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On the Rank of Semigroup of Transformations with Restricted Partial Range

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Abstract. Let $\mathcal{T}(X)$ be the full transformation semigroup on a nonempty set X. For $\emptyset \neq Z \subseteq Y \subseteq X$, let $\mathcal{T}(X, Y, Z) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Z\}$. It is not difficult to see that it is a generalized form of three well-known semigroups. This paper obtains an isomorphism theorem of $\mathcal{T}(X, Y, Z)$. In addition, when X is finite and $Z \subset Y \subset X$, the rank of the semigroup $\mathcal{T}(X, Y, Z)$ is calculated.

1. Introduction

Transformation semigroups are ubiquitous in the semigroup theory because of Cayley's Theorem which states that every semigroup is embedded in some transformation semigroup (see [1, Theorem 1.1.2]). It is well known that rank is a crucia concept in the semigroup theory. As usual, the rank of a semigroup S is the smallest number of elements required to generate S defined by $\operatorname{rank}(S) = \min\{|A| : A \subseteq S, \langle A\rangle = S\}$.

For a nonempty set X, let $\mathcal{T}(X)$ be the full transformation semigroup on X that is, the semigroup under composition of all maps from X into itself. We denote by $\mathcal{PT}(X)$ the monoid of all partial transformations of X, by $\mathcal{I}(X)$ the symmetric inverse semigroup on X, i.e., the submonoid of $\mathcal{PT}(X)$ of all injective partial transformations of X, and by $\mathcal{S}(X)$ the symmetric group on X, i.e., the subgroup of $\mathcal{I}(X)$ of all injective full transformations (permutations) of X. When X is finite, we take $X = \{1, 2, \cdots, n\}$ and write $\mathcal{PT}_n$, $\mathcal{T}_n$, $\mathcal{I}_n$, and $\mathcal{S}_n$ instead of $\mathcal{PT}(X)$, $\mathcal{T}(X)$, $\mathcal{I}(X)$, and $\mathcal{S}(X)$, respectively. For $n \geq 3$, it is well known that the rank of $\mathcal{PT}_n$,$\mathcal{T}_n$, $\mathcal{I}_n$, and $\mathcal{S}_n$ are equal to 4, 3, 3, and 2, respectively. These are well known results, and they all have found strong support. See [1, pp. 39, 41, and 211], for example.

On the other hand, Gomes and Howie proved that the rank of the semigroup of singular mappings $\mathcal{S}ing_n = \{\alpha \in \mathcal{T}_n : |X\alpha| \leq n-1\}$ is equal to $n(n-1)/2$ in [2]. This result was later generalized by Howie and McFadden [3] who showed that the rank of the semigroup $\mathcal{K}(n, r) = \{\alpha \in \mathcal{T}_n : |X\alpha| \leq r\}$ is equal to $S(n, r)$, the Stirling number of the second kind for $2 \leq r \leq n-1$. Recall that for $1 \leq r \leq n$ and $n \in \mathbb{N}^+$, the Stirling number of the second kind $S(n, r)$ is the number of r-partitions on a set of n elements, which may be defined by the recurrence relation $S(n, r) = S(n-1, r-1) + rS(n-1, r)$ with $S(n, 1) = S(n, n) = 1$. In [4], Garba considered the semigroup $\mathcal{PT}(n, r) = \{\alpha \in \mathcal{PT}_n : |X\alpha| \leq r\}$ and showed that, for $2 \leq r \leq n-1$, its rank is equal to $S(n+1, r+1)$, and showed that the rank of the semigroup $\mathcal{I}(n, r) = \{\alpha \in \mathcal{I}_n : |X\alpha| \leq r\}$ for $3 \leq r \leq n-1$, is $\binom{n}{r} + 1$. Recall that the number of ways that r objects can be chosen from n distinct objects written $\binom{n}{r}$ is given by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

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Given a nonempty subset Y of X, let

$$\overline{\mathcal{T}}(X,Y) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Y\} \text{ and } \mathcal{T}(X,Y) = \{\alpha \in \mathcal{T}(X) : X\alpha \subseteq Y\}.$$

Then $\overline{\mathcal{T}}(X,Y)$ is a subsemigroup of $\mathcal{T}(X)$ and $\mathcal{T}(X,Y)$ is a subsemigroup of $\overline{\mathcal{T}}(X,Y)$. In 1966, Magill [5] introduced and studied the semigroup $\overline{\mathcal{T}}(X,Y)$. In 1975, Symons [6] introduced the semigroup $\mathcal{T}(X,Y)$, and also described all automorphisms of $\mathcal{T}(X,Y)$. The study of semigroups $\overline{\mathcal{T}}(X,Y)$ and $\mathcal{T}(X,Y)$ [7–24] includes the aspects of regularity and Green's relations (see [7–9]), abundance and starred Green's Relations (see [10, 11]), natural partial order (see [11, 12]), congruence relation (see [13, 14]), (maximal) subsemigroup with some properties (see [15–21]), and rank (see [22, 23]), etc.

In this paper, we consider the subsemigroup $\mathcal{T}(X,Y,Z)$ of $\mathcal{T}(X)$ defined by

$$\mathcal{T}(X,Y,Z) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Z\}$$

where $\emptyset \neq Z \subseteq Y \subseteq X$, and we call it the semigroup of transformations with restricted partial range on X. Clearly, the semigroup $\mathcal{T}(X,Y,Z)$ is a generalization of semigroups $\mathcal{T}(X)$, $\overline{\mathcal{T}}(X,Y)$, and $\mathcal{T}(X,Z)$, that is,

- if $Z = Y$, then $\mathcal{T}(X,Y,Z) = \overline{\mathcal{T}}(X,Y)$;
- if $Y = X$, then $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$;
- if $Z = Y = X$, then $\mathcal{T}(X,Y,Z) = \mathcal{T}(X)$.

For the case $Z = Y = X$ it is easy to see that $\mathrm{rank}(\mathcal{T}(X,Y,Z)) = \mathrm{rank}(\mathcal{T}(X))$.

For the case $Z \subset Y = X$. Fernandes and Sanwong [22, Theorem 2.3] presented the following result.

Lemma 1.1. *[22, Theorem 2.3] Let $|X| = n$, $|Z| = k$ and $k < n$. Then* $\mathrm{rank}(\mathcal{T}(X,Z)) = S(n,k)$.

For the case $Z = Y \subseteq X$. The author [23, Theorem 1] presented the following result.

Lemma 1.2. *[23, Theorem 1] Let $|X| = n$, $|Y| = m$. Then*

$$\mathrm{rank}(\overline{\mathcal{T}}(X,Y)) = \begin{cases} 1, & \text{if } n = 1; \\ 2, & \text{if } (n,m) = (2,1) \text{ or } m = n = 2; \\ 3, & \text{if } (n,m) = (3,1) \text{ or } (n,m) = (3,2) \text{ or } m = n \geq 3; \\ 4, & \text{if } n \geq 4 \text{ and } m = 1 \text{ or } n \geq 4 \text{ and } m = n-1; \\ 5, & \text{if } n \geq 4 \text{ and } 2 \leq m \leq n-2. \end{cases}$$

The motivation of this study is to compute the rank of $\mathcal{T}(X,Y,Z)$ when X is finite and $Z \subset Y \subset X$.

Throughout this paper, we always assume that X is a chain with n ($n \geq 3$) elements, say $X = \{1 < 2 < \cdots < n\}$. Also, we assume that $\emptyset \neq Z \subset Y \subset X$. We write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first. For any sets A and B, we denote by $|A|$ the cardinality of A, and write $A \setminus B = \{a \in A : a \notin B\}$.

2. Isomorphism of $\mathcal{T}(X,Y,Z)$

In this section, we aim to prove an isomorphism theorem of $\mathcal{T}(X,Y,Z)$ when $Z \subset Y \subset X$.

Let S be a subsemigroup of $\mathcal{T}(X)$. Then $S \cap \mathcal{H}(X)$ (where $\mathcal{H}(X)$ is the set of transformations whose image has cardinality one: the constant functions) will be abbreviated to $\mathcal{H}(S)$. Symons [6, Theorem 1.1] proved the following lemma.

Lemma 2.1. *[6, Theorem 1.1] Let S,T are both subsemigroups of $\mathcal{T}(X)$ such that $\mathcal{H}(S), \mathcal{H}(T) \neq \emptyset$. If $\phi : S \to T$ is an isomorphism. Then $\mathcal{H}(S)\phi = \mathcal{H}(T)$.*

We can now present the main result of this section.

