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Hopkins-Levitzki Theorem for Krasner Hyperrings

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Abstract. In this paper, our aim is to generalize and extend the Hopkins-Levitzki theorem from non-commutative rings to Krasner hyperring. Also, we prove that a Krasner hyperring R is Noetherian if and only if it satisfies the ascending chain conditions of prime hyperideals.

1. Introduction

In recent years, the study of algebraic hyperstructures has attracted a number of authors as many papers in this aspects have been recently published. The concept of algebraic hyperstructures is a natural generalization of classical algebraic structures such as group, ring, module and etc. In the classical algebraic structure such as groups, rings and etc, we use binary operation and the composition of two elements is an element, while in an algebraic hyperstructure, the result of this composition is a set. In the literature, the theory of hyperstructure was first initiated by F. Marty [10] at the 8^{th} Congress of Scandinavian Mathematicians in 1934 and he first introduced the concept of hypergroups and presented some of their applications, with some utility in the study of groups, algebraic functions and rational fractions.

The general algebraic structure that satisfies the ring-like axioms is called the hyperring. Now, we let $(R, +, \circ)$ be an algebraic structure where $+$ and $\circ$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\circ$ is an associative hyperoperation which is distributive with respect to $+$. There are different types of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\circ$ is the usual operation, then we say that R is an additive hyperring. The concept of multiplicative hyperring was first introduced by R. Rota [12] in 1982. In this hyperring, the multiplication is a hyperoperation, while the addition is an operation, A special case of this type is the hyperring introduced by M. Krasner [9]. Also, Davvaz et.al. introduced the notions of normal, prime, maximal, and Jacobson radical of a hyperring and by considering these notions they obtained some results and they defined hyperring of fractions and hyper-valuation on a hyperring and they proved the Chinese Remainder Theorem for the case of hyperrings[6]. The concept of ordering hypergroups introduced by Davvaz et. al. [11] as a special class of Krasner hyperrings. For information concerning the structure of hyperstructure, hypergroups and hyperrings can be found in the monographs by B. Davvaz and V. Leonreanu [5].

It is well known that Hopkins-Levitzki theorem connected the ascending chain condition and descending chain condition in modules over semisimple rings [1]. In particular, every right Artinian ring is right

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Noetherian [7]. The theorem was first formulated by C. Hopkins and J. Letvitzki in 1933. It is noted that W. Krull and Y. Akizuki developed the general theory relating the structures of Artinian rings and Noetherian rings around 1940. Also, P. M. Cohn proved that if every family of prime ideals satisfied ascending chain condition, then R is Noetherian rings [2]. Also, Davvaz et. al. gave a characterization of new generalization of prime hyperideals in Krasner hyperrings by introducing 2-absorbing hyperideals and they studied study fundamental properties of 2-absorbing hyperideals on Krasner hyperrings and investigate some related results [8].

In this paper, we are going to generalize the well known Hopkins-Levitzki theorem and P. M. Cohn Theorem[2, 7], for Krasner hyperrings by using a complete part concept. We also prove that a Krasner hyperring R is Noetherian if and only if it extends the ascending condition of the prime hyperideals in Krasner hyperings.

2. The Krasner hyperring

Let H be a non-empty set. Then, a map $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* on the set H, where $P^*(H)$ denotes the set of all non-empty subsets of H. Recall that *hypergroupoid* is a non-empty set H endowed with a hyperoperation and denote it by $(H, \circ)$.

