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On Tripled Fixed Point Theorems via Measure of Noncompactness with Applications to a System of Fractional Integral Equations

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Abstract. In this paper, an extension of Darbo's fixed point theorem via the concept of operators $A(f;.)$ and θ-ω-contractions in a Banach space has been presented. We illustrate that some conditions of control functions ϑ and ω can be omitted where they applied previously in [Mohammadi et al., Mathematics, 2020, 8(4), 492]. As an application, we study the existence of solutions for a system of fractional integral equations. Finally, to support the effectiveness of our results, we present an example.

1. Introduction and preliminaries

Creating generalizations of the Banach contraction principle [12], which is a fundamental result of metric fixed-point theory, is an interesting field for researchers upon its valuable applications in many branches of mathematics and other sciences.

Measures of noncompactness inaugurates a very applicable branch of nonlinear analysis. It presents a lot of applications in operator theory. Measures of noncompactness are spaciously applied in fixed point theory and are especially useful in differential and integral equations and in fractional calculous. Its first definition investigated by Kuratowski [1]. For more details and recent development of measure of noncompactness theory, the reader can refer to [2]-[11].

The present paper is devoted to a generalization of Darbo's fixed point theorem using the class of ϑ-ω control functions as a generalization of F-contractions. The paper will provide a large class of contractions introduced by Wardowski. Its results applying to the existence solutions for systems of integral equations. This paper can also consider as a provenance of new approaches associated with the measures of noncompactness and its applications in enhancing tripled fixed point theorems for mappings defined in nonempty, bounded, closed and convex subsets of Banach spaces.

Wardowski [20] presented the notion of F-contraction, which has encountered success due to its usefulness in nonlinear analysis. A fundamental advantage of such contractions is the eventuality of combined with different existing contractions to obtain new conditions.

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Throughout this paper, let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$, $Conv(\mathcal{G})$ be the convex hull of $\mathcal{G}$, $\mathfrak{M}(\mathfrak{B})$ be the set includes all nonempty bounded subsets of the Banach space $\mathfrak{B}$ and $\mathfrak{N}(\mathfrak{B})$ be the set includes all relatively compact subsets of $\mathfrak{B}$. Also, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}_+$ be the set of all nonnegative real numbers, $\overline{B}(\alpha, r)$ be the closed ball with center α and radius r and $\overline{B}_r$ be the ball $\overline{B}(0, r)$.

Definition 1.1. *[14] $\mu : \mathfrak{M}(\mathfrak{B}) \longrightarrow \mathfrak{R}_+$ is called a measure of noncompactness in the Banach space $\mathfrak{B}$ if:*

1° $\emptyset \neq ker\mu = \{\mathcal{G} \in \mathfrak{M}(\mathfrak{B}) : \mu(\mathcal{G}) = 0\} \subseteq \mathfrak{N}(\mathfrak{B})$;

2° $\mathcal{G} \subset \mathcal{Q} \Longrightarrow \mu(\mathcal{G}) \leq \mu(\mathcal{Q})$;

3° $\mu(\overline{\mathcal{G}}) = \mu(\mathcal{G})$;

4° $\mu(Conv\mathcal{G}) = \mu(\mathcal{G})$;

5° $\mu(\lambda\mathcal{G} + (1-\lambda)\mathcal{Q}) \leq \lambda\mu(\mathcal{G}) + (1-\lambda)\mu(\mathcal{Q})$ *for all* $\lambda \in [0, 1]$;

6° *If* $\{\mathcal{G}_n\} \subseteq \mathfrak{M}(\mathfrak{B})$ *for which* $\mathcal{G}_{n+1} = \mathcal{G}_n = \overline{\mathcal{G}_n}$ *and* $\lim_{n\to\infty} \mu(\mathcal{G}_n) = 0$, *then* $\mathcal{G}_\infty = \cap_{n=1}^{\infty} \mathcal{G}_n \neq \emptyset$.

As in [20], let Ω be the family of all functions $\omega : \mathbb{R}^+ \to \mathbb{R}$ such that:

(ω1) ω is strictly increasing,
(ω2) $\lim_{n\to\infty} \lambda_n = 0$ if and only if $\lim_{n\to\infty} \omega(\lambda_n) = -\infty$, for each sequence $\{\lambda_n\}$ in $(0, +\infty)$,
(ω3) $\lim_{\lambda\to 0^+} \lambda^k \omega(\lambda) = 0$, for a real number $k \in (0, 1)$.

In this paper, we assume that (Σ, σ) is a complete metric space.

Definition 1.2. *[20] $\Upsilon : \Sigma \to \Sigma$ is called an F-contraction if*

$$\sigma(\Upsilon\iota, \Upsilon\kappa) > 0 \Longrightarrow \tau + \omega(\sigma(\Upsilon\iota, \Upsilon\kappa)) \leq \omega(\sigma(\iota, \kappa)), \tag{1}$$

for all $\iota, \kappa \in \Sigma$, where $\tau \in \mathbb{R}^+$ and $\omega \in \Omega$.

Theorem 1.3. *[20] Let $\Upsilon : \Sigma \to \Sigma$ be an F-contraction. Then Υ admits a unique fixed point z in Σ and the sequence $\{\Upsilon^n u\}$ converges to z for any point $u \in \Sigma$.*

In [21] Isik et al. generalized the result of Wardowski to multivalued mappings as follows:
Let Θ interprets the set of all functions $\vartheta : \mathbb{R} \to \mathbb{R}$ such that:

(ϑ_1) $\lim_{n\to\infty} \frac{\vartheta^n(s)}{n} < 0$ for all $s > 0$;

(ϑ_2) $\vartheta(s) < s$ for all $s \in \mathbb{R}$.

(ϑ_3) ϑ is nondecreasing upper semi-continuous.

Let (Σ, σ) be a metric space and $\Upsilon, \Gamma : \Sigma \to \mathcal{CB}(\Sigma)$. The pair (Υ, Γ) is called a generalized Wardowski type contraction pair or a (ϑ, ω)-contraction pair, whenever there exist $\vartheta \in \Theta$ and $\omega \in \Omega$ such that for all $\iota, \kappa \in \Sigma$,

$$\mathfrak{H}(\Upsilon\iota, \Gamma\kappa) > 0 \Longrightarrow \omega(\mathfrak{H}(\Upsilon\iota, \Gamma\kappa)) \leq \vartheta(\omega(M(\iota, \kappa))), \tag{2}$$

where

$$M(\iota, \kappa) = \max\{\sigma(\iota, \kappa), \sigma(\iota, \Upsilon\iota), \sigma(\kappa, \Gamma\kappa), \frac{\sigma(\iota, \Gamma\kappa) + \sigma(\kappa, \Upsilon\iota)}{2}\}.$$

Theorem 1.4. *Let $\Upsilon, \Gamma : \Sigma \to \mathcal{K}(\Sigma)$ be such that (Υ, Γ) is a (ϑ, ω)-contraction pair. If Υ, Γ or ω is continuous, then Υ and Γ have a common fixed point.*

Now we remind two significant theorems which have a main designation in fixed point theory (Schauder and Darbo's fixed point theorems).