Theorem 2.2. *Let Z_i, Y_i are both nonempty subset of X with $Z_i \subset Y_i \subset X$ for $i = 1,2$. Then $\mathcal{T}(X,Y_1,Z_1) \cong \mathcal{T}(X,Y_2,Z_2)$ if and only if $|Y_1| = |Y_2|$ and $|Z_1| = |Z_2|$.*

Proof. Let $\mathcal{T}(X,Y_1,Z_1) \cong \mathcal{T}(X,Y_2,Z_2)$ and let $\phi : \mathcal{T}(X,Y_1,Z_1) \to \mathcal{T}(X,Y_2,Z_2)$ is an isomorphism. First observe that $\mathcal{T}(X,Y_1,Z_1), \mathcal{T}(X,Y_2,Z_2)$ are both subsemigroups of $\mathcal{T}(X)$ and $\mathcal{H}(\mathcal{T}(X,Y_1,Z_1)), \mathcal{H}(\mathcal{T}(X,Y_2,Z_2)) \neq \emptyset$. Using Lemma 2.1, it follows that $\mathcal{H}(\mathcal{T}(X,Y_1,Z_1))\phi = \mathcal{H}(\mathcal{T}(X,Y_2,Z_2))$. Clearly, $|\mathcal{H}(\mathcal{T}(X,Y_1,Z_1))| = |\mathcal{H}(\mathcal{T}(X,Y_2,Z_2))|$ and $|\mathcal{H}(\mathcal{T}(X,Y_i,Z_i))| = |Z_i|$ for $i = 1,2$. Hence $|Z_1| = |Z_2|$. It is easy to compute that $|\mathcal{T}(X,Y_i,Z_i)| = |Z_i|^{|Y_i|} \cdot n^{n-|Y_i|}$ for $i = 1,2$ ($n = |X|$). By hypothesis, we have $|\mathcal{T}(X,Y_1,Z_1)| = |\mathcal{T}(X,Y_2,Z_2)|$ and so $|Z_1|^{|Y_1|} \cdot n^{n-|Y_1|} = |Z_2|^{|Y_2|} \cdot n^{n-|Y_2|}$, which can be simplified to $|Z_1|^{|Y_1|-|Y_2|} = n^{|Y_1|-|Y_2|}$ (by $|Z_1| = |Z_2|$). Since $|Z_1| < n$. It follows that $|Y_1| = |Y_2|$.

Conversely, let $|Y_1| = |Y_2|$ and $|Z_1| = |Z_2|$. Since $Z_1 \subset Y_1 \subset X$ (or $Z_2 \subset Y_2 \subset X$). Then there exist some bijections

$$f : Z_1 \to Z_2,\ \ g : Y_1 \setminus Z_1 \to Y_2 \setminus Z_2 \ \text{ and } \ h : X \setminus Y_1 \to X \setminus Y_2.$$

For each $\alpha \in \mathcal{T}(X,Y_1,Z_1)$, we define

$$x\overline{\alpha} = \begin{cases} xf^{-1}\alpha f, & \text{if } x \in Z_2; \\ xg^{-1}\alpha f, & \text{if } x \in Y_2 \setminus Z_2; \\ xh^{-1}\alpha f, & \text{if } x \in X \setminus Y_2 \text{ and } xh^{-1}\alpha \in Z_1; \\ xh^{-1}\alpha g, & \text{if } x \in X \setminus Y_2 \text{ and } xh^{-1}\alpha \in Y_1 \setminus Z_1; \\ xh^{-1}\alpha h, & \text{if } x \in X \setminus Y_2 \text{ and } xh^{-1}\alpha \in X \setminus Y_1. \end{cases}$$

It is easy to verify that $\overline{\alpha} \in \mathcal{T}(X,Y_2,Z_2)$. Define $\phi : \mathcal{T}(X,Y_1,Z_1) \to \mathcal{T}(X,Y_2,Z_2)$ by $\alpha\phi = \overline{\alpha}$ ($\alpha \in \mathcal{T}(X,Y_1,Z_1)$). Clearly, ϕ is well defined. Next, we verify that ϕ is a bijection. Let $\alpha, \beta \in \mathcal{T}(X,Y_1,Z_1)$ such that $\alpha \neq \beta$, then $x_0\alpha \neq x_0\beta$ for some $x_0 \in X$. To do this, we distinguish three cases:

Case 1: $x_0 \in Z_1$. Then $x_0 f \in Z_2$ and so $(x_0 f)\overline{\alpha} = x_0 ff^{-1}\alpha f = x_0\alpha f \neq x_0\beta f = x_0 ff^{-1}\beta f = (x_0 f)\overline{\beta}$.

Case 2: $x_0 \in Y_1 \setminus Z_1$. Then $x_0 g \in Y_2 \setminus Z_2$ and so $(x_0 g)\overline{\alpha} = x_0 gg^{-1}\alpha f = x_0\alpha f \neq x_0\beta f = x_0 gg^{-1}\beta f = (x_0 g)\overline{\beta}$.

Case 3: $x_0 \in X \setminus Y_1$. Then $x_0 h \in X \setminus Y_2$ and so

$$(x_0h)\overline{\alpha} = \begin{cases} x_0hh^{-1}\alpha f = x_0\alpha f \neq x_0\beta f = x_0hh^{-1}\beta f = (x_0h)\overline{\beta}, & \text{if } x_0\alpha,\ x_0\beta \in Z_1; \\ x_0hh^{-1}\alpha f = x_0\alpha f \neq x_0\beta g = x_0hh^{-1}\beta g = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in Z_1,\ x_0\beta \in Y_1 \setminus Z_1; \\ x_0hh^{-1}\alpha f = x_0\alpha f \neq x_0\beta h = x_0hh^{-1}\beta h = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in Z_1,\ x_0\beta \in X \setminus Y_1; \\ x_0hh^{-1}\alpha g = x_0\alpha g \neq x_0\beta f = x_0hh^{-1}\beta f = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in Y_1 \setminus Z_1,\ x_0\beta \in Z_1; \\ x_0hh^{-1}\alpha g = x_0\alpha g \neq x_0\beta g = x_0hh^{-1}\beta g = (x_0h)\overline{\beta}, & \text{if } x_0\alpha, x_0\beta \in Y_1 \setminus Z_1; \\ x_0hh^{-1}\alpha g = x_0\alpha g \neq x_0\beta h = x_0hh^{-1}\beta h = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in Y_1 \setminus Z_1, x_0\beta \in X \setminus Y_1; \\ x_0hh^{-1}\alpha h = x_0\alpha h \neq x_0\beta f = x_0hh^{-1}\beta f = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in X \setminus Y_1, x_0\beta \in Z_1; \\ x_0hh^{-1}\alpha h = x_0\alpha h \neq x_0\beta g = x_0hh^{-1}\beta g = (x_0h)\overline{\beta}, & \text{if } x_0\alpha \in X \setminus Y_1, x_0\beta \in Y_1 \setminus Z_1; \\ x_0hh^{-1}\alpha h = x_0\alpha h \neq x_0\beta h = x_0hh^{-1}\beta h = (x_0h)\overline{\beta}, & \text{if } x_0\alpha, x_0\beta \in X \setminus Y_1. \end{cases}$$

Thus, we have $\overline{\alpha} \neq \overline{\beta}$ and so ϕ is one-to-one. Since $|\mathcal{T}(X,Y_i,Z_i)| = |Z_i|^{|Y_i|} \cdot n^{n-|Y_i|}$ for $i = 1,2$ and $|Z_1| = |Z_2|$, $|Y_1| = |Y_2|$, it follows that $|\mathcal{T}(X,Y_1,Z_1)| = |\mathcal{T}(X,Y_2,Z_2)|$. Therefore, we obtain that ϕ is a bijection.

Finally, we verify that ϕ is a morphism, that is, $(\alpha\phi)(\beta\phi) = (\alpha\beta)\phi$ for all $\alpha, \beta \in \mathcal{T}(X,Y_1,Z_1)$. We distinguish five cases:

Case 1: $x \in Z_2$. Then $x\overline{\alpha} = xf^{-1}\alpha f \in Z_2$ and so $x(\alpha\phi)(\beta\phi) = x\overline{\alpha}\overline{\beta} = (x\overline{\alpha})f^{-1}\beta f = xf^{-1}\alpha ff^{-1}\beta f = xf^{-1}\alpha\beta f = x\overline{\alpha\beta} = x(\alpha\beta)\phi$.

