric with an almost paracontact manifold after Takahashi [26] intoduced pseudo- Riemannian metric in contact manifold, in particular, in Sasakian manifold. Zamkovoy in [30] proved that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n+1, n)$. In recent years paracontact geometry has become area of interest for many authors ([5], [18], [16]). On the analogy of Kenmotsu manifold, Welyczko [28] introduced the notion of para-Kenmotsu manifold. Para-Kenmotsu manifold (in short p-Kenmotsu manifold) and special para-Kenmotsu manifold (briefly sp-Kenmotsu manifold) was studied by many authors, namely: Blaga [4], Adigond and Bagewadi [1], Prakasha and Vikas [20], Sinha and Prasad [24] and many others.

A pseudo-Riemannian manifold (M, g) admits a Ricci soliton which is a generalization of Einstein metric if there exists a smooth vector field V and a constant λ such that

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

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where $\mathcal{L}_V$ denotes Lie derivative along the direction V and S denotes the Ricci curvature tensor of the manifold. The vector field V is called potential vector field and λ is called soliton constant.

The Ricci soliton is a self-similar solution of the Hamilton's Ricci flow [12] which is defined by the geometric evolution equation $\frac{\partial g(t)}{\partial t} = -2S(g(t))$ with initial condition $g(0) = g$ where $g(t)$ is a one-parameter family of metrices on M. The potential vector field V and soliton constant λ play vital roles while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Now if V is zero or Killing then the Ricci soliton reduces to Einstein manifold and the soliton is called trivial soliton.

In 2005, Fischer [10] has introduced conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg, \\ r(g) &= -1, \end{aligned}$$

where $r(g)$ is the scalar curvature of the manifold, p is scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the aforementioned conformal Ricci flow equation, Basu and Bhattacharyya [2] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation is given by

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{1}$$

In 2014, Kaimakamis and Panagiotidou [14] modified the definition of Ricci soliton where they have used $*$-Ricci tensor S^* which was introduced by Tachibana [25], in place of Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(trace\{\phi.R(X, \phi Y)\})$$

for all vector fields X and Y on M. They have used the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form. A pseudo-Riemannian metric g is called a $*$-Ricci soliton if there exists a constant λ and a vector field V such that

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0.$$

Further Majhi and Dey [17] in 2020 revised the aforementioned definition of $*$-Ricci soliton with the help of (1) and defined $*$-conformal Ricci soliton as

$$\mathcal{L}_V g + 2S^* + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{2}$$

As follows in the literature, Ricci soliton on paracontact geometry studied by many authors ([3], [6], [21]). In particular, Calvaruso and Perrone [6] explicitly studied Ricci soliton on 3-dimensional almost paracontact manifolds. Conformal Ricci solitons have been studied in many contexts: on Kenmotsu manifold [2], on 3- dimensional trans-Sasakian manifold [8], on f-Kenmotsu manifold ([13], [19]) etc. by many authors. In 2018, Ghosh and Patra [11] first studied $*$-Ricci soliton on almost contact metric manifolds. The case of $*$-Ricci soliton in para-Sasakian manifold was treated by Prakasha and Veeresha in [22]. Recently in 2019, Venkatesha, Kumara and Naik [27] considered the metric of η-Einstein para-Kenmotsu manifold as $*$-Ricci soliton and proved that the manifold is Einstein. Erken [9] in 2019 considered Yamabe solitons on 3-dimensional para-cosymplectic manifold and proved some vital results like the manifold is either η-Einstein or Ricci flat.

Motivated by above mentioned works, in this paper, we consider conformal Ricci soliton and $*$-conformal Ricci soliton in the framework of para-Kenmotsu manifold and conformal Ricci soliton in the framework of 3- dimensional para-cosymplectic manifold. We have organized this paper as follows: in first section we look back on some elementary properties of para-Kenmotsu manifolds; in later section first we prove that if a para-Kenmotsu manifold satisfies conformal Ricci soliton then $\mathcal{L}_V\xi$ is orthogonal to ξ or the manifold is Einstein, secondly we prove that an η-Einstein para-kenmotsu manifold is Einstein if it admits a conformal Ricci soliton and then we prove the same for $*$-Conformal Ricci soliton. In the next section, we consider 3-dimensional para-coysmplectic manifold with a conformal Ricci soliton and deduce some relations on the scalar curvature of the manifold and finally, we provide some examples to verify our results.