Definition 2.1. *[10] A hypergroupoid $(H, \circ)$ is said to be a semihypergroup if for all $x, y, x \in H$, we have $(x \circ y) \circ z = x \circ (y \circ z)$ which means that*

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

Definition 2.2. *[10] A non-empty subset A of a semihypergroup $(H, \circ)$ is called a subsemihypergroup if $(A, \circ)$ is a semihypergroup. In other words, a non-empty subset A of a semihypergroup $(H, \circ)$ is a subsemihypergroup if $A \circ A \subseteq A$.*

If $x \in S$ and A, B are non-empty subset of H, then we have

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \;\; A \circ x = A \circ \{x\} \;\; x \circ B = \{x\} \circ B.$$

Definition 2.3. *[5] A canonical hypergroup is an algebraic structure $(H, +)$ which satisfies the following axioms:*

1) *for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$,*
2) *for every $x, y \in H$, $x + y = y + x$,*
3) *there exists $0 \in H$ such that $0 + x = \{x\}$, for every $x \in H$*
4) *for every $x \in H$ there exists a unique element x' such that $0 \in x + x'$;*
   (we shall write $-x$ for x' and we call it as the opposite of x.
5) *$z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$.*

Definition 2.4. *[4] Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then we have*

$$A\overline{R}B \Longleftrightarrow \forall a \in A \; \exists b \in B : aRb \text{ and}$$

$$\forall b' \in B, \; \exists a' \in A : a'Rb';$$

$$A\overline{\overline{R}}B \Longleftrightarrow \forall a \in A, \forall b \in B, \;\; \text{we have } \; aRb.$$

Definition 2.5. *[4] Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. Then, the relation R is called*

1) *regular on the right (on the left) if for all x of H, from aRb, it follows that* $(a \circ x)\overline{R}(b \circ x)((x \circ a)\overline{\overline{R}}(x \circ b))$;

2) *strongly regular on the right (on the left) if for all* $x \in H$, *from aRb, it follows that* $(a \circ x)\overline{\overline{R}}(b \circ x)((x \circ a)\overline{\overline{R}}(x \circ b))$;

2) *R is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left.*

Theorem 2.6. *[3] Let* $(H, \circ)$ *be a semihypergroup and R be an equivalence relation on H. Then, the following statements hold:*

1) *If R is regular, then H/R is a semihypergroup, with respect to the following hyperoperation*

$$R(x) \otimes R(y) = \{R(t) : t \in x \circ y\}.$$

2) *If the hyperoperation* $\otimes$ *is well-defined on H/R, then R is regular.*

Corollary 2.7. *Let* $(H, \circ)$ *be a hypergroup and R is an equivalence relation on H. Then R is regular if and only if* $(H/R, \otimes)$ *is a hypergroup.*

We now give the following crucial definitions in this paper.

Definition 2.8. *[9] A Krasner hyperring is an algebraic structure* $(R, +, \cdot)$ *which satisfies the following axioms:*

1) $(R, +)$ *is a canonical hypergroup,*

2) $(R, \cdot)$ *is a semigroup having zero as a a bilateral absorbing element, i.e.,*

$$\forall x \in R, \ \ x \cdot 0 = 0 \cdot x = 0,$$

3) *The multiplication* $\cdot$ *is distributive with respect to the hyperoperation* $+$.

The following elementary facts follow easily from axioms: $-(-x) = x, -(x+y) = -x-y, -A = \{-a : a \in A\}$.

When semigroup $(R, \cdot)$ is commutative (with unit element), the Krasner hyperring $(R, +, \cdot)$ is called commutative (with unit element).

Definition 2.9. *[9] Let* $(R, +, \cdot)$ *be a Krasner hyperring and A be a nonempty subset of R. Then, A is said to be a subhyperring of R if* $(A, +, \cdot)$ *is itself a Krasner hyperring.*

Definition 2.10. *[5] A subhyperring I of a Krasner hyperring R is a left (right) hyperideal of R if* $r \cdot a \in I$ $(a \cdot r \in I)$ *for all* $r \in R$ *and* $a \in I$. *Also, I is called a hyperideal if I is both a left and a right hyperideal.*

Lemma 2.11. *[5] A non-empty subset I of a Krasner hyperring R is a left (right) hyperideal if and only if*

1) $a, b \in I$ *implies that* $a - b \subseteq I$,

2) $a \in A, \ r \in R$ *implies that* $r \cdot a \in I$ $(a \cdot r \in I)$.