Case 2: $x \in Y_2 \setminus Z_2$. Then $x\overline{\alpha} = xg^{-1}\alpha f \in Z_2$ and so $x(\alpha\phi)(\beta\phi) = x\overline{\alpha}\overline{\beta} = (x\overline{\alpha})f^{-1}\beta f = xg^{-1}\alpha ff^{-1}\beta f = xg^{-1}\alpha\beta f = x\overline{\alpha\beta} = x(\alpha\beta)\phi$.

Case 3: $x \in X \setminus Y_2$ and $xh^{-1}\alpha \in Z_1$. Then $x\overline{\alpha} = xh^{-1}\alpha f \in Z_2$, $xh^{-1}\alpha\beta \in Z_1$ and so $x(\alpha\phi)(\beta\phi) = x\overline{\alpha}\overline{\beta} = (x\overline{\alpha})f^{-1}\beta f = xh^{-1}\alpha ff^{-1}\beta f = xh^{-1}\alpha\beta f = x\overline{\alpha\beta} = x(\alpha\beta)\phi$.

Case 4: $x \in X \setminus Y_2$ and $xh^{-1}\alpha \in Y_1 \setminus Z_1$. Then $x\overline{\alpha} = xh^{-1}\alpha g \in Y_2 \setminus Z_2$, $xh^{-1}\alpha\beta \in Z_1$ and so $x(\alpha\phi)(\beta\phi) = x\overline{\alpha}\overline{\beta} = (x\overline{\alpha})g^{-1}\beta f = xh^{-1}\alpha gg^{-1}\beta f = xh^{-1}\alpha\beta f = x\overline{\alpha\beta} = x(\alpha\beta)\phi$.

Case 5: $x \in X \setminus Y_2$ and $xh^{-1}\alpha \in X \setminus Y_1$. Then $x\overline{\alpha} = xh^{-1}\alpha h \in X \setminus Y_2$ and so

$$x(\alpha\phi)(\beta\phi) = x\overline{\alpha}\overline{\beta} = \begin{cases} (x\overline{\alpha})h^{-1}\beta f = xh^{-1}\alpha hh^{-1}\beta f = xh^{-1}\alpha\beta f = x\overline{\alpha\beta} = x(\alpha\beta)\phi, & \text{if } xh^{-1}\alpha\beta \in Z_1; \\ (x\overline{\alpha})h^{-1}\beta g = xh^{-1}\alpha hh^{-1}\beta g = xh^{-1}\alpha\beta g = x\overline{\alpha\beta} = x(\alpha\beta)\phi, & \text{if } xh^{-1}\alpha\beta \in Y \setminus Z_1; \\ (x\overline{\alpha})h^{-1}\beta h = xh^{-1}\alpha hh^{-1}\beta h = xh^{-1}\alpha\beta h = x\overline{\alpha\beta} = x(\alpha\beta)\phi, & \text{if } xh^{-1}\alpha\beta \in X \setminus Y_1. \end{cases}$$

In summary, $\phi : \mathcal{T}(X, Y_1, Z_1) \to \mathcal{T}(X, Y_2, Z_2)$ is an isomorphism. Therefore, it follows that $\mathcal{T}(X, Y_1, Z_1) \cong \mathcal{T}(X, Y_2, Z_2)$, as desired. □

3. Rank of $\mathcal{T}(X, Y, Z)$

For each $p \in \mathbb{N}^+$, we denote by X_p the set $\{1 < 2 < \cdots < p\}$. If $\emptyset \neq Z \subset Y \subset X$ with $|Y| = m$, $|Z| = k$. By Theorem 2.2, we have $\mathcal{T}(X, Y, Z) \cong \mathcal{T}(X_n, X_m, X_k)$. Based on that, we shall enough to consider the semigroup $\mathcal{T}(X_n, X_m, X_k)$. For convenience, we will write $\mathcal{T}_{n,m,k}$ for the semigroup $\mathcal{T}(X_n, X_m, X_k)$, where $k < m < n$.

If $\alpha \in \mathcal{T}_{n,m,k}$, we will write $\text{im}(\alpha)$ for the image of α. The kernel of α is the equivalence $\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}$. From Fountain [25], on the semigroup S the relation $\mathscr{L}^*$ (respectively $\mathscr{R}^*$) is defined by the rule that $(a, b) \in \mathscr{L}^*$ (respectively $\mathscr{R}^*$) if and only if the elements a, b are related by the Green's relation $\mathscr{L}$ (respectively $\mathscr{R}$) in some oversemigroup of S. The intersection of the equivalences $\mathscr{L}^*$ and $\mathscr{R}^*$ is denoted by $\mathscr{H}^*$. Since $\mathcal{T}_{n,m,k}$ is a subsemigroup of $\mathcal{T}_n$, the starred Green's relations in $\mathcal{T}_{n,m,k}$ can be characterized as: For $\alpha, \beta \in \mathcal{T}_{n,m,k}$,

- $(\alpha, \beta) \in \mathscr{L}^*$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$;
- $(\alpha, \beta) \in \mathscr{R}^*$ if and only if $\ker(\alpha) = \ker(\beta)$;
- $(\alpha, \beta) \in \mathscr{H}^*$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$ and $\ker(\alpha) = \ker(\beta)$.

Moreover, we define a equivalence $\mathscr{J}^*$ by

- $(\alpha, \beta) \in \mathscr{J}^*$ if and only if $|\text{im}(\alpha)| = |\text{im}(\beta)|$.

Then $\mathscr{L}^*, \mathscr{R}^* \subseteq \mathscr{J}^*$. Let $\alpha \in \mathcal{T}_{n,m,k}$. We denote by $\mathcal{L}^*_\alpha$, $\mathcal{R}^*_\alpha$, and $\mathcal{H}^*_\alpha$ the $\mathscr{L}^*$-class, $\mathscr{R}^*$-class, and $\mathscr{H}^*$-class of α, respectively.

Let $\alpha \in \mathcal{T}_{n,m,k}$. From $X_m\alpha \subseteq X_k$ we obtain that $\text{im}(\alpha) = X_m\alpha \cup (X_n \setminus X_m)\alpha \subseteq X_k \cup (X_n \setminus X_m)\alpha$. Then $1 \leq |\text{im}(\alpha)| \leq n - m + k$. Thus $\mathcal{T}_{n,m,k}$ has $n - m + k$ $\mathscr{J}^*$-classes, namely $\mathcal{J}^*_1, \mathcal{J}^*_2, \cdots, \mathcal{J}^*_{n-m+k}$, where

$$\mathcal{J}^*_r = \{\alpha \in \mathcal{T}_{n,m,k} : |\text{im}(\alpha)| = r\}$$

for $1 \leq r \leq n-m+k$. If $|\text{im}(\alpha)| = r$ with $1 \leq r \leq n-m+k$, then there exists $s \in X_r$ such that α can be expressed as

$$\alpha = \begin{bmatrix} A_i \\ a_i \end{bmatrix}^s_{1 \leq i \leq r} \tag{1}$$

where

- $A_i\alpha = a_i$ for all $1 \leq i \leq r$;
- $\{A_1, A_2, \cdots, A_r\}$ is a r-partition of X_n such that for $1 \leq j \leq s$, $A_j \cap X_m \neq \emptyset$, and for $l \geq s + 1$, $A_l \cap X_m = \emptyset$; and
- $a_1, a_2, \cdots, a_r$ are distinct elements of X_n such that for $1 \leq j \leq s$, $a_j \in X_k$.

Lemma 3.1. *Let $n - m + k \geq 3$. Then $\mathcal{J}^*_r \subseteq \langle \mathcal{J}^*_{r+1} \rangle$ for all $1 \leq r \leq n - m + k - 2$.*

Proof. Suppose first that $1 \leq r \leq n - m + k - 2$ and $\alpha \in \mathcal{J}^*_r$. Then α can be expressed as (1). Recall that $\text{im}(\alpha) \subset X_n$, we can choose $y_0 \in X_n \setminus \text{im}(\alpha)$. If $j \geq s + 1$, we also choose $b_j \in A_j$. We distinguish two cases:

Case 1: $s = k$. By formula (1) we obtain that $\{a_1, a_2, \cdots, a_s\} = X_k$. Note that $\alpha \in \mathcal{J}^*_r$ where $1 \leq r \leq n - m + k - 2$. Then there exist $x_0 \in X_n \setminus X_m$, $i \in X_r$ such that $x_0 \in A_i$ with $|A_i| \geq 2$.