2. Some preliminaries on para-Kenmotsu manifold

A $(2n + 1)$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a vector field ξ, (1,1)-tensor field ϕ and a 1-form η satisfying the following conditions

$$i)\phi^2 = I - \eta \otimes \xi, \tag{3}$$

$$ii)\eta(\xi) = 1. \tag{4}$$

iii) ϕ induces on the 2n-dimensional distribution $\mathcal{D} \equiv ker(\eta)$, an almost paracomplex structure $\mathcal{P}$ i.e., $\mathcal{P}^2 \equiv I_{\chi(M)}$ and the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues $1, -1$ of $\mathcal{P}$ respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.
If a manifold with an almost paracontact structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n + 1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{5}$$

holds for any $X, Y \in \chi(M)$, then g is called compatible metric and the manifold (M, ϕ, ξ, η, g) is called almost paracontact metric manifold. If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{6}$$

for arbitrary vector fields X and Y, then the manifold is called almost para-Kenmotsu manifold. The normality of an almost paracontact structure (M, ϕ, ξ, η) is equivalent to vanishing of the (1,2)-torsion tensor defined by $N_\phi(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ and is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any $X, Y \in \chi(M)$. A normal almost para-Kenmotsu manifold is called para-Kenmotsu manifold.
The following properties hold on a $(2n + 1)$-dimensional para-Kenmotsu manifold

$$\begin{aligned}
\phi(\xi) &= 0, && (7)\\
\eta \circ \phi &= 0, && (8)\\
\nabla_X \xi &= X - \eta(X)\xi, && (9)\\
(\nabla_X \eta)Y &= g(X, Y) - \eta(X)\eta(Y), && (10)\\
Q\xi &= -2n\xi, && (11)\\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, && (12)\\
R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, && (13)\\
(\mathcal{L}_\xi g)(X, Y) &= 2[g(X, Y) - \eta(X)\eta(Y)], && (14)
\end{aligned}$$

for any $X, Y \in \chi(M)$ where, $\mathcal{L}$ and ∇ are the operators of Lie differentiation and covariant differentiation of g respectively. Q denotes the Ricci operator associated with the Ricci tensor S defined by $S(X, Y) = g(QX, Y)$ and R denotes the Riemannian curvature tensor.

3. A para-Kenmotsu metric as conformal Ricci soliton

In this section we consider the metric of para-Kenmotsu manifold as a conformal Ricci soliton. The following lemma will be used to prove one of the our main results.

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional para-Kenmotsu manifold. Then the Ricci operator satisfies*

$$(\mathcal{L}_\xi Q)X = -2QX - 4nX = (\nabla_\xi Q)X \tag{15}$$

for any vector field X on M.

Proof. From (14), we have $(\mathcal{L}_\xi g)(Y, Z) = 2[g(Y, Z) - \eta(Y)\eta(Z)]$ for all $Y, Z \in \chi(M)$. Covariant derivative of that along an arbitrary vector field X on M and use of the equation (10), leads to

$$(\nabla_X \mathcal{L}_\xi g)(Y, Z) = 2[2\eta(X)\eta(Y)\eta(Z) - g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] \tag{16}$$

for all $Y, Z \in \chi(M)$. Again from Yano [29], we have the following commutation formula

$$(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y, Z) = -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y), \tag{17}$$

where g is the metric connection i.e., $\nabla g = 0$. So, the above equation reduces to

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y). \tag{18}$$

for all vector fields X, Y, Z on M. Combining (16) and (18), we have

$$g((\mathcal{L}_\xi \nabla)(X, Y), Z) + g((\mathcal{L}_\xi \nabla)(X, Z), Y) = 2[2\eta(X)\eta(Y)\eta(Z) - g(X, Y)\eta(Z) - g(X, Z)\eta(Y)].$$

By a straightforward combinatorial computation, the foregoing equation yields

$$(\mathcal{L}_\xi \nabla)(Y, Z) = 2[\eta(Y)\eta(Z)\xi - g(Y, Z)\xi] \tag{19}$$

for all $Y, Z \in \chi(M)$. Taking covariant derivative of the above equation with respect to an arbitrary vector field X on M and using (9) and (10), we have

$$(\nabla_X \mathcal{L}_\xi \nabla)(Y, Z) = 2[g(X, Y)\eta(Z)\xi + g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi - g(Y, Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)].$$

From Yano [29], we have the well known commutation formula

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z). \tag{20}$$

From here we can compute

$$(\mathcal{L}_\xi R)(X, Y)Z = 2[g(X, Z)Y - g(Y, Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \tag{21}$$

for all vector fields X, Y, Z on M. Contracting (21) over X we get

$$(\mathcal{L}_\xi S)(Y, Z) = 4n[\eta(Y)\eta(Z) - g(Y, Z)]. \tag{22}$$

The Lie derivative of $S(Y, Z) = g(QY, Z)$ along the direction of ξ, yields

$$(\mathcal{L}_\xi S)(Y, Z) = (\mathcal{L}_\xi g)(QY, Z) + g((\mathcal{L}_\xi Q)Y, Z). \tag{23}$$

On the other hand, replacing X and Y by QY and Z respectively in (14) and using (11), we have

$$(\mathcal{L}_\xi g)(QY, Z) = 2[g(QY, Z) + 2n\eta(Y)\eta(Z)]. \tag{24}$$

Combining (22), (23) and (24) all together, we infer

$$(\mathcal{L}_\xi Q)Y = -2QY - 4nY \tag{25}$$

for any $Y \in \chi(M)$. Again we know that

$$\begin{aligned}(\mathcal{L}_\xi Q)Y &= \mathcal{L}_\xi(QY) - Q(\mathcal{L}_\xi Y)\\ &= \nabla_\xi(QY) - \nabla_{QY}\xi - Q(\nabla_\xi Y) + Q(\nabla_Y \xi)\\ &= (\nabla_\xi Q)Y - \nabla_{QY}\xi + Q(\nabla_Y\xi).\end{aligned}$$