Theorem 1.5. *[13] Let Π be a nonempty, bounded, closed and convex subset of a Banach space $\mathfrak{B}$. Then $\omega : \Pi \to \Pi$ possesses at least one fixed point provided that ω be a continuous and compact mapping.*

From now on, we suppose that Π is a nonempty, bounded, closed and convex subset of a Banach space $\mathfrak{B}$.

Theorem 1.6. *[15] Let μ be a MNC defined in $\mathfrak{B}$ and let $\Gamma : \Pi \to \Pi$. Assume that there exists a constant $L \in [0, 1)$ such that $\mu(\Gamma\mathcal{G}) \leq L\mu(\mathcal{G})$ for any nonempty subset $\mathcal{G}$ of Π. Then Γ admits at least a fixed point in Π provided that Γ be a continuous mapping.*

In [22], the authors extended Darbo's fixed point theorem using the concept of ϑ-ω-contractions.
From now on, a nonempty, bounded, closed and convex subset Π of a Banach space $\mathfrak{B}$ is indicated by NBCC, shortly.

Theorem 1.7. *Let $\omega \in \Omega$, $\vartheta \in \Theta$, μ be an arbitrary MNC and let $\Upsilon : \Pi \to \Pi$ be a continuous operator such that*

$$\omega\Big(\mu(\Upsilon\mathcal{G})\Big) \leq \vartheta\Big(\omega(\mu(\mathcal{G}))\Big), \tag{3}$$

for all $\mathcal{G} \subseteq \Pi$. Then Υ has at least one fixed point in Π.

The notion of $A(f; .)$ operators has been presented by Altun and Turkoglu [16].
Let $\Xi([0, \infty))$ be the collection of all functions $f : [0, \infty) \to [0, \infty)$ and let $\widetilde{\Xi}$ be the class of all operators

$$A(\bullet; \cdot) : \Xi([0, \infty)) \to [0, \infty) : f \to A(f; \cdot)$$

such that:
(1) $A(f; \iota) > 0$ for all $\iota > 0$ and $A(f; 0) = 0$.
(2) $A(f; \iota) \leq A(f; \kappa)$ for all $\iota \leq \kappa$.
(3) $\lim_{n\to\infty} A(f; \iota_n) = A(f; \lim_{n\to\infty} \iota_n)$ for all sequence $\iota_n \subseteq [0, \infty)$.
(4) $A(f; \max\{\iota, \kappa\}) = \max\{A(f; \iota), A(f; \kappa)\}$ for all $f \in \Xi([0, \infty))$ and for all $\iota, \kappa \geq 0$.

2. Main Results

In this section, we want to extend Darbo's fixed point theorem via the concept of modified ϑ-ω-contractions.
Let Ω' be the following subcollection of Ω contains all functions $\omega : \mathbb{R}^+ \to \mathbb{R}$ such that

(ω_1) ω is a continuous and strictly increasing mapping;

Let Θ' indicates the collection of all functions $\vartheta : \mathbb{R} \to \mathbb{R}$ such that:

(ϑ_1) $\vartheta(s) < s$ for all $s \in \mathbb{R}$,

(ϑ_2) ϑ is an increasing continuous mapping.

Using the above two modified class of control functions, we prove the following results. In fact, we omitted some conditions on control functions ω and ϑ than the above classes introduced in [21] and [22].

Theorem 2.1. *Let $\omega \in \Omega'$, $\vartheta \in \Theta'$, $A(\bullet; \cdot) \in \widetilde{\Xi}$, μ be an arbitrary MNC and let $\Gamma : \Pi \to \Pi$ be a continuous operator such that*

$$\omega\Big(A(f; \mu(\Gamma\mathcal{G}))\Big) \leq \vartheta\Big(\omega(A(f; \mu(\mathcal{G})))\Big), \tag{4}$$

for all $\mathcal{G} \subseteq \Pi$. Then Γ has at least one fixed point in Π.

Proof. Let $\{\Pi_n\}$ be such that $\Pi_0 = \Pi$ and $\Pi_{n+1} = \overline{Conv}(\Gamma(\Pi_n))$ for all $n \in \mathbb{N}$.

If $\mu(\Pi_N) = 0$, for some $N \in \mathbb{N}$, then Π_N is relatively compact and so, from Theorem 1.5 Γ admits a fixed point. So, we may assume that $\mu(\Pi_n) > 0$ for each $n \in \mathbb{N}$.

Explicitly, $\{\Pi_n\}_{n\in\mathbb{N}}$ is a sequence of NBCC sets such that

$$\Pi_0 \supseteq \Pi_1 \supseteq \cdots \supseteq \Pi_n \supseteq \Pi_{n+1}.$$

On the other hand,

$$\begin{aligned} \omega(A(f;\mu(\Pi_{n+1}))) &= \omega(A(f;\mu(\Gamma\Pi_n))) \\ &\leq \vartheta(\omega(A(f;\mu(\Pi_n)))) \\ &< \omega(A(f;\mu(\Pi_n))). \end{aligned} \tag{5}$$

That is, according to $(\omega 1)$ $A(f;\mu(\Pi_{n+1})) < A(f;\mu(\Pi_n))$ for all $n \in \mathbb{N}$. This further means that $A(f;\mu(\Pi_n)) \to d^* \geq 0$ as $n \to +\infty$. If $d^* > 0$, we obtain from the previous relation that

$$\omega\Big(d^*\Big) \leq \vartheta\Big(\omega(d^*)\Big) \leq \omega\Big(d^*\Big), \tag{6}$$

which is a contradiction. Hence, $\lim_{n\to+\infty} A(f;\mu(\Pi_{n+1})) = 0$.