(a) If $1 \le i \le s$. Then define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \le l \le s,\ l \ne i; \\ i, & \text{if } x \in A_i \setminus \{x_0\}; \\ s+1, & \text{if } x = x_0; \\ b_t, & \text{if } x \in A_t \text{ for } s+1 \le t \le r. \end{cases} \qquad x\gamma = \begin{cases} a_x, & \text{if } 1 \le x \le s; \\ a_i, & \text{if } s+1 \le x \le m; \\ a_t, & \text{if } x = b_t \text{ for } s+1 \le t \le r; \\ y_0, & \text{otherewise.} \end{cases} \tag{2}$$

According to the given conditions, it is easy to see that $A_i(\beta\gamma) = a_i$ for all $1 \le i \le r$. Then $\alpha = \beta\gamma$. Next, we verify that $\beta, \gamma \in \mathcal{J}_r^*$. Note that $y_0\gamma^{-1} = X_n \setminus \{1, \cdots, s, s+1, \cdots, m, b_{s+1}, \cdots, b_r\}$. Since $1 \le r \le n-m+k-2$ and $s = k$, we have $|y_0\gamma^{-1}| = n-(m+r-s) \ge n-m-r+k \ge n-m-(n-m+k-2)+k = 2$ and so $y_0\gamma^{-1} \ne \emptyset$. Clearly, $\mathrm{im}(\beta) = \{1, \cdots, s, s+1, b_{s+1}, \cdots, b_r\}$ and $\mathrm{im}(\gamma) = \{a_1, \cdots, a_r, y_0\}$. Combining formula (1), it follows that $X_m\beta, X_m\gamma \subseteq X_k$ and thus $\beta, \gamma \in \mathcal{J}_r^*$.

(b) If $s+1 \le i \le r$. Then define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \le l \le s; \\ x_0, & \text{if } x = x_0; \\ b_t, & \text{if } x \in A_t \text{ for } s+1 \le t \le r,\ t \ne i; \\ b_i \in A_i \setminus \{x_0\}, & \text{if } x \in A_i \setminus \{x_0\}. \end{cases} \qquad x\gamma = \begin{cases} a_x, & \text{if } 1 \le x \le s; \\ a_s, & \text{if } s+1 \le x \le m; \\ a_i, & \text{if } x = x_0; \\ a_t, & \text{if } x = b_t \text{ for } s+1 \le t \le r; \\ y_0, & \text{otherwise.} \end{cases}$$

Case 2: $s < k$. By formula (1) we obtain that $\{a_1, a_2, \cdots, a_s\} \subset X_k$. Then there exist $x_0 \in X_m$, $i \in X_s$ such that $x_0 \in A_i$ with $|A_i| \ge 2$.

(a) If $\mathrm{im}(\alpha) \cap X_k \subset X_k$. Then we can take $a_0 \in X_k \setminus (\mathrm{im}(\alpha) \cap X_k)$. Let β be defined as (2) and define a mapping $\gamma : X_n \to X_n$ by

$$x\gamma = \begin{cases} a_x, & \text{if } 1 \le x \le s; \\ a_i, & \text{if } x = s+1; \\ a_t, & \text{if } x = b_t \text{ for } s+1 \le t \le r; \\ a_0, & \text{otherwise.} \end{cases}$$

(b) If $\mathrm{im}(\alpha) \cap X_k = X_k$. Then there exist $k-s$ elements $a_{i_1}, \cdots, a_{i_{k-s}} \in \{a_{s+1}, \cdots, a_r\}$ such that $\{a_1, \cdots, a_s, a_{i_1}, \cdots, a_{i_{k-s}}\} = X_k$. We define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \le l \le s,\ l \ne i; \\ i, & \text{if } x \in A_i \setminus \{x_0\}; \\ s+1, & \text{if } x = x_0; \\ s+p+1, & \text{if } x \in A_{i_p} \text{ for } 1 \le p \le k-s; \\ b_t, & \text{if } x \in A_t \text{ for } s+1 \le t \le r,\ t \notin \{i_1, \cdots, i_{k-s}\}. \end{cases}$$

$$x\gamma = \begin{cases} a_x, & \text{if } 1 \le x \le s; \\ a_i, & \text{if } x = s+1 \text{ or } k+2 \le x \le m; \\ a_{i_p}, & \text{if } x = s+p+1 \text{ for } 1 \le p \le k-s; \\ a_t, & \text{if } x = b_t \text{ for } s+1 \le t \le r,\ t \notin \{i_1, \cdots, i_{k-s}\}; \\ y_0, & \text{otherwise.} \end{cases}$$

For both cases, similar to case 1 (a), it is easy to verify that $\beta, \gamma \in \mathcal{J}_{r+1}^*$ and $\alpha = \beta\gamma$. Hence, $\mathcal{J}_r^* \subseteq \langle \mathcal{J}_{r+1}^* \rangle$. □

For $k, m, n \in \mathbb{N}^+$ such that $k < m < n$, we define a mapping $\lambda : X_n \to X_n$ by

$$x\lambda = \begin{cases} k, & \text{if } k+1 \le x \le m; \\ n-1, & \text{if } x = n; \\ x, & \text{otherwise.} \end{cases} \tag{3}$$

Now we state and prove the following lemma.

Lemma 3.2. *Let λ be defined as (3). Then the following statements hold:*

(i) *for $n - m = 1$, $\mathcal{J}^*_{n-m+k-1} \subseteq \langle \mathcal{J}^*_{n-m+k} \rangle$;*

(ii) *for $n - m \geq 2$, $\mathcal{J}^*_{n-m+k-1} \subseteq \langle \mathcal{J}^*_{n-m+k} \cup \{\lambda\} \rangle$.*

Proof. Suppose first that $\alpha \in \mathcal{J}^*_{n-m+k-1}$. Then α can be expressed as (1) (Here, we take $r = n - m + k - 1$), that is

$$\alpha = \begin{bmatrix} A_i \\ a_i \end{bmatrix}^s_{1 \leq i \leq n-m+k-1} \tag{4}$$

Clearly, $\mathrm{im}(\alpha) \subset X_n$, so we can choose $y_0 \in X_n \setminus \mathrm{im}(\alpha)$. If $j \geq s + 1$, we also choose $b_j \in A_j$.

(i) Let $n - m = 1$. Then $n - m + k - 1 = k$ in (4). We distinguish two cases:

Case 1 : $s = k$. Then $\mathrm{im}(\alpha) = \{a_1, a_2, \cdots, a_s\} = X_k$ and so there exists $i \in X_s$ such that $n \in A_i$ with $|A_i| \geq 2$. We define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \leq l \leq s,\ l \neq i; \\ i, & \text{if } x \in A_i \setminus \{n\}; \\ m, & \text{if } x = n. \end{cases} \qquad x\gamma = \begin{cases} a_x, & \text{if } 1 \leq x \leq s; \\ a_i, & \text{if } s+1 \leq x \leq m; \\ y_0, & \text{if } x = n. \end{cases}$$

Case 2 : $s = k - 1$. Then $A_k = A_{s+1} = \{n\}$ and so there exist $x_0 \in X_m$, $i \in X_s$ such that $x_0 \in A_i$ with $|A_i| \geq 2$.

(a) If $a_k \in X_k$. Then $\mathrm{im}(\alpha) = X_k$. We define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \leq l \leq s,\ l \neq i; \\ i, & \text{if } x \in A_i \setminus \{x_0\}; \\ s+1, & \text{if } x = x_0; \\ s+2, & \text{if } x = n. \end{cases} \qquad x\gamma = \begin{cases} a_x, & \text{if } 1 \leq x \leq s; \\ a_i, & \text{if } x = s+1; \\ a_k, & \text{if } s+2 \leq x \leq m; \\ y_0, & \text{if } x = n. \end{cases}.$$

(b) If $a_k \notin X_k$. We may take $a_0 \in X_k \setminus \mathrm{im}(\alpha)$ and define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \leq l \leq s,\ l \neq i; \\ i, & \text{if } x \in A_i \setminus \{x_0\}; \\ s+1, & \text{if } x = x_0; \\ n, & \text{if } x = n. \end{cases} \qquad x\gamma = \begin{cases} a_x, & \text{if } 1 \leq x \leq s; \\ a_i, & \text{if } x = s+1; \\ a_k, & \text{if } x = n; \\ a_0, & \text{otherwise.} \end{cases}$$

For both cases, it is easy to verify that $\beta, \gamma \in \mathcal{J}^*_{n-m+k}$, $\alpha = \beta\gamma$, and this is clearly equivalent to $\alpha \in \langle \mathcal{J}^*_{n-m+k} \rangle$, establishing (i).