By virtue of (9) and (11) we see that $(\mathcal{L}_\xi Q)Y = (\nabla_\xi Q)Y$ for arbitrary vector field Y. Hence the result is proved. □

Theorem 3.2. *If the metric g of a para-Kenmotsu manifold (M, ϕ, ξ, η, g) of dimension > 3 represents a conformal Ricci soliton then either of the following properties holds:*

i) The Lie derivative of ξ in the direction of the potential vector field V of the soliton i.e., $\mathcal{L}_V\xi$ is orthogonal to ξ.
ii) The manifold is Einstein with Einstein constant $-2n$.

Proof. Let M be a $(2n+1)$ dimensional para-Kenmotsu manifold where $n > 1$. From (12), we have $R(X,\xi)\xi = \eta(X)\xi - X$. Now Lie derivative of the Riemannian curvature along the vector field V, yields

$$(\mathcal{L}_V R)(X,\xi)\xi = ((\mathcal{L}_V\eta)X)\xi - g(X, \mathcal{L}_V\xi)\xi + 2\eta(\mathcal{L}_V\xi)X \tag{26}$$

for all vector fields X on M. Now the covariant derivative of (1) along an arbitrary vector field $Z \in \chi(M)$ provides

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\nabla_Z S)(X, Y) \tag{27}$$

for any $X, Y \in \chi(M)$. Using (18), we can rewrite (27) as

$$g((\mathcal{L}_V\nabla)(X, Y), Z) + g((\mathcal{L}_V\nabla)(X, Z), Y) = -2(\nabla_Z S)(X, Y).$$

By a straightforward combinatorial computation and using the symmetry of the (1,2)-tensor $\mathcal{L}_V\nabla$, the aforementioned yields

$$g((\mathcal{L}_V\nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X). \tag{28}$$

Again differentiating the above equation covariantly with respect to an arbitrary vector field X of M and using (9), we can find from (11) that

$$(\nabla_X Q)\xi = -QX - 2nX \tag{29}$$

for all $X \in \chi(M)$. Making use of (15) and (29) and considering $Y = \xi$ in (28), we achieve

$$(\mathcal{L}_V\nabla)(X, \xi) = 2QX + 4nX \tag{30}$$

for any vector field X on M. Now considering covariant derivative of the last equation with respect to an arbitrary vector field Y of M and using (9), we acquire

$$(\nabla_Y \mathcal{L}_V\nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V\nabla)(X, Y) + 2\eta(Y)QX + 4n\eta(Y)X. \tag{31}$$

Now letting $Z = \xi$ in (20) and using (31) in the foregoing equation, we have

$$(\mathcal{L}_V R)(X, Y)\xi = 4n[\eta(X)Y - \eta(Y)X] + 2[(\nabla_X Q)Y - (\nabla_Y Q)X] + 2[\eta(X)QY - \eta(Y)QX] \tag{32}$$

for all $X, Y \in \chi(M)$. Considering $Y = \xi$ in the aforementioned equation and using (11) and (15) in it, we obtain

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{33}$$

Now, taking into account (1), the Lie derivative of $g(\xi, \xi) = 1$ along the direction of V leads to

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n. \tag{34}$$

Again, using (11) and letting $Y = \xi$, (1) implies

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (4n - 2\lambda + p + \frac{2}{2n+1})\eta(X). \tag{35}$$

After using (33), (34) and (35), the equation (26) reduces to

$$(2\lambda - p - 4n - \frac{2}{2n+1})\phi^2 X = 0. \tag{36}$$

Since the last equation holds for any $X \in \chi(M)$, we can conclude that $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Using this result in (34) we have, $\eta(\mathcal{L}_V \xi) = 0$. From here the following two cases have arisen

Case-I: $\mathcal{L}_V \xi$ is orthogonal to ξ.

Case-II: $\mathcal{L}_V \xi = 0$ for any vector field X of M. Then additionally using the value of λ, (35) reduces to $(\mathcal{L}_V \eta)X = 0$. Which further can be reduced to $\mathcal{L}_V \eta = 0$, since X is an arbitrary vector field on M.
On other hand, we have a renowned relation (see [29]):

$$(\mathcal{L}_V \nabla)(X, Y) = \mathcal{L}_X \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y, \tag{37}$$

which holds for arbitrary vector fields X and Y of M. Now replacing Y by ξ and using (9) and the relations $\mathcal{L}_V \xi = 0$ and $\mathcal{L}_V \eta = 0$ in the foregoing equation we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = 0.$$

