Analysis of a Tritrophic Food Chain Model with Fear Effect Incorporating Prey Refuge

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Abstract. In this work, we have introduced a tritrophic food-chain model where consumer hunt for prey with Holling type-III functional response. The birth rate of the prey population has been reduced due to the fear of predation, i.e., a fear effect is considered in the prey population. Moreover, a fraction of the prey is available to the consumer for consumption and this has been done by incorporation of prey refuge term. The predation between consumer and predator follows Beddington-DeAngelis response. Boundedness and positivity of the system prove that the proposed model is well-posed. Also, there are some parametric restrictions under which the system is permanent. Routh-Hurwitz criterion shows the local stability conditions of the equilibrium points and on the other hand Lyapunov LaSalle theorem guarantees that the locally stable equilibrium points are globally stable. Also, Matlab validates the analytical results with the help of diagrams. The occurrence of transcritical bifurcations have been shown and conditions for the existence of a limit cycle in the system through Hopf bifurcation also have been stated. Both the analytical and numerical results suggest that a certain amount of fear can make the system steady. It is also noted that the prey refuge has both stabilizing and destabilizing effect on the system.

1. Introduction

The most important biological processes in ecology and population biology is the interactions between species with their corresponding environment [9, 27, 34, 42, 47] and mathematical modelling is a useful tool to investigate the insight of these biological processes. So, the researchers from many years are developing various models to study their behaviours [5, 8, 17, 29, 32].History says, in the population biology, the first model was innovated by Malthus [26] and it was modified later by Verhulst [48]. Lotka and Volterra first independently demonstrated the prey-predator interaction in biological populations [25, 49] with Malthus growth term for the prey population and a Holling type-I functional response for the predator [16] and Rosenzweig and MacArthur [37] later studied such model to present it as a more realistic predator-prey system. Later researchers have modified the basic model with various types of functional responses to study the characteristics of tri-trophic food chain model [11, 14].

Predation rate plays a vital role in the prey-predator dynamics. Holling type-I functional response and Holling type-II functional response have been used mostly in the models. Holling type III functional response is almost similar to type II response at high prey density as for both cases saturation occurs. But at low prey density levels, it is assumed that the number of prey consumed may not follow a linearly increasing curve with prey density. This particular functional response is defined with the help of either learning time or prey switching or a combination of both phenomena. The natural improvement of a predator's searching efficiency or even of their handling efficiency with increasing prey density is defined as learning time. If there is a small amount of prey present in a system, then the predator finds it difficult to search a sufficient amount of prey and they have to develop the best ways to capture the prey. This mechanism can be observed in shrews and deer mice feeding on sawflies. At lower numbers of sawfly cocoons, per capita growth rate of deer mice follows the exponential rule as the density of cocoons increased but at a certain density of cocoons, the consumption rate of the deer mice reached a saturation amount.

Holling at the time of introducing the functional response assumed that there was no interference of the predator in one another's activities and so competition among predators for food occurs only via the depletion or consumption of resource population [16]. But in 1975, Beddington and DeAngelis et al. independently proposed, a functional response that considered the mutual interference among predator (Huisman and De Boer [20] provides mathematical derivation). In their model, it is considered that two or more predator population not only spend time to search for and process the resources but also used some time encountering between predators. All these assumptions give a functional response as: $f(N, P) = \frac{aN}{1+bN+cP}$ where P is the predator population and c is a positive constant indicating the magnitude of interference among predators. The Beddington-DeAngelis functional response is almost same as Holling type-II functional response containing an extra term which denotes mutual interference between predators.

Inclusion of spatial prey refuge can make a predator-prey system more realistic as it protects a constant proportion of prey from predation. Many research works [3, 19, 21, 38, 39, 45] have been done to observe how the prey refuge controls the system dynamics of predator-prey system. It is evident that prey species can save themselves from extinction if they successfully hide, i.e., they have a safe place to physically hide from predation. Hassel [15], in his work, showed that if a system exhibits oscillating behaviour in absence of refuge, then the inclusion of a large refuge to the model can replace the oscillatory behaviour with an asymptotically stable equilibrium. It indicates that prey refugia may have a stabilizing effect on predator-prey dynamics.

In ecology, predator-prey interaction is the most important factor as it maintains the biomass flow from one trophic to other trophic levels, as well as regulates the overall population size. The predator may have an effect on the prey population in a direct or indirect or both way. For direct effect, the predator consumes prey [46] but in the case of indirect effect, predator creates fear in prey population and force them to change their behaviour [24]. The fear effect of prey is a manifestation of sustained psychological stress because the prey species are always worried about a possible attack. In fact, in some cases fear effect work higher than direct killing to reduce prey or extinction of prey when the direct predation is absent. For example, Pangle et al., in 2007, estimated the effects of predatory spiny water fleas *(Bythotrephes longimanus)* on three different species of zooplankton in Lake Erie and Lake Michigan [30]. Their overall experiment showed that fear effects worked more than seven times higher to reduce the growth rate than the effect of direct predation. Some important aspects of prey behaviour that can change due to fear are hunting and reproduction [33, 40, 54]. Prey animals always try to shift to such places where their predation risk is lower but the availability of food is higher [35, 51]. Research reveals that due to fear of predator, prey population forage in a less amount. For example, due to fear of older cannibalistic backswimmers, younger backswimmers always try to keep themselves in a safe distance and feed in a safer place and also hunt less [43]. Also, Mule deer spend less time for hunting because there is a predation risk of mountain lions [1]. Research proves that perceived predation risk affects the reproduction rate of the scared prey and Candolin showed an interesting result in his work regarding this fact. He showed that three-spine stickleback males are able to assess both the risk of predation as well as current versus future mating probability depending on which they have to adjust their reproductive decisions [2]. Creel et al. observed that Elk changes their reproductive physiology due to the predation risk of wolves [4]. The focus of research is changed these days as the recent results prove that indirect approach on prey population is even more powerful to reduce

the population than the direct killing [36, 44, 50, 54].

Research works have proved that fear effect is strong enough to affect the population dynamics in ecological systems [44, 54]. Based on the experiments performed by Zanette et al. [54] it was observed that the fear effect on song sparrows during the entire breeding season affected the birds' reproduction even when direct killing was excluded. In these experiments, direct predation was not considered as every nest was protected with electric fencing and also seine netting and then only a recording of predator calls was used many weeks before the first egg was laid and the broadcasts were continued for 130-days breeding season. They observed that the numbers of eggs, hatchings, and even fledgelings were reduced in the next generations. It was observed that fear of predators itself reduces almost 40% reproduction in the number of offspring. Hua et al. [18], in their work, controlled vocal signals of predators and observed that bluebirds which are reproducing can adjust their breeding strategies according to the signals. Experimental results suggest that the effect of fear can control the populations like snowshoe hares [41] and dugongs [52] also. Furthermore, Laundre et al. [23] showed that releasing wolves into Yellowstone Park made the moose more alert.

In the proposed tritrophic food chain system, the main assumption is motivated by real-life biological examples. In the aquatic ecosystem, it is observed that in the absence of predators like largemouth bass, trout, turtle, etc., the average growth rate of bluegill can be increased by 27% than in its presence [53]. On the other hand, fish kairomone (bluegill sunfish) reduces the growth rate of juvenile cladocerans like Daphnia and Simocephalus [13]. In ecology, research works proved that playback of predator calls or sounds of predators (raccoon, owl, hawk etc.) during the entire breeding season of female song sparrows reduced the number of eggs and etc. even in the absence of direct killing. Moreover, mesocarnivores (raccoons) reduce their foraging activities by 66% due to the fear of large carnivores (cougar, wolf, black bear) [44].

This work is organized as follows: section 2 describes the mathematical formulation along with positive initial conditions while section 3 shows that the model is well-posed. Extinction conditions for the prey, predator and top-predator have been analyzed in section 4 and equilibria with feasibility conditions are stated in Section 5. Local stability analysis and persistence of the system are described in sections 6 and section 7 respectively. The equilibria change their stability through transcritical and Hopf bifurcation which are analyzed in section 8. Section 9 gives the global stability of the equilibrium points and section 10 provides the numerical figures which support the analytical calculations. The last section provides a brief conclusion about the system dynamics.

2. Mathematical Model: Basic Equations

Modelling a biological system in terms of mathematical equations is an easy way to obtain the basic dynamics of the system. In this work, the main purpose is to observe the system dynamics of a tritrophic food chain model in the presence of fear. It is also true that any ecological system is not so easy to describe in terms of mathematics as there are many factors present in the system which can fluctuate with time to regulate the dynamics. So, for the sake of simplicity, we need to make some assumptions to reduce complex dynamics into a simpler model.

The system is a three species food chain model consists of a prey (resource) population (X), middle-predator (consumer) population (Y) and top-predator (or, simply predator) population (Z). We incorporate the cost of fear in the prey population only. Due to fear of middle-predator, the birth rate of the prey population reduces. The modified birth rate of prey may be taken as $\dfrac{RX}{1 + K_1 Y}$ [6, 7, 28, 50], which is a monotonic decreasing function of K_1 and Y. In the model, it is assumed that the predation between prey and middle-predator follows Holling type-III functional response. Also, the predation rate of top-predator depends not only on middle-predator but also on the interference of the top-predator population. Hence we have taken Beddington-DeAngelis functional response to describe the predation term of top-predator on consumer. It is assumed that when the middle-predator is not present, the prey population grows in a logistic way with intrinsic birth rate R, carrying capacity K and natural death rate D. The parameters C and P represent the coefficients of predation of consumer on prey species and top-predator on consumer species respectively.

As the middle-predator need some time to search and handle the prey species, let T_h be the average handling time of middle-predator for each prey. The terms ξ ($0 < \xi < 1$) and σ ($0 < \sigma < 1$) are the biomass conversion coefficients and G, N are the death rates of the middle and top predators. The parameter E is a saturation constant and F is the impact of top-predator interference. The term K_1 (≥ 0) represents the level of fear that rises the anti-predator behaviour of prey [50]. So, $K_1 = 0$ represents the absence of fear among the individuals of prey popoulation. Here, it is assumed that $R > D$, otherwise if $R \leq D$, then the growth rate of prey population becomes negative and ultimately the prey population go extinct. Also, mX is taken as the capacity of refuge at time T, i.e., mX of prey able to protect themselves in a safer place ($0 < m < 1$). Therefore, $(1 - m)X$ of prey available to the middle-predator for consumption.

So, considering all the above assumptions, we get the system as:

$$
\begin{aligned}
\frac{dX(T)}{dT} &= \frac{RX}{1 + K_1 Y} - DX - \frac{(R - D)}{K}X^2 - \frac{C(1 - m)^2 X^2 Y}{1 + T_h C(1 - m)^2 X^2}, \quad X(0) > 0, \\
\frac{dY(T)}{dT} &= \frac{\xi C(1 - m)^2 X^2 Y}{1 + T_h C(1 - m)^2 X^2} - \frac{PYZ}{1 + EY + FZ} - GY, \quad Y(0) > 0, \\
\frac{dZ(T)}{dT} &= \frac{\sigma PYZ}{1 + EY + FZ} - NZ, \quad Z(0) > 0.
\end{aligned}
\tag{2.1}
$$

The model parameters $R, D, \xi, C, T_h, K, m, D, E, F, G, \sigma, N$ are all assumed as positive constants and $K_1 \geq 0$.

Using the scaling $x = \dfrac{(R - D)X}{RK}$, $y = \dfrac{(R - D)Y}{RK}$, $z = \dfrac{(R - D)Z}{RK}$ and $t=RT$, system (2.1) becomes

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \frac{x}{1 + ky} - dx - x^2 - \frac{c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2}, \quad x_0 = x(0) > 0, \\
\frac{dy(t)}{dt} &= \frac{\xi c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2} - \frac{pyz}{e + y + fz} - gy, \quad y_0 = y(0) > 0, \\
\frac{dz(t)}{dt} &= \frac{\sigma pyz}{e + y + fz} - nz, \quad z_0 = z(0) > 0,
\end{aligned}
\tag{2.2}
$$

where $k = \dfrac{KK_1 R}{R-D}$, $d = \dfrac{D}{R}$, $c = \dfrac{CRK^2}{(R-D)^2}$, $a = \dfrac{T_h CR^2 K^2}{(R-D)^2}$, $p = \dfrac{P}{ER}$, $e = \dfrac{(R-D)}{KER}$, $f = \dfrac{F}{E}$, $g = \dfrac{G}{R}$, $n = \dfrac{N}{R}$.

3. Positivity and Boundedness

Now we ensure the model (2.2) is well-posed by showing the positivity and boundedness of the system variables.