(ii) To show that $\mathcal{J}^*_{n-m+k-1} \subseteq \langle \mathcal{J}^*_{n-m+k} \cup \{\lambda\} \rangle$ for $n - m \geq 2$. We distinguish two cases:

Case 1 : $s = k$. Then $\{a_1, a_2, \cdots, a_s\} = X_k$ and there exist $x_0 \in X_n \setminus X_m$, $i \in X_{n-m+k-1}$ such that $x_0 \in A_i$ with $|A_i| \geq 2$.

(a) If $1 \leq i \leq s$. Then β, γ be defined as (2) (Here, we take $r = n - m + k - 1$). Clearly, $\alpha = \beta\gamma$ and $\beta, \gamma \in \mathcal{J}^*_{n-m+k}$.

(b) If $s + 1 \leq i \leq n - m + k - 1$. We define two mappings $\beta : X_n \to X_n$ and $\gamma : X_n \to X_n$ by

$$x\beta = \begin{cases} l, & \text{if } x \in A_l \text{ for } 1 \leq l \leq s; \\ n, & \text{if } x = x_0; \\ n-1, & \text{if } x \in A_i \setminus \{x_0\}; \\ m+t-k, & \text{if } x \in A_t \text{ for } s+1 \leq t \leq i-1; \\ m+p-k-1, & \text{if } x \in A_p \text{ for } i+1 \leq p \leq n-m+k-1. \end{cases}$$

$$x\gamma = \begin{cases} a_x, & \text{if } 1 \leq x \leq s; \\ a_s, & \text{if } s+1 \leq x \leq m; \\ a_t, & \text{if } x = m+t-k \text{ for } s+1 \leq t \leq i-1; \\ a_p, & \text{if } x = m+p-k-1 \text{ for } i+1 \leq p \leq n-m+k-1; \\ a_i, & \text{if } x = n-1; \\ y_0, & \text{if } x = n. \end{cases}$$

Clearly, $\alpha = \beta\lambda\gamma$ and $\beta, \gamma \in \mathcal{J}^*_{n-m+k}$

Case 2 : $s = k - 1$. Using a similar proof of case 2 of Lemma 3.1, $\alpha = \beta\gamma$ and $\beta, \gamma \in \mathcal{J}^*_{n-m+k}$

For both cases, $\alpha \in \langle \mathcal{J}^*_{n-m+k} \cup \{\lambda\}\rangle$, giving (ii) □

Using Lemma 3.1 and Lemma 3.2, we have the following corollary:

Corollary 3.3. *Let λ be defined as (3). Then the following statements hold:*

(i) *for $n - m = 1$, $\mathcal{T}_{n,m,k} = \langle \mathcal{J}^*_{n-m+k}\rangle$.*

(ii) *for $n - m \geq 2$, $\mathcal{T}_{n,m,k} = \langle \mathcal{J}^*_{n-m+k} \cup \{\lambda\}\rangle$.*

For $k, m, n \in \mathbb{N}^+$ such that $k < m < n$, we define a mapping $\epsilon : X_n \to X_n$ by

$$x\epsilon = \begin{cases} x, & \text{if } 1 \leq x \leq k \text{ or } m+1 \leq x \leq n; \\ k, & \text{if } k+1 \leq x \leq m. \end{cases} \tag{5}$$

Then $\epsilon \in \mathcal{J}^*_{n-m+k}$ and

$$\mathcal{H}^*_\epsilon = \left\{ \left(\begin{array}{cccc|ccc} \{1\} & \cdots & \{k-1\} & X_m \setminus X_{k-1} & \{m+1\} & \cdots & \{n\} \\ 1\sigma & \cdots & (k-1)\sigma & k\sigma & (m+1)\rho & \cdots & n\rho \end{array} \right) : \sigma \in \mathcal{S}(X_k),\ \rho \in \mathcal{S}(X_n \setminus X_m) \right\} \tag{6}$$

is a group $\mathscr{H}^*$-class containing ϵ. Clearly, $\mathcal{H}^*_\epsilon \cong \mathcal{S}(X_k) \times \mathcal{S}(X_n \setminus X_m)$.

The following lemma was proved by Toker and Ayık [26, Lemma 3].

Lemma 3.4. *[26, Lemma 3] Let $p, q \in \mathbb{N}^+$. Then*

$$\text{rank}(\mathcal{S}(X_p) \times \mathcal{S}(X_q)) = \begin{cases} 1, & \textit{if } (p,q) = (1,1), (1,2) \textit{ or } (2,1); \\ 2, & \textit{otherwise.} \end{cases}$$

If $(k, n-m) = (1,1), (1,2)$ or $(2,1)$. We know from Lemma 3.4 that there exists $\theta_{n,m,k} \in \mathcal{H}^*_\epsilon$ such that

$$\mathcal{H}^*_\epsilon = \langle \{\theta_{n,m,k}\}\rangle. \tag{7}$$

Otherwise there exist two elements $\upsilon_{n,m,k}, \upsilon\prime_{n,m,k} \in \mathcal{H}^*_\epsilon$ such that

$$\mathcal{H}^*_\epsilon = \langle \{\upsilon_{n,m,k}, \upsilon\prime_{n,m,k}\}\rangle. \tag{8}$$

Obviously, there are $\sigma_1, \sigma_2 \in \mathcal{S}(X_k)$, $\rho_1, \rho_2 \in \mathcal{S}(X_n \setminus X_m)$ such that $\upsilon_{n,m,k}, \upsilon\prime_{n,m,k}$ are expressed respectively as

$$\upsilon_{n,m,k} = \left(\begin{array}{cccc|ccc} \{1\} & \cdots & \{k-1\} & X_m \setminus X_{k-1} & \{m+1\} & \cdots & \{n\} \\ 1\sigma_1 & \cdots & (k-1)\sigma_1 & k\sigma_1 & (m+1)\rho_1 & \cdots & n\rho_1 \end{array} \right) \tag{9}$$

$$\upsilon\prime_{n,m,k} = \left(\begin{array}{cccc|ccc} \{1\} & \cdots & \{k-1\} & X_m \setminus X_{k-1} & \{m+1\} & \cdots & \{n\} \\ 1\sigma_2 & \cdots & (k-1)\sigma_2 & k\sigma_2 & (m+1)\rho_2 & \cdots & n\rho_2 \end{array} \right) \tag{10}$$

We write

$$\tau_{n,m,k} = \left(\begin{array}{cccc|ccc} \{1\} \cup (X_m \setminus X_k) & \{2\} & \cdots & \{k\} & \{m+1\} & \cdots & \{n\} \\ 1\sigma_1^{-1}\sigma_2 & 2\sigma_1^{-1}\sigma_2 & \cdots & k\sigma_1^{-1}\sigma_2 & (m+1)\rho_1^{-1}\rho_2 & \cdots & n\rho_1^{-1}\rho_2 \end{array} \right) \tag{11}$$

Clearly, $\upsilon\prime_{n,m,k} = \upsilon_{n,m,k}\tau_{n,m,k}$ and $\tau_{n,m,k} \in \mathcal{L}^*_\epsilon$. By formula (8) we get that

$$\mathcal{H}^*_\epsilon \subseteq \langle \{\upsilon_{n,m,k}, \tau_{n,m,k}\}\rangle. \tag{12}$$

Let ϵ be defined as (5), we denote by $\mathfrak{L}$ the set of a single arbitrary element from each $\mathscr{H}^*$-class contained in $\mathcal{L}^*_\epsilon$. In addition, we denote by $\mathfrak{R}$ the set of a single arbitrary element from each $\mathscr{H}^*$-class contained in $\mathcal{R}^*_\epsilon \setminus \mathcal{H}^*_\epsilon$. In fact,

- $\mathfrak{L} \subseteq \mathcal{L}^*_\epsilon$, and for all $\alpha \in \mathcal{L}^*_\epsilon$, the intersection of $\mathfrak{L}$ and $\mathcal{H}^*_\alpha$ has exactly one element;
- $\mathfrak{R} \subseteq \mathcal{R}^*_\epsilon \setminus \mathcal{H}^*_\epsilon$, and for all $\beta \in \mathcal{R}^*_\epsilon \setminus \mathcal{H}^*_\epsilon$, the intersection of $\mathfrak{R}$ and $\mathcal{H}^*_\beta$ has exactly one element.