Finally substituting this in (30), we get $S(X, Y) = -2ng(X, Y)$ for any arbitrary vector fields X and Y on M. From this we can conclude that the manifold is Einstein with Einstein constant $-2n$. □

A (2n+1)-dimensional almost para-Kenmotsu metric manifold is said to be η-Einstein para-Kenmotsu manifold if there exists two smooth functions a and b which satisfies the following relation

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{38}$$

for all $X, Y \in \chi(M)$. Clearly, if $b = 0$ then η-Einstein manifold reduces to Einstein manifold. Now considering $X = Y = \xi$ in the last equation and using (11), we have $a + b = -2n$. Contracting (38) over X and Y we get $r = (2n + 1)a + b$, where r denotes the scalar curvature of the manifold. Solving the last two equations, we get $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values we can rewrite (38) as

$$S(X, Y) = (1 + \frac{r}{2n})g(X, Y) - (2n + 1 + \frac{r}{2n})\eta(X)\eta(Y). \tag{39}$$

Theorem 3.3. *Let M be a (2n+1)-dimensional η-Einstein para-Kenmotsu manifold where $n > 1$. If the metric of the manifold represents a conformal Ricci soliton, then the manifold is Einstein.*

Proof. Let the metric g of an η-Einstein para-Kenmotsu manifold M whose dimension is greater than 3 represents a conformal Ricci soliton. Then clearly it satisfies (1) as well as (39). Combining these two relations, we have

$$(\mathcal{L}_V g)(Y,Z) = (p - 2\lambda - \frac{r}{n} - \frac{4n}{2n+1})g(Y,Z) + (4n + 2 + \frac{r}{n})\eta(Y)\eta(Z) \tag{40}$$

for all $Y, Z \in \chi(M)$. Covariant derivative of (40) with respect to an arbitrary vector field X on M and use of (18), leads to

$$\begin{aligned} g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y) =& (4n + 2 + \frac{r}{n})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] \\ &- \frac{Xr}{n}[g(Y,Z) + \eta(Y)\eta(Z)] \end{aligned} \tag{41}$$

for any vector fields X, Y and Z on M. By straightforward computation of the last equation, keeping the symmetry of $(\mathcal{L}_V\nabla)$ in mind, provides

$$\begin{aligned} 2n(\mathcal{L}_V\nabla)(X,Y) \quad = \quad & (Xr)\eta(Y)\xi - (Xr)Y + (Yr)\eta(X)\xi - (Yr)X + (Dr)g(X,Y) - (Dr)\eta(X)\eta(Y) \\ & +2(4n^2 + 2n + r)[g(X,Y)\xi - \eta(X)\eta(Y)\xi], \end{aligned} \tag{42}$$

where Dr is the gradient of r. Let us consider a local orthonormal basis of the manifold as $\{e_i\}_{i=1}^{2n+1}$. Next, setting $X = Y = e_i$ and summing over $1 \le i \le 2n+1$ in the last equation, we infer

$$n(\mathcal{L}_V\nabla)(e_i, e_i) = (\xi r)\xi + (n-1)Dr + 2n(4n^2 + 2n + r)\xi. \tag{43}$$

After considering $X = Y = e_i$ and summing over i, (28) reduces to $g((\mathcal{L}_V\nabla)(e_i, e_i), Z) = Zr - \frac{1}{2}Zr - \frac{1}{2}Zr{=}0$. Since this holds for an arbitrary vector field Z, this can be rewritten as

$$(\mathcal{L}_V\nabla)(e_i, e_i) = 0. \tag{44}$$

Comparing (43) and (44), we get $(\xi r)\xi + (n-1)Dr + 2n(4n^2 + 2n + r) = 0$. Taking inner product with ξ this implies that

$$\xi r = -2(4n^2 + 2n + r). \tag{45}$$

Again it further implies that $Dr = (\xi r)\xi$. Next substituting Y by ξ in (42), we get

$$2n(\mathcal{L}_V\nabla)(X,\xi) = (\xi r)(-X + \eta(X)\xi). \tag{46}$$

Covariant derivative of the foregoing equation with respect to an arbitrary vector field Y and using (9), (10) and (46), leads to

$$2n(\nabla_Y\mathcal{L}_V\nabla)(X,\xi) = (Y(\xi r))(-X+\eta(X)\xi) - 2n(\mathcal{L}_V\nabla)(X,Y) + (\xi r)[g(X,Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \tag{47}$$

Using the relation (47) in (20), we achieve

$$2n(\mathcal{L}_V R)(X,Y)\xi = (X(\xi r))(-Y + \eta(Y)\xi) - (Y(\xi r))(-X + \eta(X)\xi) + 2(\xi r)(\eta(Y)X - \eta(X)Y). \tag{48}$$