Proof. The proof is straightforward. □

Proposition 2.12. *Let R be a ring and* ρ *be an equivalence relation on R such that*

$$x\rho y \Longleftrightarrow x = y \text{ or } x = -y,$$

Then, the set $\overline{R} = \{\overline{x} : x \in R\}$, *where* $\overline{x}$ *is an equivalence class of x, is a Krasner hyperring by following hyperoperation and multiplication:*

$$\overline{x} \oplus \overline{y} = \{\overline{x+y}, \overline{x-y}\}, \quad \overline{x} \odot \overline{y} = \overline{xy}.$$

Also, J is a hyperideal of $\overline{R}$ *if and only if* $T = \overline{I}$, *where I is an ideal R.*

Proof. The proof is straightforward and hence it is omitted. □

Definition 2.13. *[5] Let $(R, +, \cdot)$ be a Krasner hyperring and ρ be an equivalence relation on R. Then, ρ is called left (right) regular relation, when $x_1\rho x_2$ implies that $(z + x_1)\overline{\rho}(z + x_2)$ and $\rho(z \cdot x_1) = \rho(z \cdot x_2)$ $((x_1 + z)\overline{\rho}(x_2 + z)$ and $\rho(x_1 \cdot z) = \rho(x_2 \cdot z))$. An equivalence relation ρ is called left (right) strongly regular, when $x_1\rho x_2$ implies that $(z + x_1)\overline{\overline{\rho}}(z + x_2)$ and $\rho(z \cdot x_1) = \rho(z \cdot x_2)$ $(\rho(x_1 \cdot z) = \rho(x_2 \cdot z)$ and $(x_1 + z)\overline{\overline{\rho}}(x_2 + z))$. An equivalence relation ρ is called regular (strongly regular), when it is regular (strongly regular) on the right and on the left.*

Let I and J be hyperideals of Krasner hyperring R. Then, the following equality holds,

$$IJ = \left\{ x \in R : x \in \sum_{i=1}^{n} a_i b_i,\ n \in \mathbb{N}, a_i, b_i \in R \right\}.$$

Definition 2.14. *A proper hyperideal P of a Krasner hyperring R is called prime if for every pair of hyperideals I and J of R,*

$$IJ \subseteq P \Longrightarrow I \subseteq P \ \text{ or } J \subseteq P.$$

For Krasner hyperrings, we have the following Propositions.

Proposition 2.15. *[5] Let R be a Krasner hyperring and ρ be a regular relation on R. Then, $R/\rho = \{\rho(t) : t \in R\}$ is a Krasner hyperring by the following hyperaddition and multiplication:*

$$\rho(x_1) \oplus \rho(x_2) = \{\rho(t) : t \in x_1 + x_2\},$$

$$\rho(x_1) \cdot \rho(x_2) = \rho(x_1 \cdot x_2).$$

Corollary 2.16. *Let R be a Krasner hyperring and ρ be a strongly regular relation. Then, R/ρ is a ring.*

Proposition 2.17. *Let R be a Krasner hyperring and ρ be a regular(strongly regular) relation on R. Then, T is a hyperideal (ideals) of R/ρ if and only if $T = I/\rho$, where I is a hyperideal R.*

Proof. The proof is straightforward. □

Proposition 2.18. *Let I be a hyperideal of Krasner hyperring R and the relation I^* defined as follows:*

$$x_1 I^* x_2 \Longleftrightarrow (x_1 - x_2) \cap I \neq \emptyset.$$

Then, this relation is equivalence.

Proof. Obviously, the above relation is symmetric and reflexive. Let $x_1 I^* x_2$ and $x_2 I^* x_3$. Then, $(x_1 - x_2) \cap I \neq \emptyset$ and $(x_2 - x_3) \cap I \neq \emptyset$. This implies that there are $a \in (x_1 - x_2) \cap I$ and $b \in (x_2 - x_3) \cap I$. Hence , $x_1 \in x_2 + a$ and $x_2 \in x_3 + b$. It follows that $x_1 \in x_3 + a + b$. Since $a + b \subseteq I$, there exists $d \in I$ such that $x_1 \in x_3 + d$. Therefore, we have $d \in (x_1 - x_3) \cap I$ and the relation I^* is transitive. □

Now, we form the following theorem.