Lemma 3.5. *Let $\mathcal{H}_\epsilon^*$ be defined as (6). Then the following statements hold:*

(i) $\mathcal{H}_\alpha^* \subseteq \alpha\mathcal{H}_\epsilon^*$ *for all* $\alpha \in \mathfrak{L}$.

(ii) $\mathcal{H}_\beta^* \subseteq \mathcal{H}_\epsilon^*\beta$ *for all* $\beta \in \mathfrak{R}$.

Proof. Let $\alpha \in \mathfrak{L}$. From definition of $\mathfrak{L}$ we know that $\alpha \in \mathcal{L}_\epsilon^*$ and so $\mathrm{im}(\alpha) = \mathrm{im}(\epsilon)$. Then α can be expressed as

$$\alpha = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_k & \{m+1\} & \cdots & \{n\} \\ 1\sigma & \cdots & k\sigma & (m+1)\rho & \cdots & n\rho \end{array}\right)$$

where $\{A_1, \cdots, A_k\}$ is a k-partition of X_m, $\sigma \in \mathcal{S}(X_k)$ and $\rho \in \mathcal{S}(X_n \setminus X_m)$. For each $\gamma \in \mathcal{H}_\alpha^*$, there exist $\phi \in \mathcal{S}(X_k)$, $\varphi \in \mathcal{S}(X_n \setminus X_m)$ such that γ can be expressed as

$$\gamma = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_k & \{m+1\} & \cdots & \{n\} \\ 1\phi & \cdots & k\phi & (m+1)\varphi & \cdots & n\varphi \end{array}\right).$$

Let

$$\zeta = \left(\begin{array}{ccc|ccc} \{1\} & \cdots & X_m \setminus X_{k-1} & \{m+1\} & \cdots & \{n\} \\ 1\sigma^{-1}\phi & \cdots & k\sigma^{-1}\phi & (m+1)\rho^{-1}\varphi & \cdots & n\rho^{-1}\varphi \end{array}\right).$$

Clearly, $\gamma = \alpha\zeta$ and $\zeta \in \mathcal{H}_\epsilon^*$. It is immediate that $\gamma \in \alpha\mathcal{H}_\epsilon^*$ and so $\mathcal{H}_\alpha^* \subseteq \alpha\mathcal{H}_\epsilon^*$ as required. (ii) follows in similar way. □

Let $\mathcal{H}_\epsilon^*$, $\theta_{n,m,k}$, $\upsilon_{n,m,k}$, $\upsilon\prime_{n,m,k}$, $\tau_{n,m,k}$ be respectively defined as (6), (7), (9), (10), (11), and let

$$\Theta_{n,m,k} = \begin{cases} ([\mathfrak{L} \cup \mathfrak{R}] \setminus \mathcal{H}_\epsilon^*) \cup \{\theta_{n,m,k}\}, & \text{if } (k, n-m) = (1,1), (1,2) \text{ or } (2,1); \\ ([\mathfrak{L} \cup \mathfrak{R}] \setminus \mathcal{H}_\epsilon^*) \cup \{\upsilon_{n,m,k}, \upsilon\prime_{n,m,k}\}, & \text{if } k = 1 \text{ and } n-m \geq 3; \\ ([\mathfrak{L} \cup \mathfrak{R}] \setminus [\mathcal{H}_\epsilon^* \cup \mathcal{H}_{\tau_{n,m,k}}^*]) \cup \{\upsilon_{n,m,k}, \tau_{n,m,k}\}, & \text{otherwise.} \end{cases} \tag{13}$$

Using formulas (7), (8), (12), and Lemma 3.5, we have the following corollary.

Corollary 3.6. $\mathcal{L}_\epsilon^* \cup \mathcal{R}_\epsilon^* \subseteq \langle \Theta_{n,m,k} \rangle$.

Lemma 3.7. *Let $\Theta_{n,m,k}$ be defined as (13). Then $\mathcal{J}_{n-m+k}^* \subseteq \langle \Theta_{n,m,k} \rangle$.*

Proof. For each $\alpha \in \mathcal{J}_{n-m+k}^*$, there exist a k-partition $\{A_1, \cdots, A_k\}$ of X_m, a subset $\{a_1, \cdots, a_{n-m}\}$ of $X_n \setminus X_k$ and $\delta \in \mathcal{S}(X_k)$ such that α can be expressed as

$$\alpha = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_k & \{m+1\} & \cdots & \{n\} \\ 1\delta & \cdots & k\delta & a_1 & \cdots & a_{n-m} \end{array}\right).$$

Let

$$\beta = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_k & \{m+1\} & \cdots & \{n\} \\ 1 & \cdots & k & m+1 & \cdots & n \end{array}\right), \quad \gamma = \left(\begin{array}{ccc|ccc} \{1\} & \cdots & X_m \setminus X_{k-1} & \{m+1\} & \cdots & \{n\} \\ 1\delta & \cdots & k\delta & a_1 & \cdots & a_{n-m} \end{array}\right).$$

Clearly, $\beta \in \mathcal{L}_\epsilon^*$, $\gamma \in \mathcal{R}_\epsilon^*$, where ϵ be defined as (5). It is easy to versify that $\alpha = \beta\gamma$. From Corollary 3.6, we have $\beta, \gamma \in \langle \Theta_{n,m,k} \rangle$ and thus $\alpha \in \langle \Theta_{n,m,k} \rangle$. Therefore, $\mathcal{J}_{n-m+k}^* \subseteq \langle \Theta_{n,m,k} \rangle$, as required. □

By Lemma 3.7 and Corollary 3.3, we have the following corollary.

Corollary 3.8. *Let λ, $\Theta_{n,m,k}$ be defined as (3), (13), respectively. Then the following statements hold:*

(i) *for* $n - m = 1$, $\mathcal{T}_{n,m,k} = \langle \Theta_{n,m,k} \rangle$.

(ii) *for* $n - m \geq 2$, $\mathcal{T}_{n,m,k} = \langle \Theta_{n,m,k} \cup \{\lambda\} \rangle$.

Recall that

• if $(k, n-m) = (1,1), (1,2)$ or $(2,1)$, $\theta_{n,m,k} \in \mathcal{H}_\epsilon^*$;

• if $k = 1$ and $n - m \geq 3$, $\upsilon_{n,m,k}, \upsilon\prime_{n,m,k} \in \mathcal{H}_\epsilon^*$ and $\upsilon_{n,m,k} \neq \upsilon\prime_{n,m,k}$;

• otherwise $\upsilon_{n,m,k} \in \mathcal{H}_\epsilon^*$ and $\tau_{n,m,k} \in \mathcal{L}_\epsilon^* \setminus \mathcal{H}_\epsilon^*$.

By definitions of $\mathfrak{L}$ and $\mathfrak{R}$, we have $\mathfrak{L} \cap \mathfrak{R} = \emptyset$, $|\mathfrak{L} \cap \mathcal{H}_\epsilon^*| = 1$ and $\mathfrak{R} \cap \mathcal{H}_\epsilon^* = \emptyset$. It is also easy to show that $|\mathfrak{L}| = S(m,k)$ (recall that $S(m,k)$ is the Stirling number of the second kind) and $|\mathfrak{R}| = \binom{n-k}{n-m} - 1$. Thus $|\mathfrak{L} \cup \mathfrak{R}| = S(m,k) + \binom{n-k}{n-m} - 1$. Combining formula (13), we have

$$|\Theta_{n,m,k}| = \begin{cases} |\mathfrak{L} \cup \mathfrak{R}| + 1 = \binom{n-1}{n-m} + 1, & \text{if } k = 1 \text{ and } n - m \geq 3; \\ |\mathfrak{L} \cup \mathfrak{R}| = S(m,k) + \binom{n-k}{n-m} - 1, & \text{otherwise.} \end{cases}$$

Using Corollary 3.8, we obtain the following corollary.