Contracting this over X, we have $(\mathcal{L}_V S)(Y,\xi) = 0$, where we have used $Dr = (\xi r)\xi$. Finally using $(\mathcal{L}_V S)(Y,\xi) = 0$, (39) and (40) in the Lie derivative of $S(Y,\xi) = -2n\eta(Y)$, we obtain

$$2n\Big(p - 2\lambda - \frac{4n}{2n+1} + 4n + 2\Big)\eta(Y) + \Big(1 + 2n + \frac{r}{2n}\Big)g(Y, \mathcal{L}_V\xi) = \Big(2n + 1 + \frac{r}{2n}\Big)\eta(Y)\eta(\mathcal{L}_V\xi) \tag{49}$$

for any vector field Y on M. Taking $Y = \xi$ in the last equation, we get $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Setting $Y = Z = \xi$ in (40) and using the value of λ, we obtain $\eta(\mathcal{L}_V\xi) = 0$. Using these two relations, the equation (49) can be written as

$$(2n(2n+1) + r)\mathcal{L}_V\xi = 0. \tag{50}$$

We suppose $r \neq -2n(2n+1)$ on some open set O of M. Then (50) implies that $\mathcal{L}_V\xi = 0$, which further implies with help of (9) that $\nabla_\xi V = V - \eta(V)\xi$. Using these relations along with (9), (40) and (46) in (37) we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature is constant. So, from (45), we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (50), we can infer $r \neq -2n(2n+1)$ on the entire manifold. Finally from (39), we have $S(X,Y) = -2ng(X,Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

4. A para-Kenmotsu metric as ∗-conformal Ricci soliton

In this section we assume that the metric of para-Kenmotsu manifold represents a ∗-conformal Ricci soliton. Venkatesha, Kumara and Naik[27] have deduced the expression of ∗-Ricci tensor for para-Kenmotsu manifold as

$$S^*(X,Y) = -S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y) \tag{51}$$

for all vector fields X and Y on M.

Theorem 4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g), n > 1$ be a η-Einstein para-Kenmotsu manifold. If g represents a ∗-conformal Ricci soliton, then the manifold is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. Let M be a $(2n+1)$-dimensional η-Einstein para-Kenmotsu manifold of dimension > 3 whose metric g represents a ∗-conformal Ricci soliton. So, the relations (2), (39) and (51) are satisfied. Rewriting (2) with the help of the rest two relations, we have

$$(\mathcal{L}_V g)(Y,Z) = (p - 2\lambda + \frac{r}{n} + 4n + \frac{2}{2n+1})g(Y,Z) - (4n + \frac{r}{n})\eta(Y)\eta(Z) \tag{52}$$

for all $Y, Z \in \chi(M)$. Differentiating the above equation with respect to an arbitrary vector field X of M and using (10), we achieve

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = \frac{Xr}{n}g(Y,Z) - \frac{Xr}{n}\eta(Y)\eta(Z) - (4n + \frac{r}{n})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] \tag{53}$$

for any vector fields X, Y and Z of M. Again from (18), we know $(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y)$. Using this and by a combinatorial computation, keeping in mind that $\mathcal{L}_V\nabla$ is a symmetric operator, the foregoing equation gives

$$\begin{aligned} 2n(\mathcal{L}_V\nabla)(X,Y) &= (Xr)[Y - \eta(Y)\xi] + (Yr)[X - \eta(X)\xi] - (Dr)[g(X,Y) - \eta(X)\eta(Y)] \\ &\quad -2(4n^2 + r)[g(X,Y) - \eta(X)\eta(Y)]\xi. \end{aligned} \tag{54}$$

The covariant derivative of (2) with respect to an arbitrary vector field X, yields

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = -2(\nabla_X S^*)(Y,Z). \tag{55}$$

The straightforward computation and use of the relation (18) in the equation (55), leads to

$$g((\mathcal{L}_V\nabla)(X,Y),Z) = (\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(Z,X). \tag{56}$$

Again, taking covariant derivative of (51) with respect to an arbitrary vector field Z of M and then using (10), we get

$$(\nabla_Z S^*)(X,Y) = -(\nabla_Z S)(X,Y) - g(X,Z)\eta(Y) - g(Y,Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z). \tag{57}$$

Combining (57) with (56), yields

$$g((\mathcal{L}_V\nabla)(X,Y),Z) = (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) - (\nabla_Z S)(X,Y) + 2g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z). \tag{58}$$

Now, let us consider a local orthonormal basis $\{e_i\}_{i=1}^{2n+1}$ of the manifold. Replacing $X = Y = e_i$ in (54), we have

$$2n(\mathcal{L}_V\nabla)(e_i, e_i) = -2(\xi r)\xi - 2(n-1)(Dr) - 4n(4n^2 + r)\xi. \tag{59}$$