Theorem 3.1. *All solutions of system (2.2) which start in $\mathbb{R}_+^3$ are positive for all time.*

Proof. Right hand side of system (2.2) is continuous and locally Lipschitzian on C (space of continuous functions) and hence it implies the solution $(x(t), y(t), z(t))$ of (2.2) exists and is unique on the interval $[0, \kappa)$, where $0 < \kappa \leq +\infty$ [12]. From the first equation of (2.2), we get

$$
\frac{dx(t)}{dt} = \frac{x}{1 + ky} - dx - x^2 - \frac{c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2},
$$

$$
\text{i.e., } x(t) = x_0 \exp\left[\int_0^t \left\{\frac{1}{1 + ky(s)} - d - x(s) - \frac{c(1 - m)^2 x(s)y(s)}{1 + a(1 - m)^2 x(s)^2}\right\} ds\right]
$$
$$
> 0, \text{ for } x_0 > 0.
$$

Similarly,

$$y(t) = y_0 \exp\left(\int_0^t \left[\frac{\xi c(1-m)^2 x^2(s)}{1 + a(1-m)^2 x(s)^2} - \frac{pz(s)}{e + y(s) + fz(s)} - g\right] ds\right) > 0, \text{ for } y_0 > 0,$$

$$z(t) = z_0 \exp\left(\int_0^t \left[\frac{\sigma p y(s)}{e + y(s) + fz(s)} - n\right] ds\right) > 0, \text{ for } z_0 > 0.$$

□

Theorem 3.2. *All solutions of system* (2.2) *which start in* $\mathbb{R}_+^3$ *are uniformly bounded.*

Proof. From the first equation of (2.2):

$$\begin{aligned}
\frac{dx(t)}{dt} &= \frac{x}{1 + ky} - dx - x^2 - \frac{c(1-m)^2 x^2 y}{1 + a(1-m)^2 x^2} \\
&\leq \frac{x}{1 + ky} - x^2 \\
&< x(1-x)
\end{aligned}$$

$$\Rightarrow \limsup_{t\to\infty} x(t) \leq 1.$$

Let, $N(t) = x(t) + \frac{1}{\xi} y(t) + \frac{1}{\xi\sigma} z(t)$

So,

$$\begin{aligned}
\frac{dN}{dt} &= \frac{dx}{dt} + \frac{1}{\xi}\frac{dy}{dt} + \frac{1}{\xi\sigma}\frac{dz}{dt} \\
&= \frac{x}{1 + ky} - dx - x^2 - \frac{g}{\xi}y - \frac{n}{\xi\sigma}z \\
&\leq x - \tau N, \text{ where } \tau = \min\{d, g, n\} \\
&< 1 - \tau N, \text{ (for large time)}
\end{aligned}$$

$$\therefore N(t) \leq \frac{1}{\tau}(1 - \exp(-\tau t)) + N(x_0, y_0, z_0)\exp(-\tau t).$$

As $t \to \infty$, $0 < N(t) \leq \frac{1}{\tau}$. Hence, all solutions of system (2.2) will enter into region:

$$\Delta = \left\{(x, y, z) : 0 \leq x(t) \leq 1; 0 \leq N(t) \leq \frac{1}{\tau} + \epsilon, \epsilon > 0\right\}.$$

□

4. Extinction Scenarios

　　This section provides the conditions for which prey and predators (both middle and top) will go extinct from the system in long time.

Let us adopt the following notations: $\bar{x} = \limsup_{t\to\infty} x(t)$; $\bar{y} = \limsup_{t\to\infty} y(t)$; $\bar{z} = \limsup_{t\to\infty} z(t)$. Similarly, $\underline{x} = \liminf_{t\to\infty} x(t)$; $\underline{y} = \liminf_{t\to\infty} y(t)$; $\underline{z} = \liminf_{t\to\infty} z(t)$.

Here we also use the facts that (for large time): $\bar{x} \leq 1$ and $\bar{y}, \bar{z} \leq M$ (say) (because all solutions are uniformly bounded).

The first two theorems give the extinction criterion of prey population while the later two theorems provide the extinction criterion of middle-predator population and the last theorem gives the condition for extinction of top-predator.

Theorem 4.1. *If $d > 1$, then $\lim_{t \to \infty} x(t) = 0$.*

Proof.

$$\frac{dx(t)}{dt} = \frac{x}{1 + ky} - dx - x^2 - \frac{c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2}$$
$$\leq (1 - d)x$$
$$< 0$$
$$\Rightarrow x(t) \to 0 \text{ as } t \to \infty.$$

□

Remark 4.2. *If $d > 1 \Leftrightarrow$ death rate of prey is greater than its birth rate, then the prey population will be washed out from the system as time goes (obeying Malthus growth law).*

Theorem 4.3. *If $1 + k\underline{y} > \frac{1}{d}$, then $\lim_{t \to \infty} x(t) = 0$.*

Proof. Choose ϵ such that for $0 < \epsilon < \underline{y} - \frac{(1-d)}{kd}$, there exists $T > 0$, s.t. $y(t) > \underline{y} - \epsilon$ for all $t > T$. For all $t > T$:

$$\frac{dx}{dt} = \frac{x}{1 + ky} - dx - x^2 - \frac{c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2}$$
$$< \left\{ \frac{1}{1 + ky} - d \right\} x$$
$$< \left\{ \frac{1}{1 + k(\underline{y} - \epsilon)} - d \right\} x$$
$$= -\mu x, \text{ where } \mu = d - \frac{1}{1 + k(\underline{y} - \epsilon)} > 0$$

Hence, $\lim_{t \to \infty} x(t) = 0$. □

Remark 4.4. *If the growth of prey population decreases due to fear effect of middle-predator and also the mortality rate of prey starts to increase, then ultimately the prey population will go extinct from the system in long run.*

Theorem 4.5. *If $\xi c(1 - m)^2 < g$, then $\lim_{t \to \infty} y(t) = 0$.*

Proof.

$$\frac{dy(t)}{dt} = \frac{\xi c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2} - \frac{pyz}{e + y + fz} - gy,$$
$$< \frac{\xi c(1 - m)^2 x^2 y}{1 + a(1 - m)^2 x^2} - gy$$
$$< \{\xi c(1 - m)^2 - g\} y$$
$$< 0$$
$$\Rightarrow y(t) \to 0 \text{ as } t \to \infty.$$

□

Remark 4.6. *If the growth rate of middle-predator (by consuming prey) fails to exceed the mortality rate of the population, then the middle-predator population will be washed out from the system in long run.*

Theorem 4.7. *If $\frac{p\underline{z}}{e+M+fM} > \xi c(1-m)^2 - g$, then $\lim\limits_{t\to\infty} y(t) = 0$.*

Proof. Choose ϵ such that $0 < \epsilon < \underline{z} - \frac{1}{p}\{\xi c(1-m)^2 - g\}\{e + M(1+f)\}$. Then there exists $T > 0$, such that $z > \underline{z} - \epsilon$, for all $t > T$.
For all $t > T$:

$$\frac{dy(t)}{dt} = \frac{\xi c(1-m)^2 x^2 y}{1 + a(1-m)^2 x^2} - \frac{pyz}{e + y + fz} - gy,$$

$$< \left\{ \xi c(1-m)^2 x^2 - \frac{pz}{e + y + fz} - g \right\} y,$$

$$< \left\{ \xi c(1-m)^2 - \frac{p(\underline{z}-\epsilon)}{e + M + fM} - g \right\} y,$$

$$= -\mu y, \text{ where } \mu = \frac{p(\underline{z}-\epsilon)}{e + M + fM} + g - \xi c(1-m)^2 > 0$$

Hence, $\lim\limits_{t\to\infty} y(t) = 0$. $\square$

Remark 4.8. *If the consumption rate of top-predator exceeds the overall middle-predator population, then the consumer will automatically be washed out from the system with time.*

Theorem 4.9. *If $\sigma p < n$, then $\lim\limits_{t\to\infty} z = 0$.*

Proof.

$$\frac{dz(t)}{dt} = \frac{\sigma p y z}{e + y + fz} - nz$$

$$< (\sigma p - n)z, \quad (\because e + y + fz > y)$$

$$< 0$$

$$\Rightarrow z(t) \to 0 \text{ as } t \to \infty.$$

$\square$

Remark 4.10. *If the mortality rate of top-predator is higher than its conversion rate, then the top-predator will ultimately go extinct from the system.*

5. Equilibrium Points

Here we obtain the equilibrium points of system (2.2) by solving the nullclines which are as follows:

1. Trivial Equilibrium Point: $E_0(0, 0, 0)$.
2. Axial Equilibrium Point: $E_1(1 - d, 0, 0)$.
3. Planar Equilibrium Point: $E_2(\widetilde{x}, \widetilde{y}, 0)$, where $\widetilde{x} = \frac{1}{1-m}\sqrt{\frac{g}{c\xi - ag}}$ and $\widetilde{y}$ is the positive solution of the equation:

$$A_1 y^2 + A_2 y + A_3 = 0$$

where $A_1 = kc(1-m)^2\widetilde{x}$, $A_2 = c(1-m)^2\widetilde{x} + k(d + \widetilde{x})\{1 + a(1-m)^2\widetilde{x}^2\}$ and $A_3 = (d + \widetilde{x} - 1)\{1 + a(1-m)^2\widetilde{x}^2\}$. If $A_3 > 0$, we won't get any feasible planer equilibrium point as A_1, A_2 are already positive for any feasible $\widetilde{x}$. So, we get only one planer equilibrium E_2 if $A_3 < 0$ along with $c\xi > ag$.

4. Interior Equilibrium Point $E^*(x^*, y^*, z^*)$ satisfies the following equations:

$$\frac{1}{1 + ky^*} - d - x^* - \frac{c(1 - m)^2 x^* y^*}{1 + a(1 - m)^2 x^{*2}} = 0,$$

$$\frac{\xi c(1 - m)^2 x^{*2}}{1 + a(1 - m)^2 x^{*2}} - \frac{pz^*}{e + y^* + fz^*} - g = 0,$$

$$\frac{\sigma p y^*}{e + y^* + fz^*} = n.$$

Solving we have:

$$y^* = \frac{-B_2 + \sqrt{B_2^2 - 4B_1 B_3}}{2B_1}, \quad z^* = \frac{1}{fn}[(\sigma p - n)y^* - ne]$$

where, $B_1 = kc(1 - m)^2 x^*$, $B_2 = c(1 - m)^2 x^* + k(d + x^*)\{1 + a(1 - m)^2 x^{*2}\}$, $B_3 = (d + x^* - 1)\{1 + a(1 - m)^2 x^{*2}\}$. Let, $P = \frac{p\sigma - n}{f\sigma} + g$. Then, x^* is the positive root of the equation:

$$G(x) \equiv C_1 x^5 + C_2 x^4 + C_3 x^3 + C_4 x^2 + C_5 x + C_6 = 0, \tag{5.1}$$

where

$$C_1 = f\sigma(1 - m)^4(Pa - \xi c)\{neka + f\sigma(Pa - \xi c)\},$$
$$C_2 = f\sigma(1 - m)^4(Pa - \xi c)\{nekda + f\sigma(d - 1)(Pa - \xi c)\},$$
$$C_3 = n^2 e^2 kca(1 - m)^4 + nef\sigma(1 - m)^2[Pka + (Pa - \xi c)\{k + c(1 - m)^2\}]$$
$$\quad + 2f^2\sigma^2 P(1 - m)^2(Pa - \xi c),$$
$$C_4 = nef\sigma kd(1 - m)^2(2Pa - \xi c) + 2Pf^2\sigma^2(d - 1)(1 - m)^2(Pa - \xi c),$$
$$C_5 = n^2 e^2 kc(1 - m)^2 + nef\sigma P\{k + c(1 - m)^2\} + f^2 P^2\sigma^2,$$
$$C_6 = f\sigma P\{nekd + f\sigma P(d - 1)\}$$

For E^*, y^* exists only when $d + x^* < 1$ and z^* is feasible when $y^* > \frac{ne}{\sigma p - n}$.

6. Local Stability Analysis

This section contains the local stability criterion of the equilibrium points which can be determined by the eigenvalues of the corresponding Jacobian matrices and applying Routh-Hurwitz criterion. Now, the Jacobian matrix of system (2.2) is

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{6.1}$$

where $a_{11} = \frac{1}{1 + ky} - d - 2x - \frac{2c(1 - m)^2 xy}{\{1 + a(1 - m)^2 x^2\}^2}$; $\quad a_{12} = -\frac{kx}{(1 + ky)^2} - \frac{c(1 - m)^2 x^2}{1 + a(1 - m)^2 x^2}$;

$a_{13} = 0$; $\quad a_{21} = \frac{2\xi c(1 - m)^2 xy}{\{1 + a(1 - m)^2 x^2\}^2}$; $\quad a_{22} = \frac{\xi c(1 - m)^2 x^2}{1 + a(1 - m)^2 x^2} - \frac{pz(e + fz)}{(e + y + fz)^2} - g$;

$a_{23} = -\frac{py(e + y)}{(e + y + fz)^2}$; $\quad a_{31} = 0$; $\quad a_{32} = \frac{\sigma pz(e + fz)}{(e + y + fz)^2}$; $\quad a_{33} = \frac{\sigma py(e + y)}{(e + y + fz)^2} - n.$

For $E_0 = (0, 0, 0)$:

$$J|_{E_0} = \begin{pmatrix} 1-d & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & -n \end{pmatrix}.$$

So, $\lambda_1 = 1 - d$, $\lambda_2 = -g$, $\lambda_3 = -n$. Here λ_2, λ_3 are always negative.
So we have the following theorem:

Theorem 6.1. E_0 is locally asymptotically stable (LAS) when $d > 1$.

For $E_1 = (1 - d, 0, 0)$:

$$J|_{E_1} = \begin{pmatrix} -(1-d) & -\left(k(1-d) + \frac{c(1-m)^2(1-d)^2}{1+a(1-m)^2(1-d)^2}\right) & 0 \\ 0 & \frac{\xi c(1-m)^2(1-d)^2}{1+a(1-m)^2(1-d)^2} - g & 0 \\ 0 & 0 & -n \end{pmatrix}.$$

So, $\lambda_1 = -(1 - d)$, $\lambda_2 = \dfrac{\xi c(1 - m)^2(1 - d)^2}{1 + a(1 - m)^2(1 - d)^2} - g$, $\lambda_3 = -n$. Here λ_1, λ_3 are always negative.
So we have the following theorem:

Theorem 6.2. E_1 is locally asymptotically stable (LAS) when $(\xi c - ag)(1 - m)^2(1 - d)^2 < g$.