So, we obtain that

$$\lim_{n\to\infty} \mu(\Pi_{n+1}) = 0.$$

Reviewing axiom (6°) of Definition 1.1 we subsume that the set $\Pi_\infty = \bigcap_{n=1}^{\infty} \Pi_n$ is a nonempty, closed and convex set which is fixed under Γ and is an element of $Ker\mu$. Then according to the Schauder theorem, Γ admits a fixed point. □

Taking $A(f;s) = s$, $f = I$ (the identity mapping) and $\vartheta(s) = s - \tau$, for all $s \in \mathbb{R}$, we derive that:

Corollary 2.2. *Let $\omega \in \Omega'$, τ be an arbitrary positive amount, μ be an arbitrary MNC and let $\Gamma : \Pi \to \Pi$ be a continuous operator such that*

$$\tau + \omega\Big(\mu(\Gamma\mathcal{G})\Big) \leq \omega\Big(\mu(\mathcal{G})\Big), \tag{7}$$

for all $\mathcal{G} \subseteq \Pi$. Then Γ admits at least one fixed point in Π.

Remark 2.3. *If we define $\omega(s) = \ln s$, for all $s > 0$, we derive the Darbo's fixed point theorem from the above corollary.*

Corollary 2.4. *Let τ be an arbitrary positive amount, μ be an arbitrary MNC and let $\Gamma : \Pi \to \Pi$ be a continuous operator such that*

$$\tau + \mu(\Gamma\mathcal{G}) - \frac{1}{\mu(\Gamma\mathcal{G})} \leq \mu(\mathcal{G}) - \frac{1}{\mu(\mathcal{G})}, \tag{8}$$

for all $\mathcal{G} \subseteq \Pi$. Then Γ admits at least one fixed point in Π.

Taking $\omega(s) = s - \frac{1}{s}$ and $\vartheta(s) = s - W(e^s)$ where W is the Lambert W-function [23], we obtain that:

Corollary 2.5. *Let $\Gamma : \Pi \to \Pi$ be a continuous operator such that*

$$\mu(\Gamma\mathcal{G}) - \frac{1}{\mu(\Gamma\mathcal{G})} \leq \mu(\mathcal{G}) - \frac{1}{\mu(\mathcal{G})} - W(e^{\mu(\mathcal{G}) - \frac{1}{\mu(\mathcal{G})}}),$$

for all $\mathcal{G} \subseteq \Pi$, where μ is an arbitrary MNC. Then Γ admits at least one fixed point in Π.

Taking $\omega(s) = \ln s$ and $\vartheta(s) = g(s)$ where g is the inverse of the function $s + \frac{1}{1+s^2}$, we obtain the following corollary.

Corollary 2.6. *Let $\Gamma : \Pi \to \Pi$ be a continuous operator such that*

$$\begin{aligned}\ln(\mu(\Gamma(\mathcal{G}))) \leq & \frac{\ln \mu(\mathcal{G})}{2}\\ & - \frac{(\ln \mu(\mathcal{G}))^2 - 3}{3\sqrt[3]{-(\ln \mu(\mathcal{G}))^3 - 9(\ln \mu(\mathcal{G})) + \frac{1}{2}\sqrt{-4((\ln \mu(\mathcal{G}))^2 - 3)^3 + (-2(\ln \mu(\mathcal{G}))^3 - 18(\ln \mu(\mathcal{G})) + 27)^2} + \frac{27}{2}}}\\ & - \frac{1}{3}\sqrt[3]{-(\ln \mu(\mathcal{G}))^3 - 9(\ln \mu(\mathcal{G})) + \frac{1}{2}\sqrt{-4((\ln \mu(\mathcal{G}))^2 - 3)^3 + (-2(\ln \mu(\mathcal{G}))^3 - 18(\ln \mu(\mathcal{G})) + 27)^2} + \frac{27}{2}},\end{aligned}$$

for all $\mathcal{G} \subseteq \Pi$, where μ is an arbitrary MNC. Then Γ admits at least one fixed point in Π.

3. Tripled fixed point results

The notion of tripled fixed point has been presented by Berinde and Borcut [17]. They provided some tripled fixed point results for contractive mappings in metric spaces. For more details on tripled fixed point results and corresponding topics the reader may refer to [17], [18] and [19].

Definition 3.1. *[17] We say that $(\widehat{i}, \widehat{j}, \widehat{k}) \in \mathfrak{B}^3$ is a tripled fixed point of a mapping $\Gamma : \mathfrak{B} \times \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ if $\Gamma(\widehat{i}, \widehat{j}, \widehat{k}) = \widehat{i}$, $\Gamma(\widehat{j}, \widehat{i}, \widehat{j},) = \widehat{j}$ and $\Gamma(\widehat{k}, \widehat{j}, \widehat{i},) = \widehat{k}$.*

Theorem 3.2. *[6] Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness defined in Banach spaces $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$, respectively. Also, let the function $f : [0, \infty)^n \longrightarrow [0, \infty)$ be a convex function such that $f(\widehat{i}_1, \cdots, \widehat{i}_n) = 0$ if and only if $\widehat{i}_i = 0$ for all $i = 1, 2, \cdots, n$. Then*

$$\tilde{\mu}(\mathcal{G}) = f(\mu_1(\mathcal{G}_1), \mu_2(\mathcal{G}_2), \ldots, \mu_n(\mathcal{G}_n)),$$

is a measure of noncompactness in $\mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n$, where $\mathcal{G}_i$ is the natural projection of $\mathcal{G}$ into $\mathfrak{B}_i$, for all $i = 1, 2, \ldots, n$.

Theorem 3.3. *Let $\Gamma : \Pi \times \Pi \times \Pi \to \Pi$ be a continuous function such that*

$$\omega\Big(A(f; \mu(\Gamma(\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3)))\Big) \leq \frac{1}{3}\Big[\vartheta\Big(\omega\Big(A(f; \mu(\mathcal{G}_1) + \mu(\mathcal{G}_2) + \mu(\mathcal{G}_3))\Big)\Big)\Big] \tag{9}$$

for all subsets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ of Π, where μ is an arbitrary MNC, $\mu(\mathcal{G}_2) \leq \mu(\mathcal{G}_3)$, $\vartheta \in \Theta'$ and $\omega \in \Omega'$ and ω is a sub-additive mapping. Also, let $A(\bullet; \cdot) \in \widetilde{\Xi}$ and $A(f; r + s + t) \leq A(f; r) + A(f; s) + A(f; t)$ for all $r, s, t \geq 0$. Then Γ embraces at least a tripled fixed point.