Again, substituting X and Y by e_i in equation (58) and summing over i, we get

$$(\mathcal{L}_V\nabla)(e_i, e_i) = 4n\xi. \tag{60}$$

Combining the above two relations we directly have

$$(\xi r)\xi + (n-1)(Dr) + 2n(4n^2 + 2n + r)\xi = 0. \tag{61}$$

The inner product with respect to ξ, reduces the aforementioned equation to $\xi r = -2(2n(2n+1) + r)$. As $n > 1$, using this relation in the equation (61) we easily obtain $Dr = (\xi r)\xi$. After substituting Y by ξ in (54) and using (3), we infer

$$2n(\mathcal{L}_V\nabla)(X, \xi) = (\xi r)\phi^2(X) \tag{62}$$

for all $X \in \chi(M)$. Differentiating (62) with respect to an arbitrary vector field Y and using (9), (10) and (62), we get

$$2n(\nabla_Y\mathcal{L}_V\nabla)(X, \xi) + 2n(\mathcal{L}_V\nabla)(X, Y) = (Y(\xi r))\phi^2X - (\xi r)[g(X, Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \tag{63}$$

Using this in the well known formula (20), we have

$$2n(\mathcal{L}_VR)(X, Y)\xi = (X(\xi r))\phi^2Y - (Y(\xi r))\phi^2X - 2(\xi r)[\eta(Y)X - \eta(X)Y] \tag{64}$$

for all $X, Y \in \chi(M)$. Contracting the above equation over X and using the relation $Dr = (\xi r)\xi$, we have $(\mathcal{L}_VS)(Y, \xi) = 0$. Using (39), (52) and $(\mathcal{L}_VS)(Y, \xi) = 0$ in the Lie derivative of $S(Y, \xi) = -2n\eta(Y)$, we get

$$2n\Big(p - 2\lambda + \frac{2}{2n+1}\Big)\eta(Y) + \Big(2n + 1 + \frac{r}{2n}\Big)[g(Y, \mathcal{L}_V\xi) - \eta(Y)\eta(\mathcal{L}_V\xi)] = 0. \tag{65}$$

In the last equation considering $Y = \xi$, we obtain $\lambda = \frac{p}{2} + \frac{1}{2n+1}$ as $n > 1$. Again setting $Y = Z = \xi$ in (52), we have $\eta(\mathcal{L}_V\xi) = 0$. Applying these relations, we can rewrite (65) as

$$(2n(2n+1) + r)\mathcal{L}_V\xi = 0. \tag{66}$$

We suppose $r \neq -2n(2n+1)$ on some open set O of M. Then from (66), directly we obtain $\mathcal{L}_V\xi = 0$. From (9), we deduce that $\nabla_\xi V = V - \eta(V)\xi$. Again taking $Z = \xi$ in (52) and using $\lambda = \frac{p}{2} + \frac{1}{2n+1}$, we have $\mathcal{L}_V\eta = 0$. Using these relations along with (9) and (62) in the identity (37), we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature r is constant. So, from the relation $\xi r = -2(2n(2n+1) + r)$, we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (66), we can conclude that $r = -2n(2n+1)$ on the entire manifold M. Moreover from (39), we have $S(X, Y) = -2ng(X, Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

5. A 3-dimensional para-cosymplectic metric as conformal Ricci soliton

In 2004, Dacko [7] introduced the notion of para-cosymplectic manifold. The fundamental 2-form Φ is defined on an almost paracontact metric manifold (M, ϕ, ξ, η, g) by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M. Clearly the skew-symmetricness of the 2-form Φ inherits from ϕ.

An almost paracontact metric manifold is said to be almost para- coymplectic if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$ respectively. In addition if the normality of almost para-cosymplectic manifold is fulfilled then the it is called para-cosymplectic manifold. Equivalently we can say an almost paracontact metric manifold is para-cosymplectic if the forms η and Φ are parallel with respect to the corresponding Levi-Civita connection ∇ of the metric g i.e., $\nabla\eta = 0$ and $\nabla\Phi = 0$ respectively. We recall some useful relations which are satisfied for any para-cosymplectic manifold.

$$R(X,Y)\xi = 0, \tag{67}$$
$$(\nabla_X\phi) = 0, \tag{68}$$
$$\nabla_X\xi = 0, \tag{69}$$
$$S(X,\xi) = 0, \tag{70}$$
$$Q\xi = 0, \tag{71}$$

where X is an arbitrary vector field and R, ∇, S and Q are the usual notations. For the 3-dimensional case, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{72}$$

Using this result we deduce that 3-dimensional para-cosymplectic manifold satisfies

$$S(X,Y) = \frac{r}{2}[g(X,Y) - \eta(X)\eta(Y)], \tag{73}$$
$$QX = \frac{r}{2}[X - \eta(X)\xi] \tag{74}$$

for any $X, Y \in \chi(M)$.

A vector field V is said to be conformal Killing vector field or simply conformal vector field if there is a smooth function ρ such that

$$\mathcal{L}_V g = 2\rho g. \tag{75}$$

ρ is called the conformal coefficient. If we consider the conformal coefficient ρ to be zero then the conformal vector field reduces to Killing vector field. Now we first prove some lemmas whose results are used to deduce our main result.