For $E_2 = (\widetilde{x}, \widetilde{y}, 0)$:

$$J|_{E_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

where $a_{11} = -\widetilde{x} + \dfrac{c(1 - m)^2\widetilde{x}\widetilde{y}\{a(1 - m)^2\widetilde{x}^2 - 1\}}{\{1 + a(1 - m)^2\widetilde{x}^2\}^2}$; $\quad a_{12} = -\dfrac{k\widetilde{x}}{(1 + k\widetilde{y})^2} - \dfrac{c(1 - m)^2\widetilde{x}^2}{1 + a(1 - m)^2\widetilde{x}^2}$;

$a_{21} = \dfrac{2\xi c(1 - m)^2\widetilde{x}\widetilde{y}}{\{1 + a(1 - m)^2\widetilde{x}^2\}^2}$; $\quad a_{23} = -\dfrac{p\widetilde{y}}{(e + \widetilde{y})}$; $\quad a_{33} = \dfrac{\sigma p\widetilde{y}}{(e + \widetilde{y})} - n.$

One eigenvalue will be

$$\lambda_1 = \frac{\sigma p\widetilde{y}}{(e + \widetilde{y})} - n$$

and other two will be the roots of the quadratic equation:

$$\lambda^2 + D_1\lambda + D_2 = 0,$$

where $D_1 = -a_{11}$, $D_2 = -a_{21}a_{12} > 0$.
So, E_2 will be stable if $\lambda_1 < 0$ and $D_1 > 0$ $\left(\text{i.e., } 1 - \dfrac{c(1 - m)^2\widetilde{y}\{a(1 - m)^2\widetilde{x}^2 - 1\}}{\{1 + a(1 - m)^2\widetilde{x}^2\}^2} > 0\right)$.
Therefore we have the following theorem:

Theorem 6.3. E_2 is locally asymptotically stable if $(\sigma p - n)\widetilde{y} < ne$ holds along with $c\xi^2 > (1 - m)^2(c\xi - ag)(2ag - c\xi)\widetilde{y}$.

Now, for $E^*(x^*, y^*, z^*)$:

$$J|_{E^*} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

where $a_{11} = -x^* + \dfrac{c(1-m)^2 x^* y^* \{a(1-m)^2 x^{*2} - 1\}}{\{1 + a(1-m)^2 x^{*2}\}^2}$; $a_{12} = -\dfrac{kx^*}{(1+ky^*)^2} - \dfrac{c(1-m)^2 x^{*2}}{1+a(1-m)^2 x^{*2}}$;

$a_{21} = \dfrac{2\xi c(1-m)^2 x^* y^*}{\{1+a(1-m)^2 x^{*2}\}^2}$; $a_{22} = \dfrac{py^* z^*}{(e+y^*+fz^*)^2}$; $a_{23} = -\dfrac{py^*(e+y^*)}{(e+y^*+fz^*)^2}$;

$a_{32} = \dfrac{\sigma pz^*(e+fz^*)}{(e+y^*+fz^*)^2}$; $a_{33} = -\dfrac{fnz^*}{e+y^*+fz^*}$.

Characteristic equation for $E^*(x^*, y^*, z^*)$ is

$$\lambda^3 + G_1 \lambda^2 + G_2 \lambda + G_3 = 0, \tag{6.2}$$

where $G_1 = -(a_{11} + a_{22} + a_{33})$, $G_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32}$,
$G_3 = a_{11}a_{23}a_{32} - a_{33}(a_{11}a_{22} - a_{12}a_{21})$.
Let $\Delta = G_1 G_2 - G_3$.
By Routh-Hurwitz criterion all the roots of equation (6.2) have negative real parts if $G_1 > 0$, $G_3 > 0$ and $G_1 G_2 - G_3 > 0$. Hence, we have the following theorem:

Theorem 6.4. $E^*(x^*, y^*, z^*)$ *will be LAS by Routh-Hurwitz criterion if* $G_1 > 0$, $G_3 > 0$ *and* $\Delta = G_1 G_2 - G_3 > 0$.

7. Persistence

In the ecological context, permanence means the long term survival of all species which exist initially.

Theorem 7.1. *System (2.2) is permanent if the following conditions hold:*
(i) $1 - d > 0$;
(ii) $(\xi c - ag)(1-m)^2 (1-d)^2 > g$;
(iii) $(\sigma p - n)\widetilde{y} > ne$.

Proof. Let the average Lyapunov function be $V(x, y, z) = x^{\beta_1} y^{\beta_2} z^{\beta_3}$ where β_i for $i = 1, 2, 3$ are positive. In the interior of $\mathbb{R}^3_+$, we have

$$\frac{\dot{V}}{V} = \phi(x, y, z) = \beta_1 \left[\frac{1}{1+ky} - d - x - \frac{c(1-m)^2 xy}{1+a(1-m)^2 x^2} \right]$$
$$+ \beta_2 \left[\frac{\xi c(1-m)^2 x^2}{1+a(1-m)^2 x^2} - \frac{pz}{e+y+fz} - g \right] + \beta_3 \left[\frac{\sigma py}{e+y+fz} - n \right]$$

If the system is permanent, then $\phi(x, y, z) > 0$ for all boundary equilibria of the system. The values of $\phi(x, y, z)$ at the boundary equilibria E_0, E_1 and E_2 are as follows:
$E_0 : \phi(0, 0, 0) = \beta_1(1-d) - \beta_2 g - \beta_3 n$.
$E_1 : \phi(1-d, 0, 0) = \beta_2 \left[\dfrac{\xi c(1-m)^2 (1-d)^2}{1+a(1-m)^2(1-d)^2} - g \right] - \beta_3 n$.
$E_3 : \phi(\widetilde{x}, \widetilde{y}, 0) = \beta_3 \left[\dfrac{\sigma p \widetilde{y}}{e + \widetilde{y}} - n \right]$.
Now, $\phi(0, 0, 0)$ is positive for some positive β_i for $i = 1, 2, 3$ if $1 - d > 0$. And if the inequalities stated in $(i) - (iii)$ hold, then ϕ is positive at E_0, E_1 and E_2 for some $\beta_i > 0$ for $i = 1, 2, 3$. So, system (2.2) is permanent [10] if the conditions $(i) - (iii)$ are satisfied. $\square$

Remark: Conditions (i), (ii) and (iii) guarantee the instability of the boundary equilibria of system (2.2).

8. Bifurcation Analysis

In the section, we have mainly discussed the bifurcation analysis around the equilibrium points and for that we have used **Sotomayor's Theorem** [31] and the Hopf Bifurcation Theorem [29]. To apply Sotomayor's Theorem, one of the eigenvalues of the Jacobian matrix at the bifurcating equilibrium point need to be zero. Let $V = (v_1, v_2, v_3)^T$ and $W = (w_1, w_2, w_3)^T$ be the eigenvectors of $J|_{(eq.\ point)}$ and $J|_{(eq.\ point)}^T$ corresponding to zero eigenvalue of the equilibrium point respectively.

Let $F = (F_1, F_2, F_3)^T$, where

$$F_1 = \frac{x}{1+ky} - dx - x^2 - \frac{c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2},$$

$$F_2 = \frac{\xi c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2} - \frac{pyz}{e+y+fz} - gy,$$

$$F_3 = \frac{\sigma pyz}{e+y+fz} - nz.$$

Theorem 8.1. *System (2.2) undergoes a transcritical bifurcation w.r.to the bifurcation parameter d around $E_0(0,0,0)$ if $d = 1$.*

Proof.

$$J|_{E_0} = \begin{pmatrix} 1-d & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & -n \end{pmatrix}$$

Let $d_{[TC_1]}$ be the value of d s.t $J|_{E_0}$ has a simple zero eigenvalue at $d = d_{[TC_1]}$.
So, at $d = d_{[TC_1]}$:

$$J|_{E_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & -n \end{pmatrix}.$$

Here, $\lambda_1 = -g < 0$ and $\lambda_2 = -n < 0$.
After some calculations: $V = (1,0,0)^T$ and $W = (1,0,0)^T$.
Therefore,

$$\Omega_1 = W^T.F_d(E_0, d_{[TC_1]}) = -x|_{E_0} = 0,$$

$$\Omega_2 = W^T \left[DF_d(E_0, d_{[TC_1]})V \right] = -1 \neq 0$$

$$\text{and } \Omega_3 = W^T \left[D^2F(E_0, d_{[TC_1]})(V, V) \right] = -2 \neq 0$$

By Sotomayor's Theorem, system (2.2) undergoes a transcritical bifurcation around E_0 at $d = d_{[TC_1]}$. □

Theorem 8.2. *System (2.2) undergoes a transcritical bifurcation w.r.to the bifurcation parameter g around $E_1(1 - d, 0, 0)$ if $(\xi c - ag)(1-m)^2(1-d)^2 = g$.*

Proof.

$$J|_{E_1} = \begin{pmatrix} -(1-d) & -\left(k(1-d) + \frac{c(1-m)^2(1-d)^2}{1+a(1-m)^2(1-d)^2}\right) & 0 \\ 0 & \frac{\xi c(1-m)^2(1-d)^2}{1+a(1-m)^2(1-d)^2} - g & 0 \\ 0 & 0 & -n \end{pmatrix}$$

Let $g_{[TC_2]}$ be the value of g s.t $J|_{E_1}$ has a simple zero eigenvalue at $g = g_{[TC_2]}$.
So, at $g = g_{[TC_2]}$:

$$J|_{E_1} = \begin{pmatrix} -(1-d) & -\left(k(1-d) + \frac{c(1-m)^2(1-d)^2}{1+a(1-m)^2(1-d)^2}\right) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -n \end{pmatrix}.$$

Here, $\lambda_1 = -(1 - d) < 0$ and $\lambda_2 = -n < 0$.

After some calculations: $V = (v_1, v_2, v_3)^T = \left(-\left[k + \frac{c(1-m)^2(1-d)}{1+a(1-m)^2(1-d)^2}\right], 1, 0\right)^T$ and $W = (0, 1, 0)^T$.

Therefore,

$$\Omega_1 = W^T.F_g(E_1, g_{[TC_2]}) = -y|_{E_1} = 0,$$

$$\Omega_2 = W^T\left[DF_g(E_1, g_{[TC_2]})V\right] = -1 \neq 0$$

$$\text{and } \Omega_3 = W^T\left[D^2F(E_1, g_{[TC_2]})(V, V)\right] = \frac{4\xi c(1 - m)^2(1 - d)}{\{1 + a(1 - m)^2(1 - d)^2\}^2}v_1v_2 \neq 0$$

By Sotomayor's Theorem, system (2.2) undergoes a transcritical bifurcation around E_1 at $g = g_{[TC_2]}$. $\quad\square$

Theorem 8.3. *System (2.2) undergoes a transcritical bifurcation w.r.to the bifurcation parameter n around $E_2(\widetilde{x}, \widetilde{y}, 0)$ if $(\sigma p - n)\widetilde{y} = ne$ but $c\xi^2 > (1 - m)^2(c\xi - ag)(2ag - c\xi)\widetilde{y}$.*

Proof.

$$J|_{E_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

where $a_{11} = -\widetilde{x} + \frac{c(1-m)^2\widetilde{x}\widetilde{y}\{a(1-m)^2\widetilde{x}^2-1\}}{\{1+a(1-m)^2\widetilde{x}^2\}^2}$; $a_{12} = -\frac{k\widetilde{x}}{(1+k\widetilde{y})^2} - \frac{c(1-m)^2\widetilde{x}^2}{\{1+a(1-m)^2\widetilde{x}^2\}}$;

$a_{21} = \frac{2\xi c(1-m)^2\widetilde{x}\widetilde{y}}{\{1+a(1-m)^2\widetilde{x}^2\}^2}$; $a_{23} = -\frac{p\widetilde{y}}{(e+\widetilde{y})}$; $a_{33} = \frac{\sigma p\widetilde{y}}{(e+\widetilde{y})} - n$.

Let $n_{[TC_3]}$ be the value of n s.t $J|_{E_2}$ has a simple zero eigenvalue at $n = n_{[TC_3]}$.

So, at $n = n_{[TC_3]}$:

$$J|_{E_2} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

For $c\xi^2 > (1 - m)^2(c\xi - ag)(2ag - c\xi)\widetilde{y}$ we shall get other two eigenvalues with negative real part at $n = n_{[TC_3]}$.

After some calculations:

$V = (a_{12}a_{23}, -a_{11}a_{23}, -a_{21}a_{12})^T$ and $W = (0, 0, 1)^T$.

Therefore,

$$\Omega_1 = W^T.F_n(E_2, n_{[TC_3]}) = -z|_{E_2} = 0,$$

$$\Omega_2 = W^T\left[DF_n(E_2, n_{[TC_3]})V\right] = -v_3 \neq 0$$

$$\text{and } \Omega_3 = W^T\left[D^2F(E_2, n_{[TC_3]})(V, V)\right] = \frac{2\sigma pv_3}{(e + \widetilde{y})^2}(ev_2 - f\widetilde{y}v_3) \neq 0.$$

By Sotomayor's Theorem, system (2.2) undergoes a transcritical bifurcation around E_2 at $n = n_{[TC_3]}$. $\quad\square$

- **Hopf Bifurcation at equilibrium points**

Let us consider b as bifurcation parameter of a system where the characteristic equation of an equilibrium point $E(x, y, z)$ is

$$\lambda^3 + G_1(b)\lambda^2 + G_2(b)\lambda + G_3 = 0, \tag{8.1}$$

then *Hopf Bifurcation Theorem* is stated as follows:

Theorem 8.4. *(Hopf Bifurcation Theorem) Suppose $G_i(b)$; $i = 1, 2, 3$ are continuous functions of b in $N_\epsilon(b_0)$, $(\epsilon > 0)$, $b_0 \in \mathbb{R}$ such that the characteristic equation (8.1) has*
(i) a pair of complex eigenvalues $\lambda = k(b) \pm il(b)$ (with $k(b), l(b) \in \mathbb{R}$) which become purely imaginary when $b = b_0$ and $\frac{dk}{db}\big|_{b=b_0} \neq 0$,
(ii) the other eigenvalue is negative when $b = b_0$.
Then a Hopf Bifurcation occurs around E when $b = b_0$.