Proof. Consider $\widetilde{\Gamma} : \Pi^3 \to \Pi^3$ by

$$\widetilde{\Gamma}(\widehat{i}, \widehat{j}, \widehat{k}) = (\Gamma(\widehat{i}, \widehat{j}, \widehat{k}), \Gamma(\widehat{j}, \widehat{i}, \widehat{j}), \Gamma(\widehat{k}, \widehat{j}, \widehat{i})).$$

Explicitly, $\widetilde{\Gamma}$ is continuous. The main thing that remains to be proven is that $\widetilde{\Gamma}$ satisfies all the reservations of Theorem 2.1. Let $\mathcal{G} \subset \Pi^3$ be a nonempty subset. We know that $\widetilde{\mu}(\mathcal{G}) = \mu(\mathcal{G}_1) + \mu(\mathcal{G}_2) + \mu(\mathcal{G}_3)$ is a (MNC)[14]. From (9) we have

$$
\begin{aligned}
\omega\Big(A(f;\ \widetilde{\mu}(\widetilde{\Gamma}(\mathcal{G})))\Big) &\le \omega\Big(A(f;\ \widetilde{\mu}(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)\times\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2)\times\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1)))\Big)\\
&= \omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3))+\mu(\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2))+\mu(\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1)))\Big)\\
&\le \omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)))\Big)+\omega\Big(A(f;\mu(\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2))\Big)\\
&+ \omega\Big(A(f;\mu(\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1)))\Big)\\
&\le \frac{1}{3}\Big[\vartheta\Big(\omega\Big(A(f;\ \mu(\mathcal{G}_1)+\mu(\mathcal{G}_2)+\mu(\mathcal{G}_3))\Big)\Big)\Big]\\
&+ \frac{1}{3}\Big[\vartheta\Big(\omega\Big(A(f;\mu(\mathcal{G}_2)+\mu(\mathcal{G}_1)+\mu(\mathcal{G}_2))\Big)\Big)\Big]\\
&+ \frac{1}{3}\Big[\vartheta\Big(\omega\Big(A(f;\mu(\mathcal{G}_3)+\mu(\mathcal{G}_2)+\mu(\mathcal{G}_1))\Big)\Big)\Big]\\
&\le \vartheta\Big(\omega\Big(A(f;\ \mu(\mathcal{G}_1)+\mu(\mathcal{G}_2)+\mu(\mathcal{G}_3))\Big)\Big)\\
&= \vartheta\Big(\omega(A(f;\ \widetilde{\mu}(\mathcal{G})))\Big).
\end{aligned}
$$

Now, from Theorem 2.1 it is concluded that $\widetilde{\Gamma}$ possesses at least a fixed point which infers that Γ admits at least a tripled fixed point. □

Substituting $A(f;t)=t$, $f=I$ and $\vartheta(t)=t-3\tau$ $(\tau>0)$ in Theorem 3.3 we infer:

Corollary 3.4. *Let $\Gamma:\Pi\times\Pi\times\Pi\to\Pi$ be a continuous function such that*

$$\tau+\omega[\mu(\Upsilon(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3))]\le\frac{1}{3}\omega[\mu(\mathcal{G}_1)+\mu(\mathcal{G}_2)+\mu(\mathcal{G}_3)] \tag{10}$$

for any subsets $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3$ of Π, where μ is an arbitrary (MNC), $\mu(\mathcal{G}_2)\le\mu(\mathcal{G}_3)$, and $\omega\in\Omega'$. Then Γ has at least a tripled fixed point.

Theorem 3.5. *Let $\Gamma:\Pi\times\Pi\times\Pi\to\Pi$ be a continuous function such that*

$$\omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)))\Big)\le\vartheta\Big(\omega\Big(A(f;\ \max\{\mu(\mathcal{G}_1),\mu(\mathcal{G}_2),\mu(\mathcal{G}_3)\})\Big)\Big) \tag{11}$$

for all subsets $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3$ of Π, where μ is an arbitrary MNC, $\mu(\mathcal{G}_2)\le\mu(\mathcal{G}_3)$, $\vartheta\in\Theta'$ and $\omega\in\Omega'$. Also, let $A(\bullet;\cdot)\in\widetilde{\Xi}$ and $A(f;\max\{r,s,t\})=\max\{A(f;r),A(f;s),A(f;t)\}$. Then Γ possesses at least a tripled fixed point.

Proof. Take $\widetilde{\Upsilon}:\Pi^3\to\Pi^3$ by

$$\widetilde{\Gamma}(\widehat{i},\widehat{j},\widehat{k})=(\Gamma(\widehat{i},\widehat{j},\widehat{k}),\Gamma(\widehat{j},\widehat{i},\widehat{j},),\Gamma(\widehat{k},\widehat{j},\widehat{i})).$$

The continuity of $\widetilde{\Gamma}$ is clear. We show that $\widetilde{\Gamma}$ fulfill all the reservations of Theorem 2.1. Clearly, $\widetilde{\mu}(\mathcal{G})=\max\{\mu(\mathcal{G}_1),\mu(\mathcal{G}_2),\mu(\mathcal{G}_3)\}$ is a (MNC)[14]. Let $\mathcal{G}\subset\Pi^3$ be a nonempty subset. According to (12) we infer

$$
\begin{aligned}
\omega\Big(A(f;\widetilde{\mu}(\widetilde{\Gamma}(\mathcal{G})))\Big) &\le \omega(A(f;\ \widetilde{\mu}(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)\times\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2)\times\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1)))\\
&= \omega\Big(A(f;\ \max\{\mu(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)),\mu(\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2)),\mu(\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1))\})\Big)\\
&= \max\{\omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)))\Big),\omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_2\times\mathcal{G}_1\times\mathcal{G}_2)))\Big),\\
&\qquad \omega\Big(A(f;\ \mu(\Gamma(\mathcal{G}_3\times\mathcal{G}_2\times\mathcal{G}_1)))\Big)\}\\
&\le \max\Big\{\vartheta\Big(\omega\Big(A(f;\ \max\{\mu(\mathcal{G}_1),\mu(\mathcal{G}_2),\mu(\mathcal{G}_3)\})\Big)\Big),\vartheta\Big(\omega\Big(A(f;\max\{\mu(\mathcal{G}_2),\mu(\mathcal{G}_1),\mu(\mathcal{G}_3)\}\Big)\Big),\\
&\qquad \vartheta\Big(\omega\Big((A(f;\ \max\{\mu(\mathcal{G}_3),\mu(\mathcal{G}_2),\mu(\mathcal{G}_1)\}\Big)\Big)\Big\}\\
&= \vartheta\Big(\omega\Big(A(f;\ \max\{\mu(\mathcal{G}_1),\mu(\mathcal{G}_2),\mu(\mathcal{G}_3)\})\Big)\Big)\\
&= \vartheta\Big(\omega\Big(A(f;\widetilde{\mu}(\mathcal{G}))\Big)\Big).
\end{aligned}
$$

According to Theorem 2.1 we elicit that $\widetilde{\Gamma}$ owes at least a fixed point, that is, Γ enjoys at least a tripled fixed point. $\square$

Taking $A(f;s) = s$ and $f = I$ in Theorem 3.5 we have:

Corollary 3.6. *Let $\Gamma : \Pi \times \Pi \times \Pi \to \Pi$ be a continuous function and*

$$\omega\Big(\mu(\Gamma(\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3))\Big) \leq \vartheta\Big(\omega\Big(\max\{\mu(\mathcal{G}_1), \mu(\mathcal{G}_2), \mu(\mathcal{G}_3)\}\Big)\Big) \tag{12}$$

for all subsets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ of Π, where μ is an arbitrary MNC, $\mu(\mathcal{G}_2) \leq \mu(\mathcal{G}_3)$ and $\vartheta \in \Theta'$ and $\omega \in \Omega'$. Then Υ possesses at least a tripled fixed point.