Theorem 2.19. *Let I be a hyperideal of Krasner hyperring R. Then, we define the hyperoperation $\oplus$ and the multiplication $\otimes$ on the set of all classes $R/I^* = \{I^*(x) : x \in R\}$, as follows:*

$$I^*(x_1) \oplus I^*(x_2) = \{I^*(x) : x \in x_1 + x_2\},$$

$$I^*(x_1) \otimes I^*(x_2) = I^*(x_1 \cdot x_2).$$

Consequently, R/I^* is a Krasner ring.

Proof. Suppose that $x_1 I^* x_2$ and $x \in R$. Then we have $(x_1 - x_2) \cap I \neq \emptyset$ and $(x_1 + x) \subseteq (x_2 + x) + I$. It follows that for every $t_1 \in x_1 + x$ there exist $t_2 \in x_2 + x$ and $a \in I$ such that $t_1 \in t_2 + a$. Hence $(t_1 - t_2) \cap I \neq \emptyset$ and $t_1 I^* t_2$. In the same way, for every $s_1 \in x_2 + x$ there is $s_1 \in x_1 + x$ such that $s_1 I^* s_2$. Therefore, the relation I^* is regular and by Proposition 2.15, R/I^* is a Krasner hyperring. □

3. Noetherian(Artinian) Krasner hyperrings

C. Hopkins proved that every right (left) Artinian ring is right (left) Noetherian. Also, P. M.Cohn proved that if every family of prime ideals satisfied the ascending chain condition, then R is Noetherian. In this section, we generalized Hopkins and Cohen Theorem for Krasner hyperrings.

A Krasner hyperring R satisfies the descending chain condition if every descending chain of hyperideals of R

$$T_1 \supseteq T_2 \supseteq ...$$

and there exists an integer n such that $T_n = T_{n+1} = T_{n+2} = ...$

Also, a Krasner hyperring R satisfies the ascending chain condition if every ascending chain of hyperideals of R.

$$T_1 \subseteq T_2 \subseteq ...$$

and there exists an integer n such that $T_n = T_{n+1} = T_{n+2} = ...$

Definition 3.1. *Let R be a Krasner hyperring. Then, R is said to be Noetherian, when R satisfies the ascending chain condition on hyperideals and R is said to be Artinian, when R satisfies the descending chain condition.*

Let $\overline{R}$ be a Krasner hyperring defined in the Proposition 2.12. Now, we present a construction of Noetherian (Artinian) Krasner hyperring as follows:

Proposition 3.2. *Let R be a ring. Then, $\overline{R}$ is a Noetherian (Artinian) Krasner hyperring if and only if R is a Noetherian (Artinian) ring.*

Proof. The proof is straightforward. □

In what follows, we present Noetherian Krasner hyperring such that is not Artinian Krasner hyperring.

Example 3.3. *Let $\mathbb{Z}$ be integer numbers. Then, by Proposition 3.2, the Krasner hyperring $\overline{\mathbb{Z}}$ is Noetherian but it is not an Artinian Krasner hyperring.*

Example 3.4. *Let p be a prime number. Then, we can easily see that $\mathbb{Z}_{p^\infty}$ is an Artinian ring. By Proposition 3.2, we see that $\overline{\mathbb{Z}_{p^\infty}}$ is an Artinian Krasner hyperring.*

Theorem 3.5. *Let R be Noetherian (Artinian) Krasner hyperring and I be a hyperideal. Then, R/I^* is a Noetherian (Artinian) Krasner hyperring.*