- ## Hopf Bifurcation at $E^*(x^*, y^*, z^*)$

Here, let us consider k as bifurcation parameter to check the instability of the equilibrium point E^*. The characteristic equation of system (2.2) at $E^*(x^*, y^*, z^*)$ is

$$\lambda^3 + G_1(k)\lambda^2 + G_2(k)\lambda + G_3(k) = 0, \text{ where} \tag{8.2}$$

$$G_1 = -(a_{11} + a_{22} + a_{33})$$
$$= -\left[-x^* + \frac{c(1-m)^2 x^* y^* \{a(1-m)^2 x^{*2} - 1\}}{\{1 + a(1-m)^2 x^{*2}\}^2} + \frac{py^* z^*}{(e + y^* + fz^*)^2} - \frac{fnz^*}{e + y^* + fz^*} \right],$$

$$G_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32}$$
$$= \left[-x^* + \frac{c(1-m)^2 x^* y^* \{a(1-m)^2 x^{*2} - 1\}}{\{1 + a(1-m)^2 x^{*2}\}^2} \right] \left\{ \frac{py^* z^*}{(e + y^* + fz^*)^2} - \frac{fnz^*}{e + y^* + fz^*} \right\}$$
$$- \frac{pfny^* z^{*2}}{(e + y^* + fz^*)^3} + \frac{2\xi c(1-m)^2 x^* y^*}{\{1 + a(1-m)^2 x^{*2}\}^2} \left[\frac{kx^*}{(1 + ky^*)^2} + \frac{c(1-m)^2 x^{*2}}{1 + a(1-m)^2 x^{*2}} \right]$$
$$+ \frac{\sigma p^2 z^*(e + fz^*)y^*(e + y^*)}{(e + y^* + fz^*)^4},$$

$$G_3 = a_{11}a_{23}a_{32} - a_{33}(a_{11}a_{22} - a_{12}a_{21})$$
$$= \left[-x^* + \frac{c(1-m)^2 x^* y^* \{a(1-m)^2 x^{*2} - 1\}}{\{1 + a(1-m)^2 x^{*2}\}^2} \right] \frac{\sigma p^2 z^*(e + fz^*)y^*(e + y^*)}{(e + y^* + fz^*)^4}$$
$$+ \frac{fnz^*}{e + y^* + fz^*} \left[\left\{ -x^* + \frac{c(1-m)^2 x^* y^* \{a(1-m)^2 x^{*2} - 1\}}{\{1 + a(1-m)^2 x^{*2}\}^2} \right\} \left\{ \frac{py^* z^*}{(e + y^* + fz^*)^2} - \frac{fnz^*}{e + y^* + fz^*} \right\} \right.$$
$$\left. + \frac{2\xi c(1-m)^2 x^* y^*}{\{1 + a(1-m)^2 x^{*2}\}^2} \left\{ \frac{kx^*}{(1 + ky^*)^2} + \frac{c(1-m)^2 x^{*2}}{1 + a(1-m)^2 x^{*2}} \right\} \right]$$

Theorem 8.5. *If E^* exists with the feasibility conditions, then a simple Hopf bifurcation occurs at unique $k = k_0$, where k_0 is the unique positive root of the equation: $G_1(k)G_2(k) - G_3(k) = 0$ with $G_1(k_0), G_2(k_0) > 0$.*

Proof. For $k = k_0$, the characteristic equation of system (2.2) at E^* is $(\lambda^2 + G_2)(\lambda + G_1) = 0$
which gives roots: $\lambda_1 = i\sqrt{G_2}$, $\lambda_2 = -i\sqrt{G_2}$ and $\lambda_3 = -G_1$. So, there exists a pair of purely imaginary eigenvalues and a strictly negative real eigenvalue. Also, $G_i(k)$ are continuous functions of k.
So, for k in a neighbourhood of k_0, the roots have the form:
$\lambda_1 = p_1(k) + ip_2(k)$, $\lambda_2 = p_1(k) - ip_2(k)$, $\lambda_3 = -p_3(k)$; $p_j(k)$ are real for $j = 1, 2, 3$.

Next to check the transversality condition: $\frac{d}{dk}[Re(\lambda_j(k))]\big|_{k=k_0} \neq 0$, for $j = 1, 2$.
Put $\lambda(k) = p_1(k) + ip_2(k)$ in (8.2), we get

$$(p_1 + ip_2)^3 + G_1(p_1 + ip_2)^2 + G_2(p_1 + ip_2) + G_3 = 0. \tag{8.3}$$

Taking derivative w.r.to k, we get

$$3(p_1 + ip_2)^2(\dot{p}_1 + i\dot{p}_2) + 2G_1(p_1 + ip_2)(\dot{p}_1 + i\dot{p}_2) + G_2(\dot{p}_1 + i\dot{p}_2) + \dot{G}_1(p_1 + ip_2)^2 + \dot{G}_2(p_1 + ip_2) + \dot{G}_3 = 0.$$

Comparing real and imaginary parts:

$$S_1 \dot{p}_1 - S_2 \dot{p}_2 + S_3 = 0, \tag{8.4}$$

$$S_2 \dot{p}_1 + S_1 \dot{p}_2 + S_4 = 0, \tag{8.5}$$

where $S_1 = 3(p_1^2 - p_2^2) + 2G_1 p_1 + G_2$; $\quad S_2 = 6p_1 p_2 + 2G_1 p_2$;
$S_3 = \dot{G}_1(p_1^2 - p_2^2) + \dot{G}_2 p_1 + \dot{G}_3$; $\quad S_4 = 2\dot{G}_1 p_1 p_2 + \dot{G}_2 p_2$.

From (8.4) and (8.5):

$$\dot{p}_1 = -\frac{S_2 S_4 + S_1 S_3}{S_1^2 + S_2^2}. \tag{8.6}$$

Now, $S_3 = \dot{G}_1(p_1^2 - p_2^2) + \dot{G}_2 p_1 + \dot{G}_3 \neq \dot{G}_1(p_1^2 - p_2^2) + \dot{G}_2 p_1 + \dot{G}_1 G_2 + \dot{G}_2 G_1$
At $k = k_0$:
Case-(1): $p_1 = 0$; $p_2 = \sqrt{G_2}$.
So, $S_1 = -2G_2$; $S_2 = 2G_1 \sqrt{G_2}$; $S_3 \neq G_1 \dot{G}_2$; $S_4 = \dot{G}_2 \sqrt{G_2}$
and $S_2 S_4 + S_1 S_3 \neq 2G_1 G_2 \dot{G}_2 - 2G_1 G_2 \dot{G}_2 = 0$.
Case-(2): $p_1 = 0$; $p_2 = -\sqrt{G_2}$.
So, $S_1 = -2G_2$; $S_2 = -2G_1 \sqrt{G_2}$; $S_3 \neq G_1 \dot{G}_2$; $S_4 = -\dot{G}_2 \sqrt{G_2}$
and $S_2 S_4 + S_1 S_3 \neq 2G_1 G_2 \dot{G}_2 - 2G_1 G_2 \dot{G}_2 = 0$.

$$\therefore \frac{d}{dk}[Re(\lambda_i(k))]\Big|_{k=k_0} = -\frac{S_2 S_4 + S_1 S_3}{S_1^2 + S_2^2} \neq 0.$$

Also, $\lambda_3 = -p_3 = -G_1(k_0) < 0$. Hence the theorem. $\square$

9. Global Stability

In this section we discuss the global stability of those equilibrium points which are locally asymptotically stable (LAS) under some parametric conditions.

Theorem 9.1. *The trivial equilibrium* $E_0(0,0,0)$, *if LAS, is globally asymptotically stable (GAS) also.*

Proof. Consider the Lyapunov function as $V_1(x, y, z) = x + \frac{1}{\xi}y + \frac{1}{\xi\sigma}z$ Here, $V_1(x, y, z)$ is a positive definite function for all (x, y, z) except $(0, 0, 0)$.
Now time derivative of V_1 computed along the solutions of system (2.2) is given by:

$$
\begin{aligned}
\frac{dV_1}{dt} &= \frac{dx}{dt} + \frac{1}{\xi}\frac{dy}{dt} + \frac{1}{\xi\sigma}\frac{dz}{dt} \\
&= \left[\frac{x}{1+ky} - dx - x^2 - \frac{c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2}\right] + \frac{1}{\xi}\left[\frac{\xi c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2} - \frac{pyz}{e+y+fz} - gy\right] \\
&\quad + \frac{1}{\xi\sigma}\left[\frac{\sigma pyz}{e+y+fz} - nz\right] \\
&\leq x - dx - \frac{g}{\xi}y - \frac{n}{\xi\sigma}z \\
&\leq (1-d)x
\end{aligned}
$$

So, $\frac{dV_1}{dt} < 0$ when it is LAS. Also $\frac{dV_1}{dt} = 0$ when $(x, y, z) = (0, 0, 0)$. Hence, $\frac{dV_1}{dt}$ is negative definite when it is LAS. As the only solution of model (2.2) that satisfies $x = 0$, $y = 0$ and $z = 0$ is the equilibrium, LaSalle theorem [22] implies global asymptotic stability of E_0. $\square$

Theorem 9.2. *The axial equilibrium* $E_1(1-d, 0, 0)$ *is globally asymptotically stable if* $g - \xi c(1-m)^2(1-d)^2 > 0$.

Proof. From 1st equation of system (2.2) we have

$$\frac{dx}{dt} = \frac{x}{1+ky} - dx - x^2 - \frac{c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2}$$

$$\leq (1-d)x\left\{1 - \frac{x}{1-d}\right\}$$

Hence, $\lim\limits_{t\to\infty} x \leq (1-d)$.

Now consider

$$\frac{d}{dt}\left(y + \frac{1}{\sigma}z\right) = \frac{dy}{dt} + \frac{1}{\sigma}\frac{dz}{dt}$$

$$= \frac{\xi c(1-m)^2 x^2 y}{1+a(1-m)^2 x^2} - gy - \frac{nz}{\sigma}$$

$$< -\{g - \xi c(1-m)^2(1-d)^2\}y - \frac{nz}{\sigma}$$

$$= -Py - \frac{nz}{\sigma}, \ \left(\text{let, } P = g - \xi c(1-m)^2(1-d)^2\right)$$

$$\leq -\kappa\left(y + \frac{1}{\sigma}z\right), \ (\text{where, } \kappa = \min\{P, n\})$$

Thus, $y(t) + \frac{1}{\sigma}z(t) \leq \left(y_0 + \frac{1}{\sigma}z_0\right)\exp(-\kappa t)$ and the system is dissipative.

From above, we have, $\lim\limits_{t\to\infty} y = \lim\limits_{t\to\infty} z = 0$. And in the limit form, $x(t)$ is a positive solution of the equation $\dot{x}(t) = (1-d)x\left\{1 - \frac{x}{1-d}\right\}$. As, $x_0 > 0$, the theorem is proved. $\square$

Theorem 9.3. *If the equilibrium $E_2(\widetilde{x}, \widetilde{y}, 0)$ exists and is locally asymptotically stable, then it is globally asymptotically stable if $1 - d + \widetilde{x} < 0$, $\xi c(1-m)^2\widetilde{x} < g$ and $d\widetilde{x} + \frac{\overline{y}}{\xi}\left(\frac{p}{f} + g\right) - \frac{n}{\xi\sigma} < 0$.*

Proof. Consider the Lyapunov function as:

$$V_2(x, y, z) = \left[x - \widetilde{x} - \widetilde{x}\log\left(\frac{x}{\widetilde{x}}\right)\right] + \frac{1}{\xi}\left[y - \widetilde{y} - \widetilde{y}\log\left(\frac{y}{\widetilde{y}}\right)\right] + \frac{1}{\xi\sigma}z$$

Here, $V_2(x, y, z)$ is a positive definite function for all (x, y, z) except $(\widetilde{x}, \widetilde{y}, 0)$.