Corollary 3.7. *Let $\Gamma : \Pi \times \Pi \times \Pi \to \Pi$ be a continuous function such that*

$$\tau + \omega\Big(\mu(\Gamma(\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3))\Big) \leq \omega\Big(\max\{\mu(\mathcal{G}_1), \mu(\mathcal{G}_2), \mu(\mathcal{G}_3)\}\Big) \tag{13}$$

for all subsets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ of Π, where μ is an arbitrary (MNC), $\tau > 0$ and ω is as in Theorem 2.1. Then Γ has at least a tripled fixed point.

4. Application

The fractional integral of a function $f \in L_1(a, b)$ with respect to another function g of order α is,

$$I^{\alpha}_{a+,g} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{\Big(g(x) - g(t)\Big)^{1-\alpha}} dt, \ \alpha > 0, \ -\infty \leq a < b \leq \infty$$

which is defined for every continuous function $f(t)$ and for any monotone function $g(t)$ having a continuous derivative [24].

Analogous to the above definition, the fractional integral for a continuous function $h(t, s, r)$ in three variables on $\mathbb{R}^3$, w.r.t. monotone functions g, h, i of order α is given by,

$$\begin{aligned} & I^{\beta}_{a+,g,h,i} h(X, Y, Z) \\ & = \frac{1}{\Gamma(\alpha)^3} \int_a^X \int_a^Y \int_a^Z \frac{g'(t) h'(s) i'(r) h(t, s, r)}{\Big(g(X) - g(t)\Big)^{1-\alpha} \Big(h(Y) - h(s)\Big)^{1-\alpha} \Big(i(Z) - i(r)\Big)^{1-\alpha}} dsdtdr \end{aligned}$$

where $\alpha > 0$, $\Gamma(Z) = \int_0^{\infty} t^{Z-1} e^{-t} dt$, for all $Z > 0$ and $X, Y, Z \in [a, b]$ where $-\infty \leq a < b \leq \infty$.

Now, we discus the existence of solutions for the following system of fractional integral equations:

$$\begin{cases} \sigma_1(\mu_1,\mu_2,\mu_3) = h(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3), \\ \int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\left(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\right)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3) \\ \sigma_2(\mu_1,\mu_2,\mu_3) = h(\mu_1,\mu_2,\mu_3,\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3), \\ \int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\left(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3)\right)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3) \\ \sigma_3(\mu_1,\mu_2,\mu_3) = h(\mu_1,\mu_2,\mu_3,\sigma_3(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_1(\mu_1,\mu_2,\mu_3), \\ \int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\left(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_3(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_1(\zeta_1,\zeta_2,\zeta_3)\right)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3) \end{cases} \tag{14}$$

where $\mu_i \in [0,a]$.

Let $C([0,a]^3)$ be the space of all bounded continuous real functions on $[0,a]$ equipped with

$$\|\sigma\| = \sup\{|\sigma(\mu_1,\mu_2,\mu_3)| : \mu_1,\mu_2,\mu_3 \in [0,a]\}.$$

Let

$$\omega(\sigma,\epsilon) = \sup\{|\sigma(\mu_1,\mu_2,\mu_3) - \sigma(\mu_1',\mu_2',\mu_3')| : \mu_1,\mu_2,\mu_3,\mu_1',\mu_2',\mu_3' \in [0,a], |\mu_i - \mu_i'| \le \epsilon, \quad i = 1,2,3\},$$

for all $\sigma \in C([0,a]^3)$.

Now, let $\omega(\mathcal{G},\varepsilon) = \sup\{\omega(\sigma,\varepsilon) : \sigma \in \mathcal{G}\}$. Let

$$\omega(\mathcal{G}) = \lim_{\epsilon\to 0}\Big\{\sup_{\sigma\in\mathcal{G}} \omega(\sigma,\epsilon)\Big\},$$

be the Hausdorff measure of noncompactness for all bounded set $\mathcal{G}$ of $C([0,a]^3)$.

Theorem 4.1. *Let:*

(i) *The functions $A, B, C : I \to \mathbb{R}_+$ are C^1 and nondecreasing. Also, $A', B', C' \ge 0$,*

(ii) *The function $h : [0,a]^3 \times C([0,a])^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and*

$$\Big|h\Big(\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3,C\Big) - h\Big(\mu_1,\mu_2,\mu_3,\varsigma_1,\varsigma_2,\varsigma_3,D\Big)\Big| \le k\Big(\max_{1\le i\le 3}\Big\{|\sigma_i(\mu_1,\mu_2,\mu_3) - \varsigma_i(\mu_1,\mu_2,\mu_3)|\Big\} + \Big|C - D\Big|\Big), \tag{15}$$

for some $k \in (0,1)$,

(iii) *$g : [0,a]^6 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous,*

(iv) *The inequality*

$$kr + kK\frac{[A(a) - A(0)]^\alpha}{\alpha}\frac{[B(a) - B(0)]^\alpha}{\alpha}\frac{[C(a) - C(0)]^\alpha}{\alpha} + M \le r$$

enjoys a positive solution r_0, where $M = \max\Big\{\Big|h(\mu_1,\mu_2,\mu_3,0,0,0,0)\Big| : \mu_i \in [0,a]\Big\}$, *and*

$$K = \sup\{|g(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)))| \\ : \mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3 \in [0,a], \|\sigma_i\| \le r, \quad i = 1,2,3\}.$$

Then the system of integral equations (14) *enjoys at least a solution in* $(C([0,a]^3))^3$.