Proof. Suppose that R is a Noetherian Krasner hyperring and $T_1 \subseteq T_2 \subseteq ... \subseteq T_n \subseteq T_{n+1}...$ is a ascending chain of hyperideals of the Krasner hyperring R/I^*. Then, we have $T_n = I_n/I^*$, where $I \subseteq I_n$ are hyperideals of the Krasner hyperring R. Let $I^*(x_1) \in I_n/I^*$. Then, we arrive $I^*(x_1) = I^*(x_2)$, where $x_2 \in I_{n+1}$. Indeed, $I_n/I^* \subseteq I_{n+1}/I^*$. Hence, $(x_1 - x_2) \cap I \neq \emptyset$. It follows that there exists an element $a \in I$ such that $a \in x_1 - x_2$, and so $x_1 \in x_2 + a$. Thus, $x_1 \in I_{n+1}$ and there is ascending chain $I_1 \subseteq I_2 \subseteq ....I_n \subseteq I_{n+1} \subseteq ....$ Since R is a Noetherian Krasner hyperring this chain stop after a finite number steps. Therefore, R/I^* is a Noetherian Krasner hyperring. □

Theorem 3.6. *Let I be a hyperideal of Krasner hyperring R such that I and R/I^* be Noetherian (Artinian) Krasner hyperring. Then, R is a Noetherian (Artinian) Krasner hyperring.*

Proof. Suppose that $\{I_n\}_{n=1}^{\infty}$ is a family of hyperideals such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$ is an ascending chain of hyperideals of R. Then, we have $(I_1 + I)/I^* \subseteq (I_2 + I)/I^* ... \subseteq (I_n + I)/I^* ...$ and $(I_1 \cap I) \subseteq (I_2 \cap I) \subseteq ... \subseteq (I_n \cap I) \subseteq ...$ are ascending chain hyperideals of R/I^* and I, respectively. Hence there are $k_1, k_2 \in \mathbb{N}$, such that for every $n \geq k = max\{k_1, k_2\}$,

$$(I_n + I)/I^* = (I_k + I)/I^*, \ \ (I_n \cap I) = (I_k \cap I).$$

Let $x \in I_n$. Then, we have $I^*(x) \in (I_n + I)/I^*$. It follows that $I^*(x) \in (I_k + I)/I^*$ and there exists $x_1 \in I_k + I$ such that $I^*(x) = I^*(x_1)$. Hence, $(x - x_1) \cap I \neq \emptyset$. This implies that $x \in x_1 + a$ and $x_1 \in c + d$, where $a, d \in I$, $c \in I_k$. Thus, we have $x \in c + a + d \subseteq c + I$ and there is $w \in I$ such that $x \in c + w$. Hence, $w \in x - c \subseteq I_n \cap I$. This implies that $w \in I \cap I_k$. Therefore, for every $n \geq k$, $I_n = I_k$. This completes the proof. □

Definition 3.7. *Let $(R, +, \cdot)$ be a hyperring. We now define the relation γ as follows:*

$$a\gamma b \Longleftrightarrow \{a, b\} \subseteq u,$$

where $u \in U$ and

$$U = \left\{ u \in R : u \in \sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij},\ n, m \in \mathbb{N}, x_{ij} \in R \right\}.$$

We denote the transitive closure of γ by γ^*. The equivalence relation γ^* is called the fundamental equivalence relation in R. The equivalence class of the element a denoted (also this class is called the fundamental class of a) by $\gamma^*(a)$.

By the distributive law, every set which is the value of a polynomial in elements of R is an element of U.

Theorem 3.8. *[5] Let $(R, +, \cdot)$ be a hyperring. Then, the relation γ^* is a transitive closure of γ and is the smallest equivalence relation such that the quotient R/γ^* is a ring by following hyperaddition $\oplus$ and multiplication $\odot$:*

$$\gamma^*(a) \oplus \gamma^*(b) = \{\gamma^*(c) : c \in a + b\},$$

$$\gamma^*(a) \odot \gamma^*(b) = \gamma^*(a \cdot b),$$

where $\gamma^*(a), \gamma^*(b) \in R/\gamma^*$.