Lemma 5.1 ([29]). *If a n-dimensional Riemannian manifold admits a conformal vector field V then we have*

$$(\mathcal{L}_V S)(X,Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X,Y), \tag{76}$$
$$\mathcal{L}_V r = 2(n-1)\Delta\rho - 2\rho r \tag{77}$$

for any vector fields X and Y, where D and Δ denote the gradient and Laplacian operator of g respectively and r represents the scalar curvature of the manifold.

Lemma 5.2. *If the metric g of a 3-dimensional para-cosymplectic manifold represents a conformal Ricci soliton then the following properties hold*

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}, \tag{78}$$
$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}. \tag{79}$$

Proof. As the vector field ξ is a unit vector field, we have $g(\xi,\xi) = 1$. Taking Lie derivative of the previous relation with respect to vector field V, we have $(\mathcal{L}_V g)(\xi,\xi) + 2\eta(\mathcal{L}_V\xi) = 0$. Using (1), (4) and (73), we acquire

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}.$$

Taking Lie derivative of (4) along the direction of the vector field V and using (78), we achieve

$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}.$$

□

Lemma 5.3. *For a 3-dimensional para-cosymplectic manifold, we have*

$$\xi(r) = 0. \tag{80}$$

Proof. For proof we refer to [9]. □

Theorem 5.4. *If the metric g of a 3-dimensional para-cosymplectic manifold $(M^3, \phi, \xi, \eta, g)$ which admits a conformal vector field V, represents a conformal Ricci soliton then the scalar curvature of the manifold is Harmonic and the manifold is Ricci flat.*

Proof. Combining (1) and (75) for 3-dimensional para-cosymplectic manifold, we have

$$\Big(2\rho + 2\lambda - p - \frac{2}{3}\Big)g(X, Y) + 2S(X, Y) = 0$$

for any $X, Y \in \chi(M)$. Contracting the above equation, we get

$$\rho = \frac{1}{6}(3p - 6\lambda - 2r + 2). \tag{81}$$

Using (81) in (76) and (77), we get

$$(\mathcal{L}_V S)(X, Y) = \frac{1}{3}g(\nabla_X Dr, Y) - \frac{1}{3}(\Delta r)g(X, Y), \tag{82}$$

$$\mathcal{L}_V r = -\frac{1}{3}(3p - 6\lambda - 2r + 2)r - \frac{4}{3}(\Delta r). \tag{83}$$

Taking Lie derivative of (73) in the direction of the vector field V and using (1), (73), (82) and (83), we have

$$g(\nabla_X Dr, Y) = -\Big(\Delta r + \frac{r^2}{2}\Big)g(X, Y) + \Big[\frac{r}{2}(3p - 6\lambda + r + 2) + 2(\Delta r)\Big]\eta(X)\eta(Y) - \frac{3r}{2}\Big[((\mathcal{L}_V\eta)X)\eta(Y) + \eta(X)((\mathcal{L}_V\eta)Y)\Big]. \tag{84}$$

Covariant derivative of (80) along an arbitrary vector field X, yields $g(\nabla_X Dr, \xi) = 0$. Now setting $X = Y = \xi$ in the equation (84) and using the aforementioned relation along with the equation (79), we get

$$\Delta r = 0. \tag{85}$$

Hence the scalar curvature r of the manifold is Harmonic.
Now considering $Y = \xi$ in (84) and using the relation $g(\nabla_X Dr, \xi) = 0$, (85), (79), we obtain the following relation

$$r((\mathcal{L}_V\eta)X) = r\Big(\frac{p}{2} + \frac{1}{3} - \lambda\Big)\eta(X) \tag{86}$$

for an arbitrary vector field X on M. Making use of the last equation, (74) and (85) in (84), we achieve

$$\nabla_X Dr = -rQX \tag{87}$$

for any arbitrary $X \in \chi(M)$. Now contracting it with respect to X, we get $\Delta r = -r^2$ and combining with (85), we infer $r = 0$ i.e., the manifold is Ricci flat. □

6. Examples

In this section we provide some examples to verify our outcomes.

Example 6.1. *We consider the manifold as $M = \{(x, y, x) \in \mathbb{R}^3\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields are defined by*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point on M. The metric g is defined by

$$g(e_1, e_1) = g(e_3, e_3) = 1, \qquad g(e_2, e_2) = -1, \qquad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let $\xi = e_3$. Then the 1-form η is defined by $\eta(X) = g(X, e_3)$, for arbitrary $X \in \chi(M)$, then we have the following relations

$$\eta(e_1) = 0, \qquad \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field ϕ as

$$\phi e_2 = e_1, \qquad \phi e_1 = e_2, \qquad \phi e_3 = 0,$$

then it satisfies

$$\begin{aligned} \phi^2(X) &= X - \eta(X)e_3, \\ g(\phi X, \phi Y) &= -g(X, Y) + \eta(X)\eta(Y) \end{aligned}$$

for arbitrary $X, Y \in \chi(M)$. Thus (ϕ, ξ, η, g) defines an almost paracontact metric structure on M. We can now easily conclude

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_3] = e_1.$$

Let ∇ be the Levi-Civita connection of g. Then the Koszul's formula for arbitrary $X, Y, Z \in \chi(M)$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using this we obtain