Proof. Let $\Gamma : C([0,a]^3) \times C([0,a]^3) \times C([0,a]^3) \longrightarrow C([0,a]^3)$ be such that

$$\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_1,\mu_2,\mu_3) = h(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3), \\ \int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3). \tag{16}$$

Let $\mu,\mu',\mu'' \in I$ be fixed and $\{\mu_n\}$ and $\{\mu_n'\}$ and $\{\mu_n''\}$ be sequences in I such that $\mu_n \to \mu$, $\mu_n' \to \mu'$ and $\mu_n'' \to \mu''$ as $n \to \infty$. Without loss of generality, we can choose $\mu_n \ge \mu$, $\mu_n' \ge \mu'$ and $\mu_n'' \ge \mu''$. Then

$$\Big|\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_n,\mu_n',\mu_n'') - \Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu,\mu',\mu'')\Big| \\ \le \Big|h\Big(\mu_n,\mu_n',\mu_n'',\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n''), \\ \int_0^{\mu_n''}\int_0^{\mu_n'}\int_0^{\mu_n} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\ - h\Big(\mu,\mu',\mu'',\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''), \\ \int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3))\Big| \\ \le k \max_{i=1,2,3}\{|\sigma_i(\mu_n,\mu_n',\mu_n'') - \sigma_i(\mu,\mu',\mu'')|\} \\ + k\Big|\int_0^{\mu_n''}\int_0^{\mu_n'}\int_0^{\mu_n} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3) \\ - \int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3))\Big| \\ \le k \max_{i=1,2,3}\{|\sigma_i(\mu_n,\mu_n',\mu_n'') - \sigma_i(\mu,\mu',\mu'')|\} + kE + kF + kG,$$

where

$$E =$$

$$\Big| \int_0^{\mu_n''} \int_0^{\mu_n'} \int_0^{\mu_n} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3)$$

$$- \int_0^{\mu''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3))))\Big|$$

$$\leq \Big| \int_0^{\mu_n''} \int_0^{\mu_n'} \int_{\mu}^{\mu_n} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3)$$

$$+ \int_0^{\mu''} \int_{\mu'}^{\mu_n'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3$$

$$+ \int_{\mu''}^{\mu_n''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big|$$

$$\leq \Big| K \frac{[A(\mu_n)-A(\mu)]^{\alpha}}{\alpha} \frac{[B(\mu_n')-B(0)]^{\alpha}}{\alpha} \frac{[C(\mu_n'')-C(0)]^{\alpha}}{\alpha}$$

$$+ K \frac{[A(\mu_n)-A(\mu)]^{\alpha}-[A(\mu_n)-A(0)]^{\alpha}}{\alpha} \frac{[B(\mu_n')-B(\mu)]^{\alpha}}{\alpha} \frac{[C(\mu_n'')-C(\mu'')]^{\alpha}-[C(\mu_n'')-C(0)]^{\alpha}}{\alpha}$$

$$+ K \frac{[[A(\mu_n)-A(\mu)]^{\alpha}-[A(\mu_n)-A(0)]^{\alpha}}{\alpha} \frac{[B(\mu_n')-B(\mu')]^{\alpha}-[B(\mu_n')-B(0)]^{\alpha}}{\alpha} \frac{[C(\mu_n'')-C(\mu'')]^{\alpha}}{\alpha} \Big|,$$

$$F =$$

$$\Big| \int_0^{\mu''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3)$$

$$- \int_0^{\mu''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big|$$

$$\leq K\Big(\int_0^{\mu''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3$$

$$- \int_0^{\mu''} \int_0^{\mu'} \int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)}{(A(\mu_n)-A(\zeta_1))^{1-\alpha}(B(\mu_n')-B(\zeta_2))^{1-\alpha}(C(\mu_n'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3$$

$$\leq K \frac{[A(\mu)-A(0)]^{\alpha}}{\alpha} \frac{[B(\mu')-B(0)]^{\alpha}}{\alpha} \frac{[C(\mu'')-C(0)]^{\alpha}}{\alpha}$$

$$- K \frac{[[A(\mu_n)-A(\mu)]^{\alpha}-[A(\mu_n)-A(0)]^{\alpha}}{\alpha} \frac{[B(\mu_n')-B(\mu')]^{\alpha}-[B(\mu_n')-B(0)]^{\alpha}}{\alpha} \frac{[C(\mu_n'')-C(\mu'')]^{\alpha}-[C(\mu_n'')-C(0)]^{\alpha}}{\alpha}$$

and

$$
\begin{aligned}
&G = \\
&\Big|\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3 \\
&-\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3\Big|
\end{aligned}
$$

which in it

$$
\begin{aligned}
Q_{r_0}^{\varepsilon} = \sup\{&|g(\mu_n,\mu_n',\mu_n'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_n,\mu_n',\mu_n''),\sigma_2(\mu_n,\mu_n',\mu_n''),\sigma_3(\mu_n,\mu_n',\mu_n'')) \\
&-g(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''))| \\
&:\mu_n,\mu_n',\mu_n'',\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\in[0,a],\sigma_i\in C([0,a]^3)\}.
\end{aligned}
$$

Now, as $n\to\infty$, continuity of A, B and C yield that $E\to 0$.

Also, taking $n\to\infty$ and using the continuity of A, B and C, it is observed that $F\to 0$.

On the other hand, from continuity presumption of g on the compact set $[0,a]^6\times\mathbb{R}^3$ we conclude that $G\longrightarrow 0$ as $n\to\infty$.

Thus, $\sigma_1,\sigma_2,\sigma_3\in C([0,a]^3)$ gives $\Gamma(\sigma_1,\sigma_2,\sigma_3)\in C([0,a]^3)$. So, the mapping $\Gamma : C([0,a]^3)^3\to C([0,a]^3)$ is well defined.

Also,

$$
\begin{aligned}
&|\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_1,\mu_2,\mu_3)| \\
&=\Big|h(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3), \\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\le\Big|h(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3) \\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3 \\
&-h(\mu_1,\mu_2,\mu_3,0,0,0,0)\Big| \\
&+\Big|h(\mu_1,\mu_2,\mu_3,0,0,0,0)\Big|.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&\Big|h\Big(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3),\\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3\\
&-h(\mu_1,\mu_2,\mu_3,0,0,0,0)\Big|\\
&\le k\max\{\Big|\sigma_1(\mu_1,\mu_2,\mu_3))\Big|,\Big|\sigma_2(\mu_1,\mu_2,\mu_3),\Big|\sigma_3(\mu_1,\mu_2,\mu_3)\Big|\}\\
&+k\Big|\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3\Big|\\
&\le k\max\{\|\sigma_1\|,\|\sigma_2\|,\|\sigma_3\|\}\\
&+k\Big|\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3\Big|\\
&\le k\max\{\|\sigma_1\|,\|\sigma_2\|,\|\sigma_3\|\}\\
&+kK\frac{[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|h\Big(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3),\\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}}d\zeta_1 d\zeta_2 d\zeta_3)\\
&-h(\mu_1,\mu_2,\mu_3,0,0,0,0)\Big|\\
&\le k\max\{\|\sigma_1\|,\|\sigma_2\|,\{\|\sigma_3\|\}+kK\frac{[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha}.
\end{aligned}$$