Now time derivative of V_2 computed along the solutions of system (2.2) is given by

$$\frac{dV_2}{dt} = \left(1 - \frac{\widetilde{x}}{x}\right)\frac{dx}{dt} + \frac{1}{\xi}\left(1 - \frac{\widetilde{y}}{y}\right)\frac{dy}{dt} + \frac{1}{\xi\sigma}\frac{dz}{dt}$$

$$= (x - \widetilde{x})\left[\frac{1}{1+ky} - d - x - \frac{c(1-m)^2 xy}{1+a(1-m)^2 x^2}\right] + \frac{1}{\xi}(y - \widetilde{y})\left[\frac{\xi c(1-m)^2 x^2}{1+a(1-m)^2 x^2} - \frac{pz}{e+y+fz} - g\right]$$

$$+ \frac{1}{\xi\sigma}\left[\frac{\sigma pyz}{e+y+fz} - nz\right]$$

$$\leq x\{1 - d + \widetilde{x}\} + \left\{c(1-m)^2\widetilde{x} - \frac{g}{\xi}\right\}y + \left\{d\widetilde{x} + \frac{\widetilde{y}}{\xi}\left(\frac{p}{f} + g\right) - \frac{n}{\xi\sigma}\right\}$$

So, $\frac{dV_2}{dt} < 0$ when $1 - d + \widetilde{x} < 0$, $\xi c(1-m)^2\widetilde{x} < g$ and $d\widetilde{x} + \frac{\widetilde{y}}{\xi}\left(\frac{p}{f} + g\right) - \frac{n}{\xi\sigma} < 0$. Also $\frac{dV_2}{dt} = 0$ when $(x, y, z) = (\widetilde{x}, \widetilde{y}, 0)$. Hence, by LaSalle theorem [22] E_2 is GAS when the stated conditions are fulfilled. $\square$

Theorem 9.4. *If E^* exists and is locally asymptotically stable, then it is globally asymptotically stable if $x^* < d$, $p\xi\sigma y^* < ne\xi$ and $1 + dx^* + \frac{gy^*}{\xi} + \frac{nz^*}{\xi\sigma} - \frac{x^*}{1+kM} < 0$.*

Parameter	m	a	p	e	f	σ	ξ	c
Value	0.5	5	0.05	0.5	0.1	0.3	0.8	5

Table 1: Parametric values used in numerical simulation

Proof. Consider an appropriate Lyapunov function:

$$V_3(x, y, z) = \left[x - x^* - x^* \log\left(\frac{x}{x^*}\right)\right] + \frac{1}{\xi}\left[y - y^* - y^* \log\left(\frac{y}{y^*}\right)\right] + \frac{1}{\xi\sigma}\left[z - z^* - z^* \log\left(\frac{z}{z^*}\right)\right].$$

Here $V_3(x, y, z)$ is a positive definite function for all (x, y, z) except (x^*, y^*, z^*).
The time derivative of V_2 computed along the solutions of system (2.2) is given by

$$\frac{dV_3}{dt} = \left(1 - \frac{x^*}{x}\right)\frac{dx}{dt} + \frac{1}{\xi}\left(1 - \frac{y^*}{y}\right)\frac{dy}{dt} + \frac{1}{\xi\sigma}\left(1 - \frac{z^*}{z}\right)\frac{dz}{dt}$$

$$= (x - x^*)\left[\frac{1}{1 + ky} - d - x - \frac{c(1 - m)^2 xy}{1 + a(1 - m)^2 x^2}\right] + \frac{1}{\xi}(y - y^*)\left[\frac{c(1 - m)^2 x^2}{1 + a(1 - m)^2 x^2} - \frac{pz}{e + y + fz} - g\right]$$

$$+ \frac{1}{\xi\sigma}(z - z^*)\left[\frac{\sigma py}{e + y + fz} - n\right]$$

$$< x - (d - x^*)x + dx^* - \frac{x^*}{1 + kM} - \frac{g}{\xi}y - \frac{c(1 - m)^2 y^* x^2}{1 + a(1 - m)^2} - x^2 + \frac{py^* z}{\xi(e + y + fz)} + \frac{gy^*}{\xi}$$

$$- \frac{nz}{\xi\sigma} - \frac{\sigma p z^* y}{\xi\sigma(e + y + fz)} + \frac{nz^*}{\xi\sigma}$$

$$< -(d - x^*)x + \left(1 + dx^* + \frac{gy^*}{\xi} + \frac{nz^*}{\xi\sigma} - \frac{x^*}{1 + kM}\right) + z\left(\frac{py^*}{e\xi} - \frac{n}{\xi\sigma}\right)$$

So, $\frac{dV_3}{dt} < 0$ when the stated conditions are satisfied. Also $\frac{dV_3}{dt} = 0$ when $(x, y, z) = (x^*, y^*, z^*)$. Hence $\frac{dV_3}{dt}$ is negative definite under some parametric restrictions and LaSalle theorem [22] implies global asymptotic stability of E^*. □

10. Numerical Simulation

Numerical simulation help us to analyze the system dynamics with the help of some pictorial diagrams. Here we vary some of the parameters to show the impact of those particular on the model system. Let us fix some ecological parameters as described in Table 1.

In absence or even absence of fear, for $d = 1.2$, $g = 0.5$ and $n = 0.03$, Figure 1 shows that all the populations in the system are going extinct with time and the trajectories converge to the trivial equilibrium $E_0(0, 0, 0)$. As usual (in agreement with Malthus growth), if the death rate exceeds birth rate, then the prey population cannot sustain in the system and they wash out resulting in the extinction of middle-predator and top-predator populations. So, none of the fear effect and refuge parameters can control the stability of the trivial equilibrium point. Only the parameter 'd' is sufficient to decide whether all species are going extinct or not. But if we consider $d = 0.2$ along with $g = 0.5$ and $n = 0.03$, we get a consumer and top-predator free system for absence or even for presence of fear term and the trajectories converge to $E_1(0.8, 0, 0)$ (see Figure 2).

As the stability criterion of E_0 holds when $d > 1$ ($\Leftrightarrow$ death rate > birth rate for prey) and $E_1(1 - d, 0, 0)$ exists only when $d < 1$. Thus at $d_{[TC]} = 1$, E_1 and E_0 coincide each other and a transcritical bifurcation occurs around E_0 (see Figure 3).

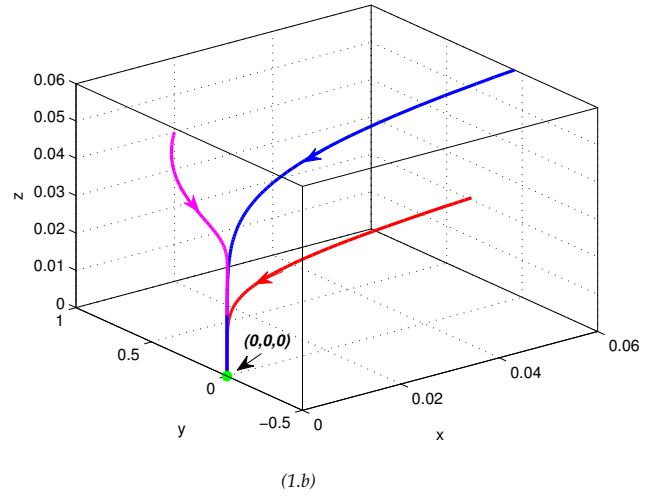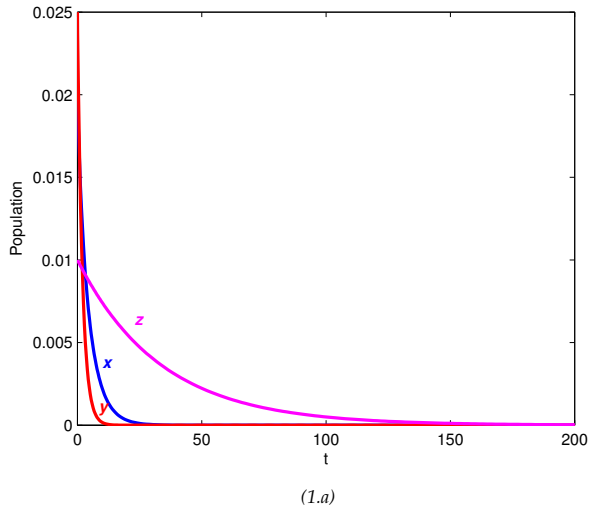Keeping $d = 0.2$ and $n = 0.03$ fixed, if we decrease the death rate coefficient of middle-predator (g) to 0.006,

(1.a) *(1.b)*
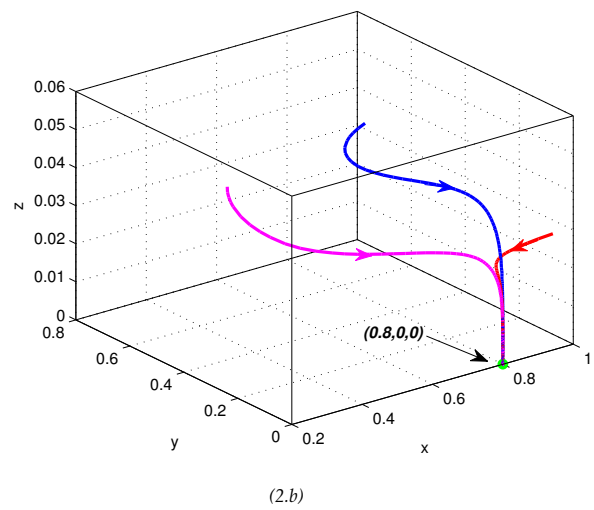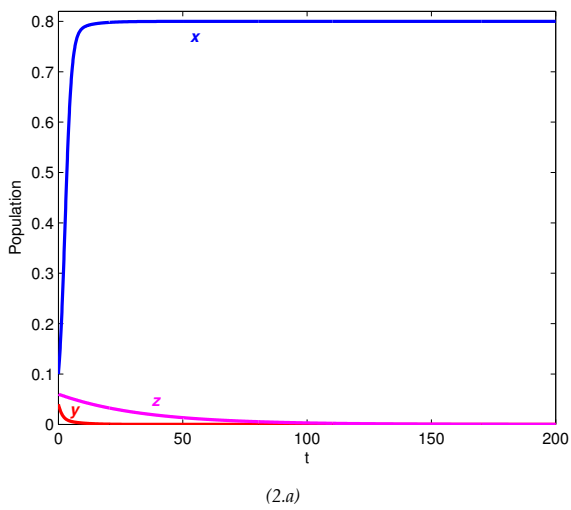
Figure 1: Stable behaviour of E_0.

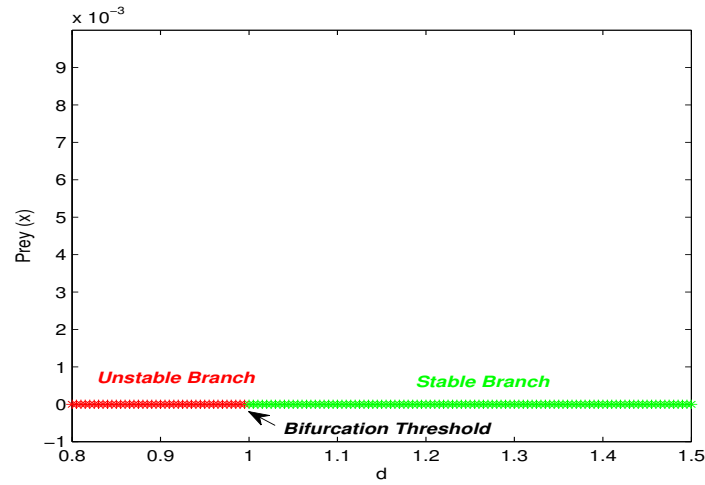*(2.a)* *(2.b)*

Figure 2: Stable behaviour of E_1.

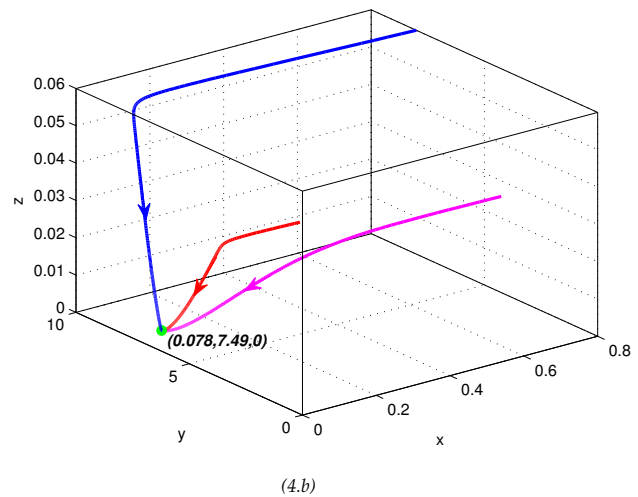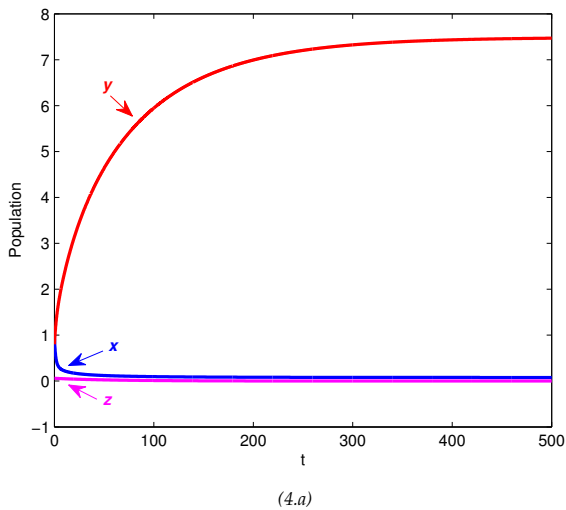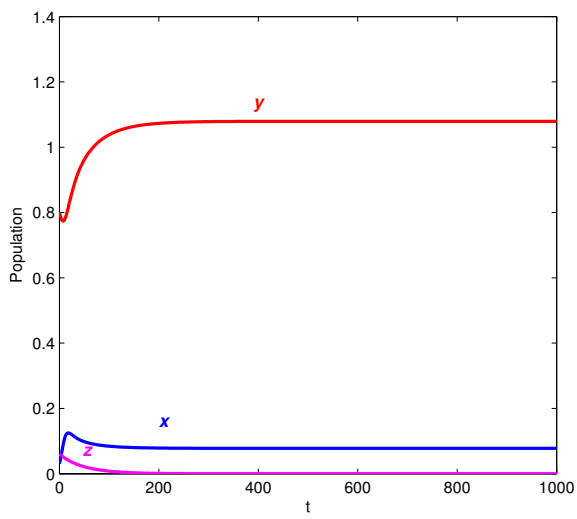Figure 3: Transcritical Bifurcation around E_0 taking d as bifurcation parameter.