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = e_3, && \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

From here we can easily verify that the relation (6) is satisfied. Hence the considered manifold is para-Kenmotsu manifold. The components of the Riemannian curvature tensor are given by

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2, && R(e_1, e_2)e_2 = e_1, && R(e_1, e_2)e_3 = 0, \\ &R(e_1, e_3)e_1 = e_3, && R(e_1, e_3)e_2 = 0, && R(e_1, e_3)e_3 = -e_1, \\ &R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -e_3, && R(e_2, e_3)e_3 = -e_2. \end{aligned}$$

And the components of Ricci tensor and $*$-Ricci tensor are given by

$$\begin{aligned} &S(e_1, e_1) = -2, && S(e_2, e_2) = 2, && S(e_3, e_3) = -2, \\ &S^*(e_1, e_1) = 1, && S^*(e_2, e_2) = -1, && S^*(e_3, e_3) = 0. \end{aligned}$$

From here we can easily deduce that the scalar curvature of the manifold $r = -6$ and $S(X, Y) = -2g(X, Y)\ \forall X, Y \in \chi(M)$. Let us define a vector field by

$$V = (x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Then we can obtain

$$(\mathcal{L}_V g)(e_1, e_1) = 2, \qquad (\mathcal{L}_V g)(e_2, e_2) = -2, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (1) and using the result $r = -6$ we deduce $\lambda = \frac{p}{2} + \frac{19}{3}$. So g defines a conformal Ricci soliton on this para-Kenmotsu manifold for $\lambda = \frac{p}{2} + \frac{19}{3}$.
Again Contracting (51) we get, $r^* = -r - 4 = 2$ (as $r = -6$). Now contracting (2) and using the previous result we obtain $\lambda = \frac{p}{2} - \frac{5}{3}$. So, g defines a $*$-conformal Ricci soliton on this para-Kenmotsu manifold for $\lambda = \frac{p}{2} - \frac{5}{3}$.

Example 6.2. *Let us consider the set $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ as our manifold where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below*

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v},$$

are linearly independent at each point of M. We define the metric g as

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 5\} \\ -1, & \text{if } i = j \text{ and } i, j \in \{3, 4\} \\ 0, & \text{otherwise.} \end{cases}$$

Let η be a 1-form defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as

$$\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = e_1, \qquad \phi(e_4) = e_2, \qquad \phi(e_5) = 0.$$

Then it satisfies the relations $\phi^2(X) = X - \eta(X)\xi$ and $\eta(\xi) = 1$, where $\xi = e_5$ and X is an arbitrary vector field on M. So, (M, ϕ, ξ, η, g) defines an almost paracontact structure on M.
We can now deduce that

$$\begin{array}{llll} [e_1, e_2] = 0, & [e_1, e_3] = 0, & [e_1, e_4] = 0, & [e_1, e_5] = e_1, \\ [e_2, e_1] = 0, & [e_2, e_3] = 0, & [e_2, e_4] = 0, & [e_2, e_5] = e_2, \\ [e_3, e_1] = 0, & [e_3, e_2] = 0, & [e_3, e_4] = 0, & [e_3, e_5] = e_3, \\ [e_4, e_1] = 0, & [e_4, e_2] = 0, & [e_4, e_3] = 0, & [e_4, e_5] = e_4, \\ [e_5, e_1] = -e_1, & [e_5, e_2] = -e_2, & [e_5, e_3], = -e_3, & [e_5, e_4] = -e_4. \end{array}$$

Let ∇ be the Levi-Civita connection of g. Then Koszul's formula is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

for arbitrary $X, Y, Z \in \chi(M)$. Using this we get

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2, = 0 & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = e_5, & \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) is an almost para-Kenmotsu manifold. The previous outcomes can easily be verified using this example.

7. Conclusion

In this article, we have used the methods of local Riemannian or semi-Riemannian geometry to interpretation solutions of (1) and (2) and impregnate Einstein metrics in a large class of metrics of conformal Ricci solitons and $*$-conformal Ricci solitons on paracontact geometry, specially on para-Kenmotsu and para-cosymplectic manifold. Our results will not only play an indispensable and incitement role in paracontact geometry but also it has significant and motivational contribution in the area of further research of complex geometry, specially on Kähler and para-Kähler manifold etc. and we can think about the physical interpretation of conformal Ricci solitons and $*$-conformal Ricci solitons also in differential geometry. There are some questions which arise from our article to study in further research:

(i) Are the results of theorem 3.2 and theorem 3.3 true if we assume the dimension of the manifold as 3?
(ii) Does theorem 4.1 hold without assuming η-Einstein condition?
(iii) If we consider the dimension more than 3, then is theorem 5.4 true?
(iv) What can we say about theorem 5.4 if we assume vector field V is not conformal?
(v) Which results of the our paper are also true in nearly Kenmotsu manifolds or f-Kenmotsu manifolds or f-cosymplectic manifolds?

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