From the above calculations, we have

$$\begin{aligned}
&\|\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_1,\mu_2,\mu_3)\|\\
&\le k\max\{\|\sigma_1\|,\|\sigma_2\|,\|\sigma_3\|\}+kK\frac{[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha}+M\\
&\le k\max\{\|\sigma_1\|,\|\sigma_2\|,\|\sigma_3\|\}+kK\frac{[A(a)-A(0)]^\alpha}{\alpha}\frac{[B(a)-B(0)]^\alpha}{\alpha}\frac{[C(a)-C(0)]^\alpha}{\alpha}+M.
\end{aligned}\tag{17}$$

Applying inequality (17) and supposition (*iv*), we conclude that the function Γ brings $(\bar{B}_{r_0})^3$ into $\bar{B}_{r_0}$.

Now, we will investigate that Γ is a continuous function on $(\bar{B}_{r_0})^3$. So, fix $\varepsilon>0$ and take $\sigma_1,\sigma_2,\sigma_3,\varsigma_1,\varsigma_2,\varsigma_3\in$

$\bar{B}_{r_0}$ arbitrarily such that $\|\sigma_i - \varsigma_i\| \leq \varepsilon$ for all $i = 1, 2, 3$. Then, for all $\mu_1, \mu_2, \mu_3 \in [0, a]$, we infer

$$\begin{aligned}
&\Big|\Gamma(\sigma_1, \sigma_2, \sigma_3)(\mu_1, \mu_2, \mu_3) - \Gamma(\varsigma_1, \varsigma_2, \varsigma_3)(\mu_1, \mu_2, \mu_3)\Big| \\
&\leq \Big|h\Big(\mu_1, \mu_2, \mu_3, \sigma_1(\mu_1, \mu_2, \mu_3), \sigma_2(\mu_1, \mu_2, \mu_3), \sigma_3(\mu_1, \mu_2, \mu_3), \\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu_1, \mu_2, \mu_3, \varsigma_1(\mu_1, \mu_2, \mu_3), \varsigma_2(\mu_1, \mu_2, \mu_3), \varsigma_3(\mu_1, \mu_2, \mu_3), \\
&\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\varsigma_1(\zeta_1,\zeta_2,\zeta_3),\varsigma_2(\zeta_1,\zeta_2,\zeta_3),\varsigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\leq k \max_{i=1,2,3}\{|\sigma_i(\mu_1, \mu_2, \mu_3) - \varsigma_i(\mu_1, \mu_2, \mu_3)|\} \\
&+ k\Big|\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- \int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_2)-B(\zeta_2))^{1-\alpha}(C(\mu_3)-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\leq k \max_{i=1,2,3}\{\|\sigma_i - \varsigma_i\|\} + kQ^{\varepsilon}_{r_0} \frac{[A(a)-A(0)]^{\alpha}}{\alpha} \frac{[B(a)-B(0)]^{\alpha}}{\alpha} \frac{[C(a)-C(0)]^{\alpha}}{\alpha},
\end{aligned}$$

where

$$\begin{aligned}
Q^{\varepsilon}_{r_0} &= \sup\{|g(\mu_1, \mu_2, \mu_3, \zeta_1, \zeta_2, \zeta_3, \sigma_1(\zeta_1, \zeta_2, \zeta_3), \sigma_2(\zeta_1, \zeta_2, \zeta_3), \sigma_3(\zeta_1, \zeta_2, \zeta_3)) \\
&\quad - g(\mu_1, \mu_2, \mu_3, \zeta_1, \zeta_2, \zeta_3, \sigma_1(\zeta_1, \zeta_2, \zeta_3), \sigma_2(\zeta_1, \zeta_2, \zeta_3), \sigma_3(\zeta_1, \zeta_2, \zeta_3))| \\
&\quad : \mu_1, \mu_2, \mu_3 \in [0, a], \|\sigma_i\|, \|\varsigma_i\| \leq r_0, \|\sigma_i - \varsigma_i\| \leq \varepsilon\}.
\end{aligned}$$

From continuity presumption of g on the compact set $[0, a]^6 \times [-r_0, r_0]^3$ we conclude that $Q^{\varepsilon}_{r_0} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus, $\Big|\Gamma(\sigma_1, \sigma_2, \sigma_3)(\mu_1, \mu_2, \mu_3) - \Gamma(\varsigma_1, \varsigma_2, \varsigma_3)(\mu_1, \mu_2, \mu_3)\Big| \to 0$ as $\varepsilon \longrightarrow 0$, i.e., Γ is a continuous function on $(\bar{B}_{r_0})^3$.

Now, we prove that Γ assures all the reservations of Theorem 3.5. Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be nonempty and bounded subsets of $\bar{B}_{r_0}$. Also, let $\varepsilon > 0$ be fixed. Also, Let $\mu, \mu', \mu'', \mu_1, \mu_1', \mu_1'' \in I$ be fixed such that $\mu_1 \geq \mu$, $\mu_1' \geq \mu'$ and $\mu_1'' \geq \mu''$ (without loss of generality) with $|\mu - \mu_1| \leq \varepsilon$, $|\mu' - \mu_1'| \leq \varepsilon$, $|\mu'' - \mu_1''| \leq \varepsilon$, and $\sigma_j \in \mathcal{G}_j$ for all $j = 1, 2, 3$.

$$
\begin{aligned}
&\Big|\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_1,\mu_1',\mu_1'') - \Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu,\mu',\mu'')\Big| \\
&\le \Big|h\Big(\mu_1,\mu_1',\mu_1'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu,\mu',\mu'',\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''), \\
&\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\le k \max_{i=1,2,3}\{|\sigma_i(\mu_1,\mu_1',\mu_1'') - \sigma_i(\mu,\mu',\mu'')|\} + k\Delta,
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Delta = &\Big|h\Big(\mu_1,\mu_1',\mu_1'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu_1,\mu_1',\mu_1'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\le \Big|h\Big(\mu_1,\mu_1',\mu_1'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu,\mu',\mu'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&+ \Big|h\Big(\mu,\mu',\mu'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu,\mu',\mu'',\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''), \\
&\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| = I + II.
\end{aligned}
$$