Corollary 3.9.

$$\operatorname{rank}(\mathcal{T}_{n,m,k}) \leq \begin{cases} S(m,k) + n - k - 1, & \text{if } n - m = 1; \\ \binom{n-1}{n-m} + 2, & \text{if } k = 1 \text{ and } n - m \geq 3; \\ S(m,k) + \binom{n-k}{n-m}, & \text{otherwise.} \end{cases}$$

Let A is a generating set of $\mathcal{T}_{n,m,k}$, and let $\alpha \in \mathcal{J}_{n-m+k}^*$. Then there are $\alpha_1, \cdots, \alpha_s \in A$ $(s \geq 2)$ such that

$$\alpha = \alpha_1 \alpha_2 \cdots \alpha_s. \tag{14}$$

Then we can claim that

• for all i, $\alpha_i \in \mathcal{J}_{n-m+k}^*$ (if not, there exists j such that $\alpha_j \notin \mathcal{J}_{n-m+k}^*$, then $|\operatorname{im}(\alpha_j)| < n - m + k$ and so $|\operatorname{im}(\alpha)| = |\operatorname{im}(\alpha_1\alpha_2\cdots\alpha_s)| < n - m + k$, contradicting the fact that $\alpha \in \mathcal{J}_{n-m+k}^*$);

• $\ker(\alpha) = \ker(\alpha_1)$ (By $\ker(\alpha_1) \subseteq \ker(\alpha_1\alpha_2\cdots\alpha_s) = \ker(\alpha)$ and $|\operatorname{im}(\alpha_1)| = |\operatorname{im}(\alpha)|$);

• $\operatorname{im}(\alpha) = \operatorname{im}(\alpha_s)$ (By $\operatorname{im}(\alpha) = \operatorname{im}(\alpha_1\alpha_2\cdots\alpha_s) \subseteq \operatorname{im}(\alpha_s)$ and $|\operatorname{im}(\alpha_s)| = |\operatorname{im}(\alpha)|$).

Hence, we proved the following:

Lemma 3.10. *Let A is a generating set of $\mathcal{T}_{n,m,k}$, and let $\alpha \in \mathcal{J}_{n-m+k}^*$. Then there are $\alpha_1, \cdots, \alpha_s \in A$ $(s \geq 2)$ such that $\alpha = \alpha_1\alpha_2\cdots\alpha_s$ and the following statements hold:*

(i) *for all i, $\alpha_i \in \mathcal{J}_{n-m+k}^*$.*

(ii) $\ker(\alpha) = \ker(\alpha_1)$.

(iii) $\operatorname{im}(\alpha) = \operatorname{im}(\alpha_s)$.

Lemma 3.11. *Let A is a generating set of $\mathcal{T}_{n,m,k}$, and let $\alpha \in \mathcal{J}_{n-m+k}^*$ such that $\operatorname{im}(\alpha) = X_k \cup (X_n \setminus X_m)$. Then $A \cap \mathcal{H}_\alpha^* \neq \emptyset$.*

Proof. Note that $\operatorname{im}(\alpha) = X_k \cup (X_n \setminus X_m)$ and $X_m\alpha \subseteq X_k$. Then $X_m\alpha = X_k$ and $(X_n \setminus X_m)\alpha = X_n \setminus X_m$. Since A is a generating set of $\mathcal{T}_{n,m,k}$, there are $\alpha_1, \cdots, \alpha_s \in A$ $(s \geq 2)$ such that formula (14) holds. From Lemma 3.10 we obtain that $\ker(\alpha) = \ker(\alpha_1)$. In fact, $\operatorname{im}(\alpha) = \operatorname{im}(\alpha_1)$ (if not, there exists $a \in \operatorname{im}(\alpha_1)$ such that $a \notin \operatorname{im}(\alpha) = X_k \cup (X_n \setminus X_m)$. Then $a \in X_m \setminus X_k$ and so there exists $a\prime \in X_n \setminus X_m$ such that $a\prime\alpha_1 = a$. Note that $X_m\alpha_i \subseteq X_k$ for all i. Thus, $a\prime\alpha = a\prime\alpha_1\alpha_2\cdots\alpha_s = a\alpha_2\cdots\alpha_s \in X_k$, this contradicts the fact that $(X_n \setminus X_m)\alpha = X_n \setminus X_m$). Then $\alpha_1 \in \mathcal{H}_\alpha^*$. Note that $\alpha_1 \in A$. Hence, $A \cap \mathcal{H}_\alpha^* \neq \emptyset$ as required. □

Lemma 3.12. *Let $k = 1$, $n - m \geq 3$ and let A is a generating set of $\mathcal{T}_{n,m,k}$. Then $|A \cap \mathcal{H}_\epsilon^*| \geq 2$ where $\mathcal{H}_\epsilon^*$ be defined as (5).*

Proof. From formula (5) we know that $\operatorname{im}(\epsilon) = X_k \cup (X_n \setminus X_m)$. By Lemma 3.11, we have $A \cap \mathcal{H}_\epsilon^* \neq \emptyset$, in other words $|A \cap \mathcal{H}_\epsilon^*| \geq 1$.

To show that $|A \cap \mathcal{H}_\epsilon^*| \geq 2$. Assume that $|A \cap \mathcal{H}_\epsilon^*| = 1$. Then there exists an element β of $\mathcal{T}_{n,m,k}$ such that $A \cap \mathcal{H}_\epsilon^* = \{\beta\}$. By formula (6), there exists $\rho_1 \in \mathcal{S}(X_n \setminus X_m)$ such that

$$\beta = \left(\begin{array}{c|ccc} X_m & \{m+1\} & \cdots & \{n\} \\ 1 & (m+1)\rho_1 & \cdots & n\rho_1 \end{array} \right)$$

For $k = 1$, $n - m \geq 3$, from Lemma 3.4 we know that $\langle A \cap \mathcal{H}_\epsilon^* \rangle = \langle\{\beta\}\rangle \subset \mathcal{H}_\epsilon^*$ if β is unique element of A. Then $\langle A \rangle \neq \mathcal{T}_{n,m,k}$, contradicting the hypothesis of the lemma. Now, let $\alpha \in \mathcal{H}_\epsilon^*$. Since A is a generating set of $\mathcal{T}_{n,m,k}$, there are $\alpha_1, \cdots, \alpha_s \in A$ $(s \geq 2)$ such that formula (14) holds. We can assert that, for all i, $\alpha_i \in \mathcal{H}_\epsilon^*$ (if not, there exists j such that $\alpha_j \notin \mathcal{H}_\epsilon^*$, from Lemma 3.10 we know that $\alpha_j \in \mathcal{J}_{n-m+k}^*$. Then there exist some distinct elements $a_1, \cdots, a_{n-m} \in X_n \setminus X_k$ with $\{a_1, \cdots, a_{n-m}\} \neq X_n \setminus X_m$ such that

$$\alpha_j = \left(\begin{array}{c|ccc} X_m & \{m+1\} & \cdots & \{n\} \\ 1 & a_1 & \cdots & a_{n-m} \end{array} \right)$$

Clearly, if $j = s$, then, by Lemma 3.10, $\{1\} \cup (X_n \setminus X_m) = \mathrm{im}(\alpha) = \mathrm{im}(\alpha_s) = \mathrm{im}(\alpha_j) = \{1, a_1, \cdots, a_{n-m}\}$. This is a contradiction; otherwise, from $\{a_1, \cdots, a_{n-m}\} \neq X_n \setminus X_m$ we know that there exists $l \in \{m+1, \cdots, n\}$ such that $a_l \in X_m$. Then $(X_m \cup \{l\})\alpha_j\alpha_{j+1} = \{1, a_l\}\alpha_{j+1} = \{1\}$ and so $|\mathrm{im}(\alpha_j\alpha_{j+1})| < n - m + k$. Hence $|\mathrm{im}(\beta)| = |\mathrm{im}(\alpha_1\alpha_2 \cdots \alpha_s)| < n - m + k$. This is also a contradiction). Since $A \cap \mathcal{H}_\epsilon^* = \{\beta\}$, we have $\alpha_1, \cdots, \alpha_s \in \langle\{\beta\}\rangle$. It is immediate that $\alpha \in \langle\{\beta\}\rangle \subset \mathcal{H}_\epsilon^*$ and so $\mathcal{H}_\epsilon^* \setminus \langle\{\beta\}\rangle$ cannot be generated by the set A, contradicting the hypothesis of the lemma. □