Definition 3.9. *Let $(R, +, \cdot)$ is a semihypergroup and C be a nonempty subset of C. Then, C is called a complete part of R if for any nonzero natural number n and for $x_{ij} \in R$, the following implication holds:*

$$C \cap \left(\sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij} \right) \neq \emptyset \Longrightarrow \sum_{i=1}^{n} \prod_{j=1}^{m_i} x_{ij} \subseteq C.$$

Example 3.10. *Let R be a ring and $\overline{R}$ be a Krasner hyperring constructed in Proposition 2.12. Then, every hyperideal of $\overline{R}$ is a complete part.*

Proposition 3.11. *Let R be a Krasner hyperring and $\{I_n\}_{n\in\mathbb{N}}$ be a chain of hyperideals such that are complete part. Then, $\bigcup_{n\geq 1} I_n$ is a hyperideal such that is a compete part.*

Proof. The proof is straightforward we omit the proof. □

Now, we determine the necessary and sufficient condition for the Noetherian Krasner hyperring R.

Theorem 3.12. *Let R be a Krasner hyperring such that every hyperideals are complete part. Then, R/γ^* is a left (right) Noetherian ring if and only if R is a left (right) Noetherian Krasner hyperring.*

Proof. Suppose that R is a left (right) Krasner hyperring and $I_1/\gamma^* \subseteq I_2/\gamma^* \subseteq ... \subseteq I_n/\gamma^* \subseteq ....$ is an ascending chain of left (right) hyperideals. Let $a \in I_n$. Then, we have $\gamma^*(a) \in I_n/\gamma^*$. Since $I_n/\gamma^* \subseteq I_{n+1}/\gamma^*$, there is $\gamma^*(b) \in I_{n+1}/\gamma^*$ such that $\gamma^*(a) = \gamma^*(b)$. Hence there exist $x_0, x_1, ..., x_{n+1} \in R$ such that $x_0 = a, x_{n+1} = b$ and $x_i\gamma x_{i+1}$, for $1 \leq i \leq n + 1$. It follows that there exists a sum of hyperproducts P_i such that $\{x_i, x_{i+1}\} \subseteq P_i$. For $i = n$, $\{x_{n-1}, x_n\} \cap P_n$. Hence $I_{n+1} \cap P_n \neq \emptyset$ and this implies that $P_n \subseteq I_{n+1}$. Since I_{n+1} is a complete part, $\{x_{n-1}, x_n\} \subseteq P_{n-1} \subseteq I_{n+1}$. After a finite number of steps, we obtain that $a = x_1 \in I_{n+1}$. Hence, for every $n \geq 1$,

$I_n \subseteq I_{n+1}$.Then there is ascending chain $I_1 \subseteq I_2 \subseteq .... \subseteq I_n \subseteq I_{n+1}....$ Since R is a Noetherian Krasner hyperring, there is $n \in \mathbb{N}$, such that $I_k = I_n$. Therefore, we have shown that R/γ^* is a Noetherian ring.

Conversely, suppose that R/γ^* is a left (right) Noetherian ring and $I_1 \subseteq I_2 \subseteq ..... \subseteq I_n \subseteq I_{n+1} \subseteq ...$ is an ascending chain of left (right) hyperideals. Hence, there is ascending chain of left (right) hyperideals $I_1/\gamma^* \subseteq I_2/\gamma^* \subseteq ... \subseteq I_n/\gamma^* \subseteq ...$ of left (right) hyperideals R/γ^*. Since R/γ^* is a left (right) Noetherian ring, implies that there exists $n \in \mathbb{N}$ such that for every $k \geq n$, $I_k/\gamma^* = I_n/\gamma^*$. By a similar argument for every $k \geq n$, $I_n = I_k$ and R is a left (right) Krasner hyperring. □

We now consider the characterization of Artinian hyperrings.

Theorem 3.13. *Let R be a left (right) Krasner hyperring such that every left (right) hyperideals are complete parts. Then, R is a left (right) Artinian hyperring if and only if R/γ^* is a left (right) Artinian ring.*

Proof. The proofs are similar to the proofs of Theorems 3.12. □

As Hopkins theorem stated that a left-Artinian ring with unit is also left-Noetherian [7]. In Theorem 3.14, we generalized the Hopkins theorem for Krasner hyperring with unit.