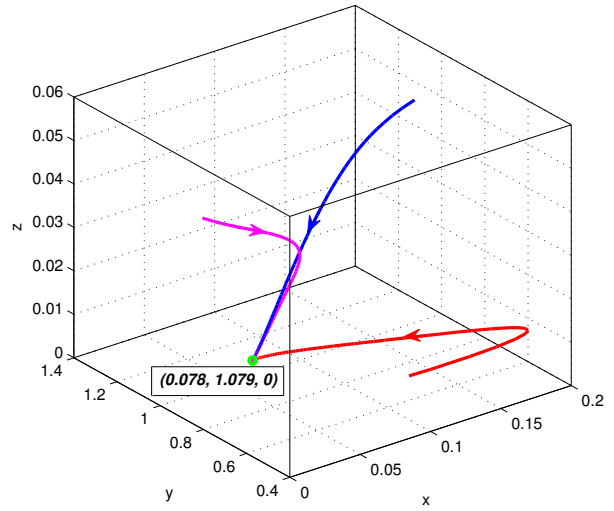Figure 4: Stable behaviour of E_2 without any fear effect.

then the trajectories converges to a top-predator free system E_2 where the prey and consumer populations exist as a steady state. In absence of fear effect, i.e., for $k = 0$, $E_2(\widetilde{x}, \widetilde{y}, 0) \equiv (0.078, 7.49, 0)$ (see Figure 4) but if we consider the fear coefficient k as 1.5, E_2 becomes $(0.078, 1.079, 0)$ (see Figure 5). So, it is observed that the fear coefficient has an important impact on middle-predator's growth and the middle-predator population decreases rapidly with increasing fear coefficient.

It is observed that for a threshold value of g, E_2 coincides with E_1. For $g > g_{[TC]}$, E_1 is stable and becomes unstable when the value of g is lower than $g_{[TC]} = 0.356$. Also, E_2 exists only when $g < g_{[TC]}$. So, a transcritical bifurcation occurs around E_1 at $g = g_{[TC]}$ (see Figure 6).

From this state, if we fix $d = 0.2$ and g as 0.006, then for a decreasing value of n, we get a system where all population exist as a steady state. For $k = 0$ and $n = 0.01$, the interior equilibrium point be $E^* = (0.279, 1.639, 3.194)$ (Figure 7). Again, in presence of fear term, for $k = 1.5$ and $n = 0.01$, E^* is $(0.0878, 1.0099, 0.0497)$ (Figure 8). So, it is evident that introduction of fear (k) in prey population affect the growth of all population. Increasing value of k decreases the prey population resulting in the decreasing

(5.a)   *(5.b)*

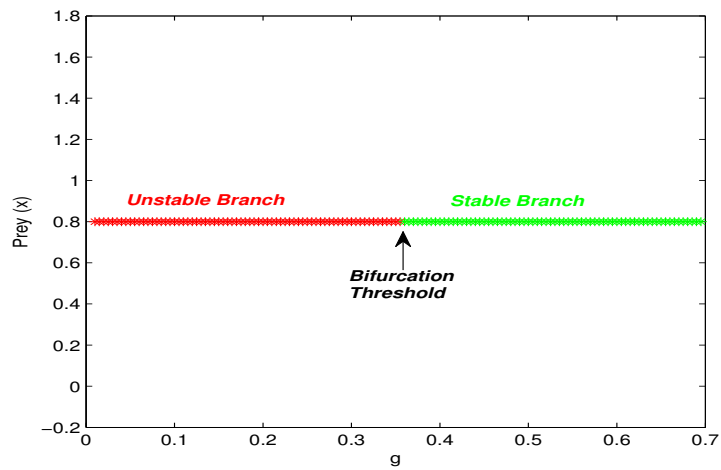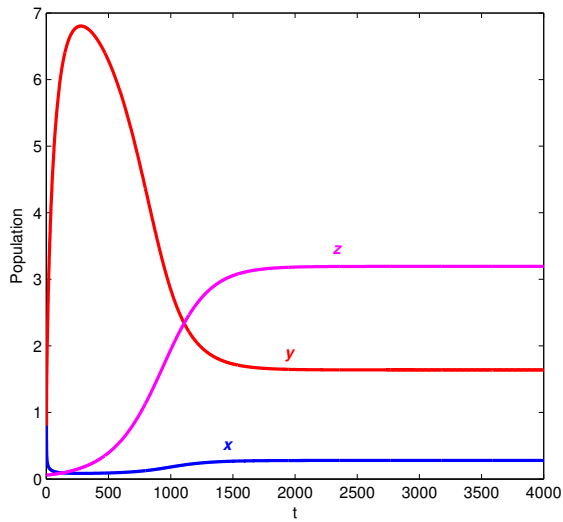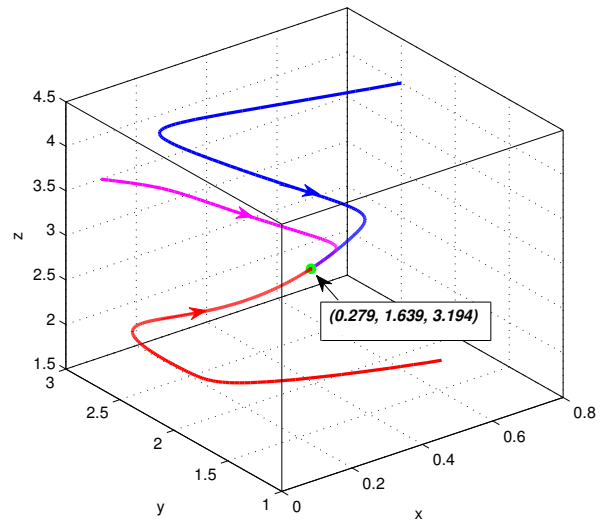Figure 5: Stable behaviour of E_2 in presence of fear ($k = 1.5$).

Figure 6: Transcritical Bifurcation around E_1 taking g as bifurcation parameter.
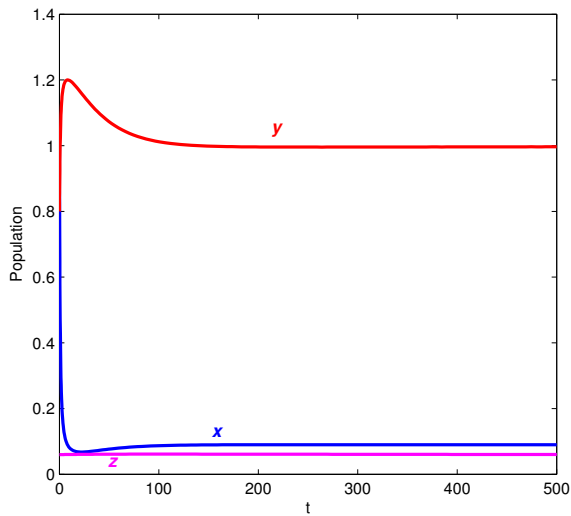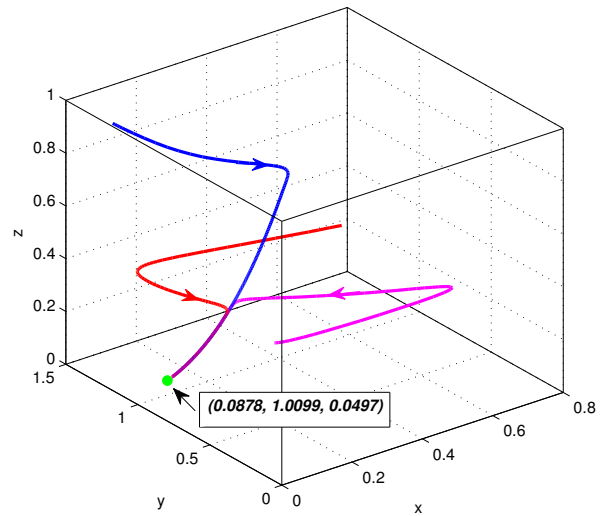
(7.a) *(7.b)*

Figure 7: Stable behaviour of E^* without any fear effect.

(8.a) *(8.b)*

Figure 8: Stable behaviour of E^* in presence of fear ($k = 1.5$).

growth of both middle and top-predator populations.

It is known that the planer equilibrium point E_2 is stable (locally) when $(\sigma p - n)\widetilde{y} < ne$ and $(1 - m)^2(\xi c - ag)(2ag - \xi c)\widetilde{y} < c\xi^2$ hold. Numerical calculations already give that in absence of fear when $n = 0.03$ we get a top-predator free steady state but a lesser value of n ($n = 0.01$) gives a system where all populations live simultaneously. So, for some threshold value of n, E^* and E_2 coincide each other. The parametric values give that at $n = n_{[TC]} = 0.0141$, a transcritical bifurcation occurs around E_2 and it loses its stability when n becomes lower than $n_{[TC]}$ (Figure 9). Now if we consider the impact of fear in the system, then for $k = 1.5$,
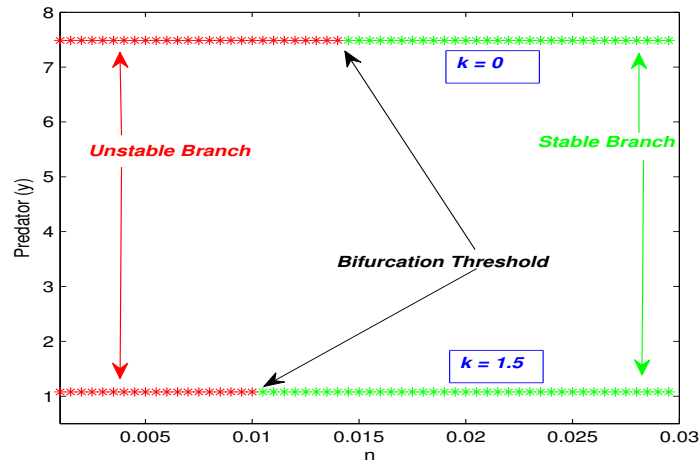
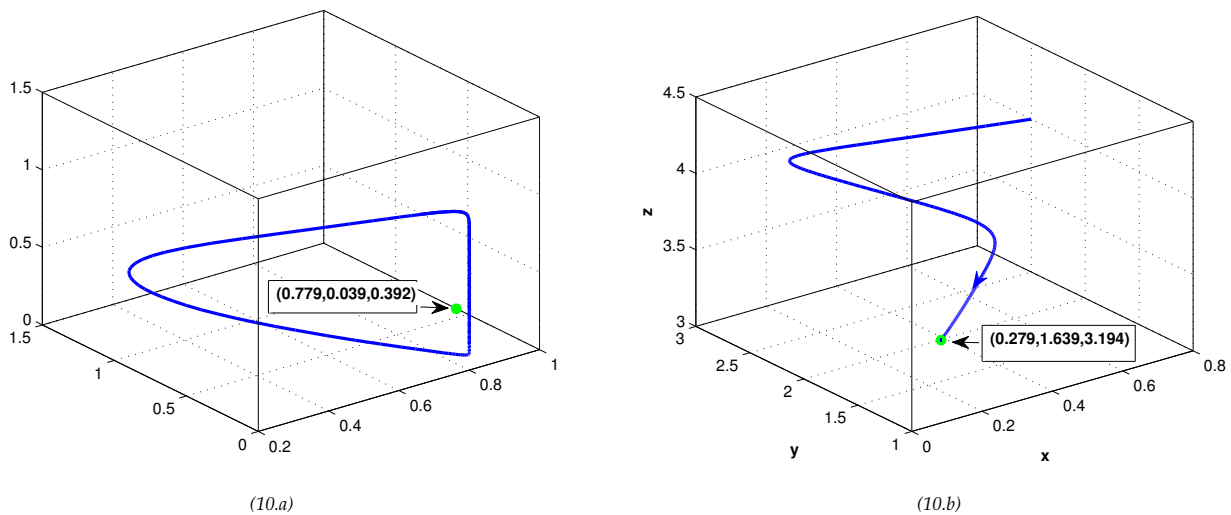Figure 9: Transcritical Bifurcation around E_2 taking n as bifurcation parameter: $k = 0$ and $k = 1.5$.

(10.a)   *(10.b)*

Figure 10: For $k = 0$: (10.a) Occurrence of limit cycle around E^* for $p > p_{[H]}$; (10.b) Stable behaviour around E^* for $p < p_{[H]}$.

we get the threshold value of n as $n_{[TC]} = 0.0102$. Thus, the fear coefficient has a stabilizing effect in the system even in the absence of top-predator.

When $k = 0$, $n = 0.01$ if we start to increase the consumption rate of middle-predator (p) gradually, then it is observed that after crossing $p_{[H]} = 0.125735$, a stable limit cycle occurs around the unstable equilibrium point and hence, a supercritical Hopf bifurcation occurs at $p = p_{[H]}$ (as 1st Lyapunov coefficient (L.E), $l_1 = -0.063329 < 0$) around $E^*_{p_{[H]}}(x^*_{p_{[H]}}, y^*_{p_{[H]}}, z^*_{p_{[H]}}) \equiv (0.662, 0.258, 2.143)$. Figure (10.a) depicts that at $p = 0.5$, a stable limit cycle occurs around the unstable interior point $E^*(0.779, 0.039, 0.392)$ while Figure (10.b) depicts that at $p = 0.05$ the trajectory converges to stable $E^*(0.279, 1.639, 3.194)$.