For each $\kappa \in \mathcal{J}_{n-m+k}^*$, let

$$\mathcal{L}_\kappa^{*E} = \{\beta \in \mathcal{L}_\kappa^* : (\forall i, j \in X_k, i \neq j)\ i\beta \neq j\beta\}. \tag{15}$$

Lemma 3.13. *Let A is a generating set of $\mathcal{T}_{n,m,k}$, and let $\alpha \in \mathcal{J}_{n-m+k}^*$ such that* $\mathrm{im}(\alpha) \neq X_k \cup (X_n \setminus X_m)$. *Then $A \cap \mathcal{L}_\kappa^{*E} \neq \emptyset$.*

Proof. Let $\beta \in \mathcal{L}_\kappa^{*E}$. By (15), $i\beta \neq j\beta$ for all $i, j \in X_k$ $(i \neq j)$. Observe that $\beta \in \mathcal{L}_\alpha^*$ and $\mathrm{im}(\alpha) \neq X_k \cup (X_n \setminus X_m)$. Then $\mathrm{im}(\beta) \neq X_k \cup (X_n \setminus X_m)$ and so there exist a k-partition $\{B_1, \cdots, B_k\}$ $(i \in B_i, i = 1, \cdots, k)$ of X_m, $(n-m)$-element subset $\{a_1, \cdots, a_{n-m}\}$ $(\{a_1, \cdots, a_{n-m}\} \neq X_n \setminus X_m)$ of $X_n \setminus X_k$ and $\sigma \in \mathcal{S}(X_k)$ such that β can be expressed as

$$\beta = \left(\begin{array}{ccc|ccc} B_1 & \cdots & B_k & \{m+1\} & \cdots & \{n\} \\ 1\delta & \cdots & k\delta & a_1 & \cdots & a_{n-m} \end{array} \right).$$

Since A is a generating set of $\mathcal{T}_{n,m,k}$, there are $\beta_1, \cdots, \beta_s \in A$ $(s \geq 2)$ such that $\beta = \beta_1\beta_2 \cdots \beta_s$. By Lemma 3.10, $\mathrm{im}(\beta_s) = \mathrm{im}(\beta)$. Then there exists a k-partition $\{C_1, C_2, \cdots, C_k\}$ of X_m such that $X_n/\ker(\beta_s) = \{C_1, C_2, \cdots, C_k, \{m+1\}, \cdots, \{n\}\}$. Now, we assert that there exists $\rho \in \mathcal{S}(X_k)$ such that $i\rho \in C_i$ where $i = 1, \cdots, k$ (if not, there are $p, q, j \in X_k$ $(p \neq q)$ such that $p, q \in C_j$. Clearly, $p, q \in \mathrm{im}(\beta_{s-1})$ and so $|\mathrm{im}(\beta)| = |\mathrm{im}(\beta_1\beta_2 \cdots \beta_s)| < n - m + k$. This is a contradiction). Thus, $\beta_s \in \mathcal{L}_\kappa^{*E}$. However, $\beta_s \in A$. Therefore $A \cap \mathcal{L}_\kappa^{*E} \neq \emptyset$, as required. □

Lemma 3.14. *Let $n - m \geq 2$, and let $\alpha \in \mathcal{J}_{n-m+k-1}^*$ such that $X_m\alpha = X_k$ and $(X_n \setminus X_m)\alpha \cap X_k = \emptyset$. Then $\alpha \notin \langle \mathcal{J}_{n-m+k}^* \rangle$.*

Proof. Assume that $\alpha \in \langle \mathcal{J}_{n-m+k}^* \rangle$. Then there are $\alpha_1, \cdots, \alpha_s \in \mathcal{J}_{n-m+k}^*$ $(s \geq 2)$ such that formula (14) holds. We can assert that $(X_n \setminus X_m)\alpha_i = X_n \setminus X_m$ for all $1 \leq i \leq s-1$ (if not, there exists $1 \leq j \leq s-1$ such that $(X_n \setminus X_m)\alpha_j \neq (X_n \setminus X_m)$, that is, there exists $x_0 \in X_n \setminus X_m$ such that $x_0\alpha_j \in X_m$. (a) If $j = 1$. Then $x_0\alpha = x_0\alpha_1 \cdots \alpha_s \in X_m\alpha_2 \cdots \alpha_s \subseteq X_k$ and so contradicting the fact that $(X_n \setminus X_m)\alpha \cap X_k = \emptyset$; (b) If $j = 2$. By (a), we have $(X_n \setminus X_m)\alpha = (X_n \setminus X_m)\alpha_1 \cdots \alpha_s = (X_n \setminus X_m)\alpha_2 \cdots \alpha_s$. However, $x_0\alpha_2 \cdots \alpha_s \in X_k$, this contradicts the fact that $(X_n \setminus X_m)\alpha \cap X_k = \emptyset$; Introduce contradictions in this way). By $\alpha_s \in \mathcal{J}_{n-m+k}^*$, there exists a $(n-m)$-element subset V of $X_n \setminus X_k$ such that $(X_n \setminus X_m)\alpha = (X_n \setminus X_m)\alpha_1 \cdots \alpha_s = (X_n \setminus X_m)\alpha_s = V$. From $X_m\alpha = X_k$, it follows that $\alpha \in \mathcal{J}_{n-m+k}^*$, contradicting the fact that $\alpha \in \mathcal{J}_{n-m+k-1}^*$. □

Theorem 3.15. *Let $|X| = n$, $|Y| = m$ and $|Z| = k$ such that $k < m < n$. Then*

$$\operatorname{rank}(\mathcal{T}(X,Y,Z)) = \begin{cases} S(m,k) + n - k - 1, & \text{if } n - m = 1; \\ \binom{n-1}{n-m} + 2, & \text{if } k = 1 \text{ and } n - m \geq 3; \\ S(m,k) + \binom{n-k}{n-m}, & \text{otherwise.} \end{cases}$$

Proof. Since Theorem 2.2 and Corollary 3.9, we only need to prove that

$$|A| \geq \begin{cases} S(m,k) + n - k - 1, & \text{if } n - m = 1; \\ \binom{n-1}{n-m} + 2, & \text{if } k = 1 \text{ and } n - m \geq 3; \\ S(m,k) + \binom{n-k}{n-m}, & \text{if } (k, n-m) = (1,2) \text{ or } k \geq 2 \text{ and } n - m \geq 2. \end{cases}$$

for any generating set A of $\mathcal{T}_{n,m,k}$. Let

$$\Sigma = \{\mathcal{H}^*_\alpha : \alpha \in \mathcal{J}^*_{n-m+k},\ \operatorname{im}(\alpha) = X_k \cup (X_n \setminus X_m)\},\quad \Lambda = \{\mathcal{L}^{*E}_\beta : \beta \in \mathcal{J}^*_{n-m+k},\ \operatorname{im}(\beta) \neq X_k \cup (X_n \setminus X_m)\}.$$

With above notation, we have the following simple observation:

$$|\Sigma| = |\mathfrak{L}| = S(m,k),\quad |\Lambda| = |\mathfrak{R}| = \binom{n-k}{n-m} - 1 \text{ and } (\bigcup_{P\in\Sigma} P) \cap (\bigcup_{Q\in\Lambda} Q) = \emptyset \tag{16}$$

We distinguish three cases:

Case 1 : $n - m = 1$. Combining Lemma 3.11, Lemma 3.13 and formula (16), we have $|A| \geq S(m,k) + n - k - 1$.

Case 2 : $k = 1$ and $n - m \geq 3$. Combining Lemma 3.12, Lemma 3.13 and formula (16), $|A| \geq \binom{n-1}{n-m} + 1$. In fact, $|A| \geq \binom{n-1}{n-m} + 2$ (if not, $A \subseteq \mathcal{J}^*_{n-m+k}$. By Lemma 3.14, $\alpha \notin \langle \mathcal{J}^*_{n-m+k}\rangle$ where α be defined as in Lemma 3.14. Hence, $\alpha \notin \langle A\rangle$, contradicting the fact that A is a generating set of $\mathcal{T}_{n,m,k}$).

Case 3 : $(k, n-m) = (1,2)$ or $k \geq 2$ and $n - m \geq 2$. Combining Lemma 3.11, Lemma 3.13 and formula (16), $|A| \geq S(m,k) + \binom{n-k}{n-m} - 1$. Using a similar proof of case 2, $|A| \geq S(m,k) + \binom{n-k}{n-m}$. □

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