Theorem 3.14. *Let R be a left (right) Artinian Krasner hyperring with unit such that every left (right) hyperideals be complete part. Then, R is a left (right) Noetherian Krasner hyperring.*

Proof. Suppose that R is a left (right) Artinian Krasner hyperring with unit. By Theorem 3.8, R/γ^* is a ring with unit and by Theorem 3.13, R/γ^* is a left (right) Artinian ring with unit. Hopkins-Theorem [7] implies that R/γ^* is a left (right) Noetherian ring. Therefore, by Theorem 3.13, R is a left (right) Krasner hyperring. □

P. M. Cohn proved that a commutative ring R is Noetherian if and only if every ascending chain of prime ideals must be stopped [2]. In the Theorem 3.16, we generalize the well known P.M. Cohn theorem for Krasner hyperring.

Proposition 3.15. *Let R be a Krasner hyperring such that every hyperideals be complete part. Then, T is a prime hyperideal of R/γ^* if and only if $T = P/\gamma^*$ and P is a prime hyperideal of R.*

Proof. The proof is straightforward we omit the proof. □

Theorem 3.16. *Let every hyperideals of commutative Krasner hyperring R be a complete part and R satisfies the ascending chain condition on prime hyperideals. Then, R is a Noetherian Krasner hyperring.*

Proof. Suppose that R be a commutative Krasner hyperring such that $T_1 \subseteq T_2 \subseteq T_3 \subseteq ... \subseteq T_n \subseteq T_{n+1} \subseteq ...$ is an ascending chain of prime ideals of commutative ring R/γ^*. Then by Proposition 3.15, for every $n \geq 1$ $T_n = P_n/\gamma^*$, where P_n are prime hyperideals of R. Let $a \in P_n$. This implies that $\gamma^*(a) \in P_n/\gamma^*$. Then, $\gamma^*(a) = \gamma^*(b)$, for some $b \in P_{n+1}$. Hence, there exist $x_0, x_1, ..., x_n \in R$ such that $x_0 = a, x_{n+1} = b$ and $x_i \gamma x_{i+1}$. This implies that $\{x_i, x_{i+1}\} \subseteq \xi_i$, where ξ_i is a sum of hyperproducts and $0 \leq i \leq n$. Hence $P_{n+1} \cap \xi_n \neq \emptyset$ and it follows that $\xi_n \subseteq P_{n+1}$. Thus, we have $x_n \in P_{n+1}$. After a finite number of steps, we obtain $a \in P_{n+1}$ and there is an ascending chain of prime hyperideals $P_1 \subseteq P_2 \subseteq ... \subseteq P_n \subseteq P_{n+1} \subseteq .....$ By our hypothesis, there exists $k \in \mathbb{N}$, such that for every $n \geq k$, $P_n = P_k$. Hence, we have $P_k/\gamma^* = P_n/\gamma^*$, for every $n \geq k$. Therefore, every ascending chain of prime ideals of commutative ring R/γ^* must be stopped after a finite number. By applying P. M . Cohen Theorem, we see that R/γ^* is a Noetherian ring and by Theorem 3.12, R is a Noetherian Krasner hyperring. Thus our proof is completed. □

4. Conclusions and Future Works

The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. A special case of this type is the hyperring introduced by Krasner [9]. The applications of hyperstructures have been extensively considered by P. Corsini and V. Fotea [4]. As for the corresponding algebraic structures-the rings, in this paper we define and present several important properties of the prime hyperideal, ascending(descending) chain conditions of hyperideals and regular (strongly regular) relations in a Krasner hyperring. Our study has focused on prime hyperideals, complete parts and the fundamental relations in Noetherian (Artinian) hyperrings, concluded with a generalization of the well known theorems given by C. Hopkins, J. Levizki , W. Krull, Y. Akizuki and P.M. Cohn in ring theory. Our future research will concentrate on some new results which are related with the homological concepts of Krasner hyperrings.

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