In presence of fear coefficient ($k = 0.5$), taking $g = 0.006$, $n = 0.01$, $d = 0.2$, it is observed that there is a threshold value of p above which the system exhibits a stable limit cycle around E^*. For these parameter values we get $p = p^k_{[H]} = 0.401471$ (1st L.E $= -0.1865 < 0$) with the coordinate $E^*_{p^k_{[H]}}(x^*_{p^k_{[H]}}, y^*_{p^k_{[H]}}, z^*_{p^k_{[H]}}) \equiv (0.748, 0.0496, 0.482)$ and thus the system undergoes a supercritical Hopf bifurcation around E^* at $p =$
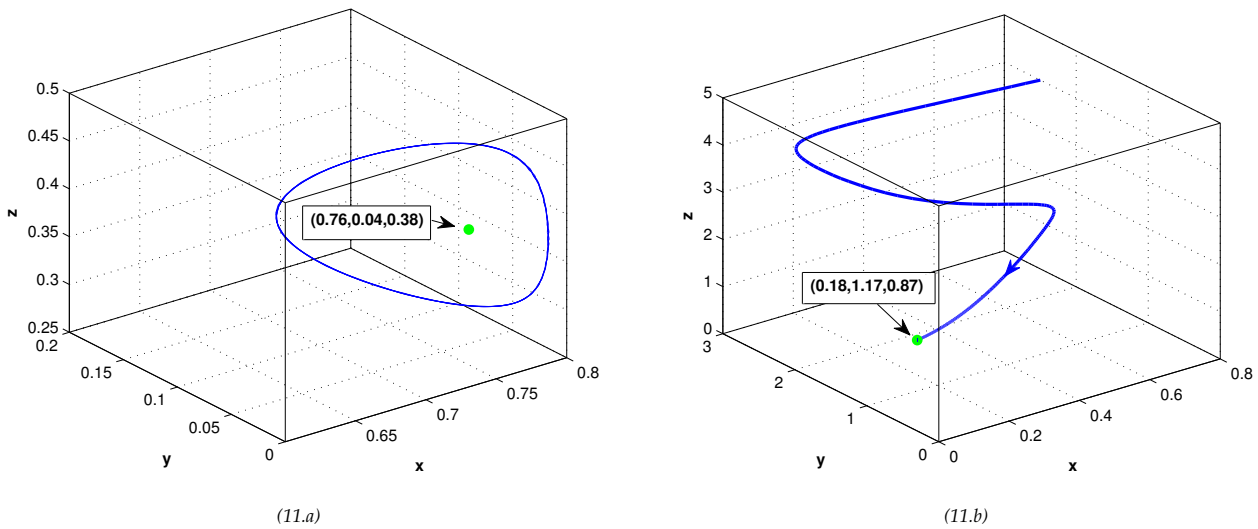
(11.a)   *(11.b)*

Figure 11: For $k = 0.5$: (11.a) Occurrence of limit cycle around E^* for $p > p_{[H]}^k$; (11.b) Stable behaviour around E^* for $p < p_{[H]}^k$.
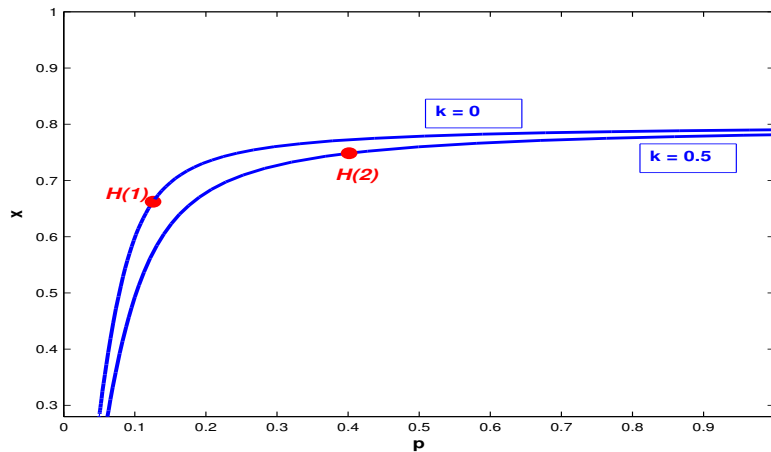

Figure 12: Hopf bifurcation thresholds in absence of fear ($k = 0$) and in presence of fear ($k = 0.5$). $H(1) : (p_{[H]}, x^*_{p_{[H]}})$ and $H(2) : (p^k_{[H]}, x^*_{p^k_{[H]}})$.

$p_{[H]}^k$. Figure (11.a) depicts that at $p = 0.5$, a stable limit cycle occurs around the unstable interior point $E^*(0.7599, 0.0384, 0.3797)$ while Figure (11.b) depicts that at $p = 0.05$ the trajectory converges to stable $E^*(0.1785, 1.1734, 0.8672)$.

Comparing the Hopf thresholds from calculations and Figure 12, it is observed that $p_{[H]} < p_{[H]}^k$. Thus, increasing consumption rate delay the system oscillation in the presence of fear coefficient. In figure 13, we have taken the scenarios when $k = 0$, 5.5 and $k = 10.5$ along with $p = 0.05$, $g = 0.006$ and $n = 0.001$. With the increasing value of k, it is observed that the growth rate of prey (x) has decreased, i.e., prey density is reduced in the presence of fear. Now if prey density starts to decrease, it results in a reduction of consumer's as well as top-predators densities. The pictures reflect the fact that the components of all population are decreased with increasing fear coefficient.

The fear coefficient (k) is an important parameter to control the system dynamics. In figure 14, we have drawn time series plots for different values of k taking $p = 0.5$ and $n = 0.01$. For $k = 0.5$, the trajectory forms
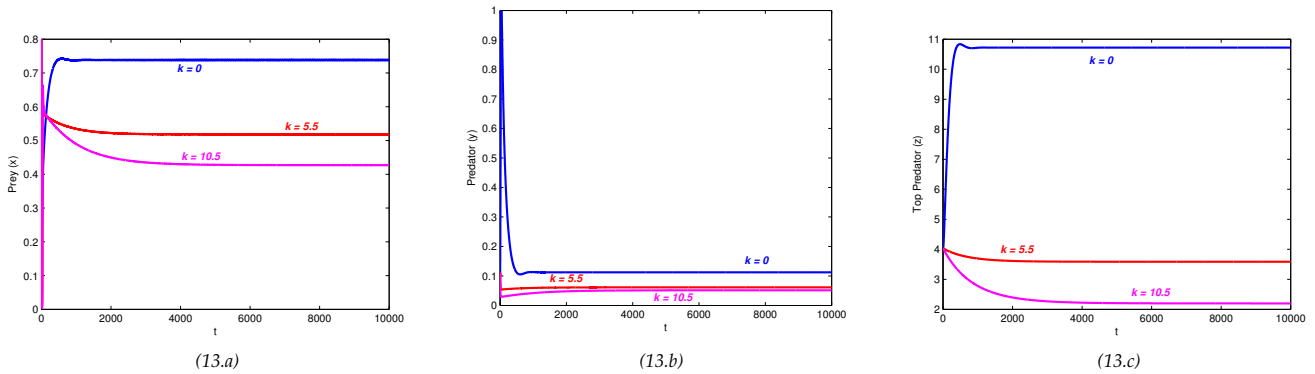
(13.a)   *(13.b)*   *(13.c)*

Figure 13: Time series plot of E^* for different value of k.
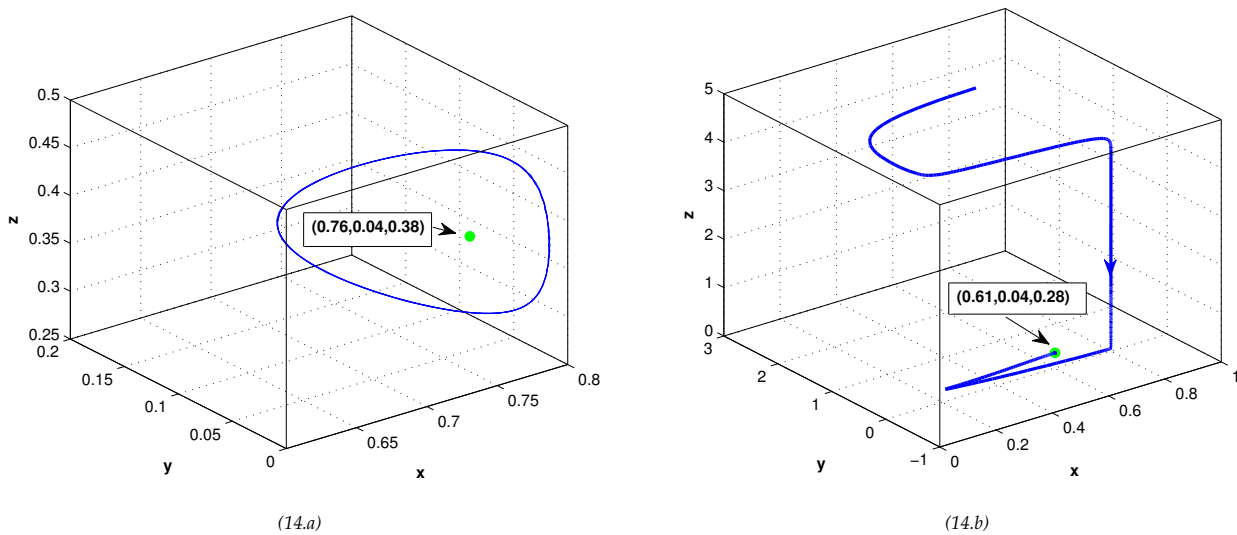

(14.a)   *(14.b)*

Figure 14: Change of behaviour of E^* for different value of k : − (14.a): oscillating behaviour for $k = 0.5$ and (14.b): stable behaviour for $k = 1.5$;

a stable limit cycle around the unstable equilibrium point $(0.7599, 0.0384, 0.3797)$ but for $k = 5.5$ the system converges to stable equilibrium point $(0.6086, 0.0377, 0.2796)$. So, the system undergoes a Hopf bifurcation at $k = k_{[H]} = 0.559147$, i.e., a stable limit cycle ($1^{st} L.E = −0.19423 < 0$) occurs for $k < k_{[H]}$ but disappears for $k > k_{[H]}$. It implies a sufficient amount of fear is necessary for a steady coexistent state.

  Now we analyze how the prey refuge makes an impact on the system. Figures 15 depicts the scenario of the equilibrium point by varying prey refuge in presence of fear effect ($k = 2.5$). Figure 16 shows that the prey population decreases whether the fear term is included or not but the slope of the curves become less sharp in the presence of fear than the case of without fear. As the growth rate is decreased due to fear of predation, so, a lesser number of prey is there in the system for consumption than the case of excluding fear effect. It effects the growth of middle-predator and top-predator population also because we have considered them to be specialist predators. The middle-predator and top-predator population show a significant declination in presence of fear term for increasing prey refuge. Though it is true that if a large number of prey is going as refuge, then the middle-predator population becomes small and the population becomes almost same for both the cases. This is true for the growth of top-predator population also.
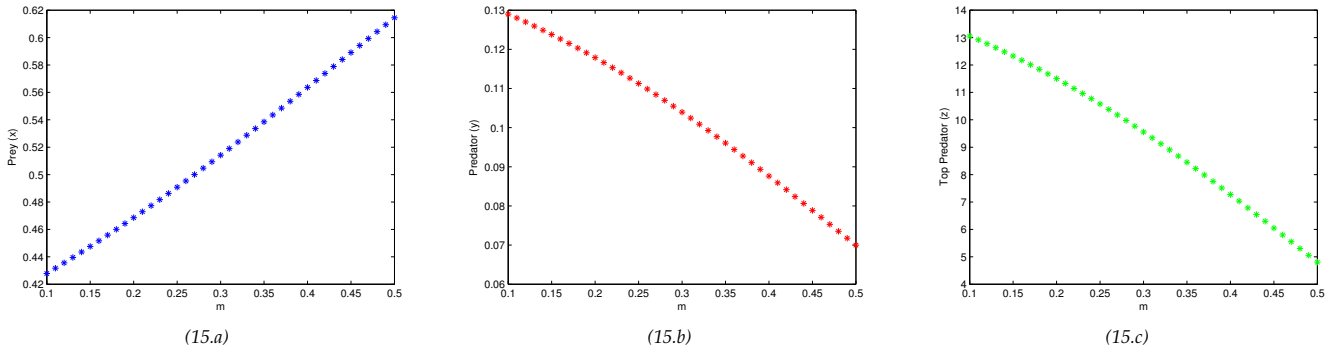
(15.a)   *(15.b)*   *(15.c)*

Figure 15: Influence of prey refuge on E^* in presence of fear effect ($k = 2.5$).

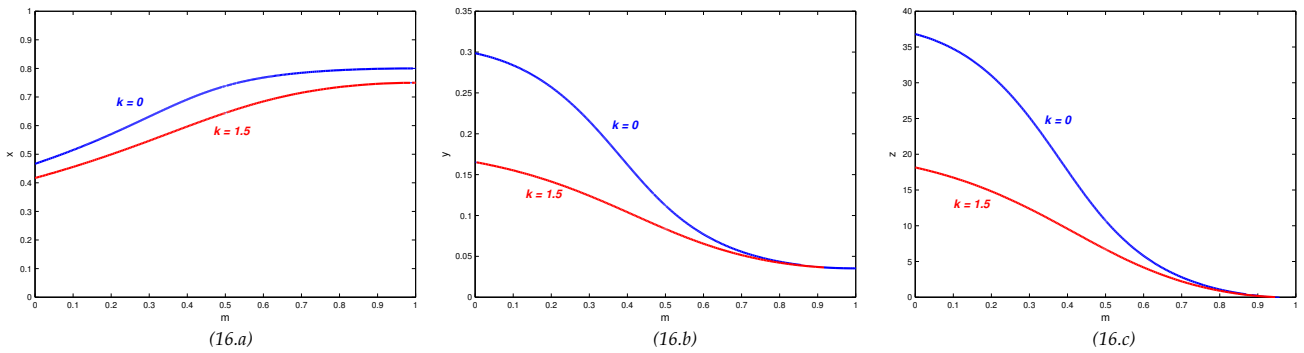*(16.a)*   *(16.b)*   *(16.c)*

Figure 16: Comparison of the components of E^* for $k = 0, 1.5$ while varying the prey refuge (m).

(17.a)   *(17.b)*
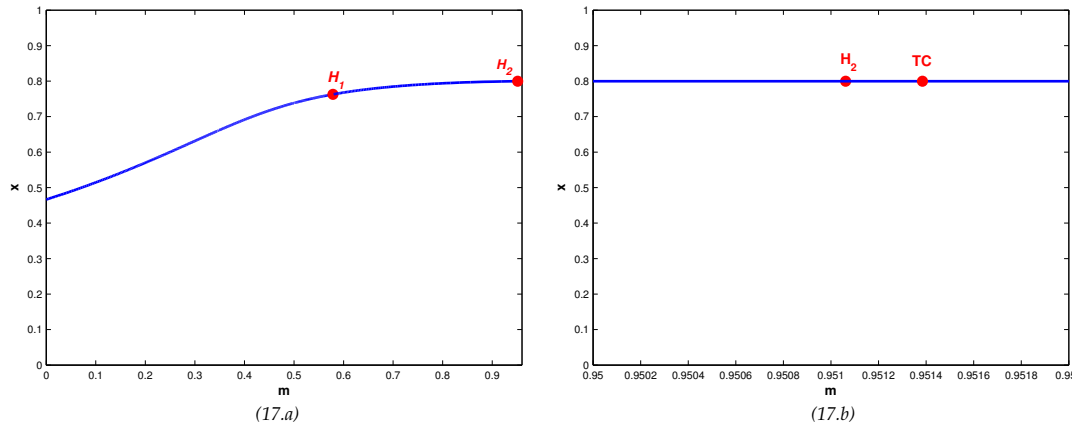
Figure 17: For $k = 0$: Bifurcation thresholds in m-x plane. Here "H_1" and "H_2" denote Hopf thresholds and "TC" denotes the transcritical threshold where E^* and E_2 coincide.

(18.a)   *(18.b)*   *(18.c)*
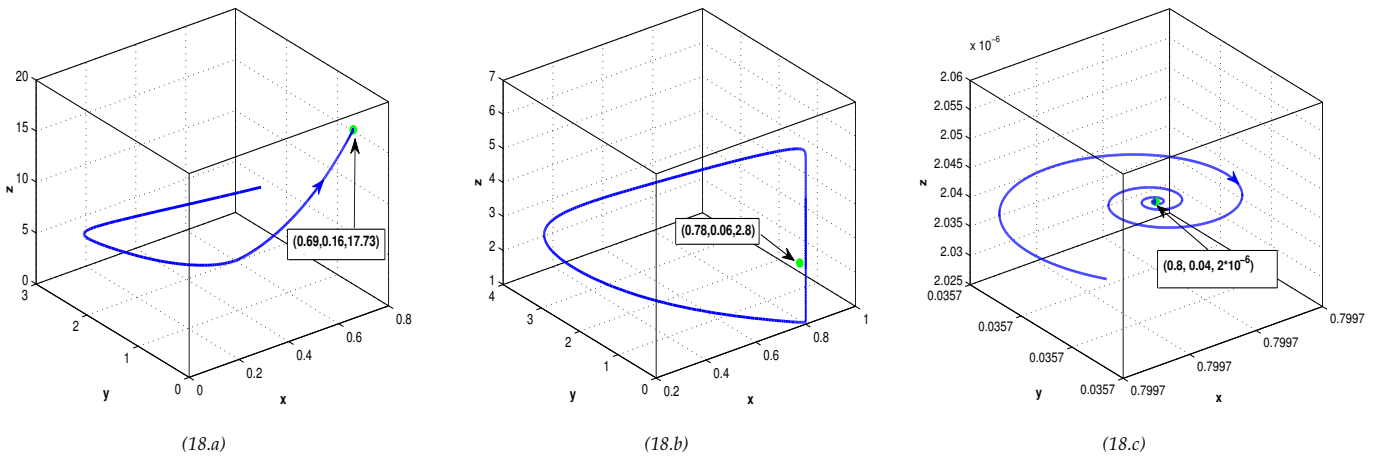
Figure 18: Stabilizing and destabilizing behaviour of system in absence of fear ($k = 0$) for different m: (18.a) Stable behaviour for $m = 0.4$, (18.b) Oscillating behaviour for $m = 0.7$ and (18.c) Stable behaviour for $m = 0.951384$.

The prey refuge (m) can control the system dynamics as well. Here we take $p = 0.05$, $g = 0.006$ and $n = 0.001$. Whether the fear effect is considered or not, for some threshold value of m the system can exhibit oscillating behaviour but it is observed that the instability does not last longer. So, m has a stabilizing as well as destabilizing effect. When there is no fear effect, it is observed that for $m < m_{[H]}^1$, we get interior steady state where all population coexist but when $m \in (m_{[H]}^1, m_{[H]}^2) = (0.578612, 0.951063)$ stable limit cycle occurs and at $m = m_{[H]}^1$, $m_{[H]}^2$ the system undergoes a Hopf bifurcation around E^*. Figure 18 shows that for $m = 0.4$ the trajectory converges to $(0.69, 0.16, 17.73)$ while for $m = 0.7$ stable limit cycle occurs around unstable $(0.78, 0.06, 2.8)$ and for $m = 0.951384$ the trajectory converges to $(0.8, 0.04, 2.04 * 10^{-6})$. When $m > m_{[H]}^2$, E^* again becomes stable and ultimately at $m = m_{[TC]} = 0.951385$, E^* coincides with E_2 and the equilibrium point be $(0.799665, 0.035714, 0)$. Thus the system undergoes a transcritical bifurcation around E^* at $m = m_{[TC]}$ (Figures 17,18).

Same thing can be observed if we even consider the fear coefficient in the system. Taking $k = 0.5$, it is obtained from figures 19 and 20, for $m < m_{[H]}^1 = 0.806773$, the trajectory converges to E^* whereas
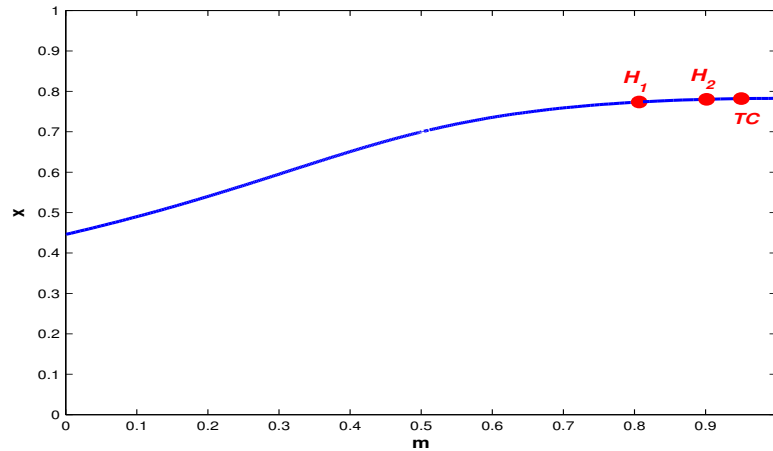
Figure 19: For $k = 1.5$: Bifurcation thresholds in m-x plane. Here "H_1" and "H_2" denote Hopf thresholds and "TC" denotes the transcritical threshold where E^* and E_2 coincide.
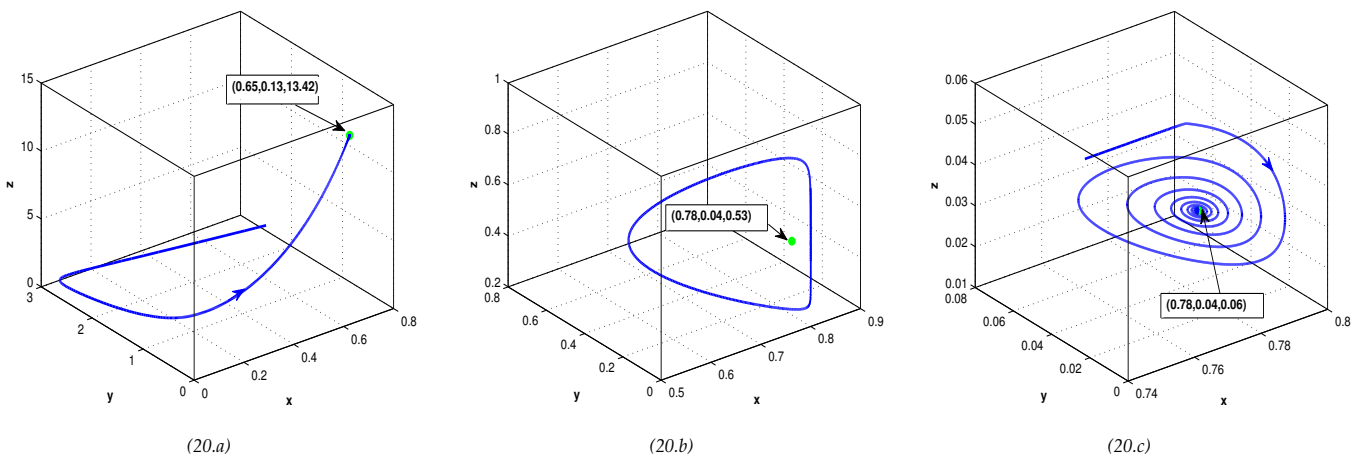

(20.a)   *(20.b)*   *(20.c)*

Figure 20: Stabilizing and destabilizing behaviour of system in presence of fear ($k = 1.5$) for different m: (20.a) Stable behaviour for $m = 0.4$, (20.b) Oscillating behaviour for $m = 0.85$ and (20.c) Stable behaviour for $m = 0.93$.

for $m^1_{[H]} < m < m^2_{[H]}$, oscillating behaviour occurs but after crossing $m^2_{[H]} = 0.901476$, the system again becomes stable. Figure 20 shows that for $m = 0.4$ the trajectory converges to $(0.65, 0.13, 13.42)$ while for $m = 0.85$ stable limit cycle occurs around unstable $(0.78, 0.04, 0.53)$ and for $m = 0.93$ the trajectory converges to $(0.78, 0.04, 0.06)$. So, at $m = m^1_{[H]}$, $m^2_{[H]}$ Hopf bifurcation occurs and in $(m^1_{[H]}, m^2_{[H]})$ stable limit cycle occurs around unstable equilibrium point and ultimately at $m = m_{[TC]} = 0.950294$, it coincides with E_2 with component $(0.782114, 0.035714, 0)$. So, at $m = m_{[TC]}$, a transcritical bifurcation occurs around E^*. So, it is observed that inclusion of fear term decreases the range of refuge parameter where oscillation occurs in the system.

11. Conclusion

Prey-predator interaction is a basic phenomenon of the biological system and this interaction often balance the food web. It is evident that the predator can live by consuming the prey population and the searching strategies depend on many factors. Sometimes, it is assumed that the predation term depends only on the

resource population size and their availability but it is also true that not only the prey but the predator density and their interference can also affect the predation term. In this work, a tritrophic food chain model is considered where the predation terms between prey-consumer and consumer-predator have been considered by Holling type-III and Beddington-DeAngelis functional responses respectively. At the time of hunting, predators usually expose themselves to a higher risk of predation and also they prefer to consume with higher fitness. But their behaviour effect other species' consumption strategies, i.e., the interactions between the forager's predator and forager's food. Zenette et al. [54] in their work have proved that reproduction of song sparrow has reduced at a large rate due to fear of predation risk. In 2016, Suraci et al. [44] have concluded that the fear of being consumed by large carnivores can decrease foraging time as well as foraging behaviour of mesocarnivores by almost 66%. Here it is assumed that only the birth rate of prey population is affected due to fear of predation. Also, only a fraction of prey is available to the middle-predators as prey refuge term is present in the system. It is observed from the results that both the fear effect and prey refuge play vital roles to control the system dynamics. The fear factor has a stabilizing effect as increasing value of k can turn the oscillating behaviour of the population into a stable state. If the fear coefficient starts to increase, the prey species decreases and it affects the other population as both the middle and top-predator population start to decrease with increasing fear coefficient. On the other hand, if a larger amount of prey successfully hides themselves, then the prey population starts to increase resulting in the declination of middle-predator and top-predator populations. The interesting part is that if we observe the influence of prey refuge in presence of fear, then it is obtained that the prey increases for a higher value of refuge but the amount is lesser than the case for $k = 0$. Also, the middle-predator and top-predator decrease with increasing m but the amount be always lesser than the case without any fear effect. Moreover, the prey refuge has a stabilizing as well as a destabilizing effect. When there is a small amount of refuge present in the system, the interior point is found to be as a steady state but for a moderate value of refuge parameter, the system loses its stability through a Hopf bifurcation. Further for a higher value of refuge, the system becomes stable again and ultimately the top-predator goes extinct and we obtain a steady state consists of prey and consumer populations only. Moreover, the results reveal that inclusion of fear term decreases the range of refuge parameter where oscillation occurs in the system.

As days go, more researchers are showing interest in fear effect problems but most of the cases are limited within two dimensions. Here, we have taken a step further by considering three-dimensional problem and incorporation of prey refuge has made the model more realistic. It can be concluded from the analytical and numerical results that the model with prey refuge and fear effect exhibits a rich dynamics. But, this model can be refined further. We can consider the fear term as a periodic function. Moreover, the prey refuge term can also be taken as predator dependent function. Also, the consumption of food is not a process of a fraction of time (i.e., not an instantaneous process); the predators take some time to digest the consumed food, which is called 'gestation delay'. So, the future work can be dealt with some models considering all these facts to make the models more realistic.

Acknowledgements

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