We can calculate that

$$
\begin{aligned}
G_{A,B,C,a} = &\Big| \int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \Big| \\
&\leq K \frac{[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&\leq K \frac{[A(a)-A(0)]^\alpha}{\alpha}\frac{[B(a)-B(0)]^\alpha}{\alpha}\frac{[C(a)-C(0)]^\alpha}{\alpha}.
\end{aligned}
$$

Let

$$
\begin{aligned}
\Omega_{r_0}(h,\varepsilon) = \sup\{&|h(\mu_1,\mu_2,\mu_3,u,v,w,z) - h(\mu_1',\mu_2',\mu_3',u,v,w,z)| : \mu_1,\mu_2,\mu_3,\mu_1',\mu_2',\mu_3' \in [0,a], |\mu_i - \mu_i'| \leq \varepsilon, \\
&\|u\|,\|v\|,\|w\| \leq r_0, |z| \leq G_{A,B,C,a}\}.
\end{aligned}
$$

As h is uniform continuous on the compact set $[0,a]^3\times[-r_0,r_0]^3\times[-G_{r_0},G_{r_0}]$, we conclude that $\Omega_{r_0}(h,\varepsilon) \longrightarrow 0$.

Also,

$$
\begin{aligned}
&II = \\
&\Big|h\Big(\mu,\mu',\mu'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu,\mu',\mu'',\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''), \\
&\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\leq k \max_{i=1,2,3}\{|\sigma_i(\mu_1,\mu_1',\mu_1'') - \sigma_i(\mu,\mu',\mu'')|\} \\
&+ k\Big|\int_0^{\mu_1''}\int_0^{\mu_1'}\int_\mu^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3) \\
&+ \int_0^{\mu''}\int_{\mu'}^{\mu_1'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&+ \int_{\mu''}^{\mu_1''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \\
&\leq k \max_{i=1,2,3}\{|\sigma_i(\mu_1,\mu_1',\mu_1'') - \sigma_i(\mu,\mu',\mu'')|\} \\
&+ k\Big|K\frac{[A(\mu_1)-A(\mu)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\frac{[A(\mu_1)-A(\mu)]^\alpha-[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(\mu)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(\mu'')]^\alpha-[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\frac{[[A(\mu_1)-A(\mu)]^\alpha-[A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(\mu')]^\alpha-[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(\mu'')]^\alpha}{\alpha}\Big|.
\end{aligned}
$$

Using condition 15 we have

$$
\begin{aligned}
\Delta &= \Big|\Gamma(\sigma_1,\sigma_2,\sigma_3)(\mu_1,\mu_1',\mu_1'') - \Gamma(\varsigma_1,\varsigma_2,\varsigma_3)(\mu_1,\mu_1',\mu_1'')\Big| \\
&= \Big|h\Big(\mu_1,\mu_1',\mu_1'',\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1''), \\
&\int_0^{\mu_1''}\int_0^{\mu_1'}\int_0^{\mu_1} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu_1,\mu_1',\mu_1'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu_1,\mu_1',\mu_1''),\sigma_2(\mu_1,\mu_1',\mu_1''),\sigma_3(\mu_1,\mu_1',\mu_1'')\Big)}{(A(\mu_1)-A(\zeta_1))^{1-\alpha}(B(\mu_1')-B(\zeta_2))^{1-\alpha}(C(\mu_1'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3 \\
&- h\Big(\mu,\mu',\mu'',\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu''), \\
&\int_0^{\mu''}\int_0^{\mu'}\int_0^{\mu} \frac{A'(\zeta_1)B'(\zeta_2)C'(\zeta_3)g\Big(\mu,\mu',\mu'',\zeta_1,\zeta_2,\zeta_3,\sigma_1(\mu,\mu',\mu''),\sigma_2(\mu,\mu',\mu''),\sigma_3(\mu,\mu',\mu'')\Big)}{(A(\mu)-A(\zeta_1))^{1-\alpha}(B(\mu')-B(\zeta_2))^{1-\alpha}(C(\mu'')-C(\zeta_3))^{1-\alpha}} d\zeta_1 d\zeta_2 d\zeta_3\Big| \qquad (18) \\
&\leq \Omega_{r_0}(h,\varepsilon) + \\
&k \max_{i=1,2,3}\{|\sigma_i(\mu_1,\mu_1',\mu_1'') - \sigma_i(\mu,\mu',\mu'')|\} \\
&+ k\Big|K\frac{[A(\mu_1)-A(\mu)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\frac{[A(\mu_1)-A(\mu)]^\alpha - [A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(\mu)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(\mu'')]^\alpha - [C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\frac{[[A(\mu_1)-A(\mu)]^\alpha - [A(\mu_1)-A(0)]^\alpha}{\alpha}\frac{[B(\mu_1')-B(\mu')]^\alpha - [B(\mu_1')-B(0)]^\alpha}{\alpha}\frac{[C(\mu_1'')-C(\mu'')]^\alpha}{\alpha}\Big|..
\end{aligned}
$$

Since in (18), σ_i was an arbitrary element of $\mathcal{G}_i$ for $i = 1,2,3$, we obtain that

$$
\begin{aligned}
&\Omega(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3),\varepsilon) \\
&\leq \Omega_{r_0}(h,\varepsilon) + k\max\{\Omega(\mathcal{G}_1,\varepsilon),\Omega(\mathcal{G}_2,\varepsilon),\Omega(\mathcal{G}_3,\varepsilon)\} \\
&+ k\Big|K\tfrac{[A(\mu_1)-A(\mu)]^\alpha}{\alpha}\tfrac{[B(\mu_1')-B(0)]^\alpha}{\alpha}\tfrac{[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\tfrac{[A(\mu_1)-A(\mu)]^\alpha-[A(\mu_1)-A(0)]^\alpha}{\alpha}\tfrac{[B(\mu_1')-B(\mu)]^\alpha}{\alpha}\tfrac{[C(\mu_1'')-C(\mu'')]^\alpha-[C(\mu_1'')-C(0)]^\alpha}{\alpha} \\
&+ K\tfrac{[[A(\mu_1)-A(\mu)]^\alpha-[A(\mu_1)-A(0)]^\alpha}{\alpha}\tfrac{[B(\mu_1')-B(\mu')]^\alpha-[B(\mu_1')-B(0)]^\alpha}{\alpha}\tfrac{[C(\mu_1'')-C(\mu'')]^\alpha}{\alpha}\Big|..
\end{aligned}
$$

As h, A, B and C are uniform continuous on the compact sets $[0,a]^3\times[-r_0,r_0]^3\times[-G_{r_0},G_{r_0}]$, $[0,a]$, $[0,a]$ and $[0,a]$, respectively, we conclude that $\Omega_{r_0}(h,\varepsilon)\longrightarrow 0$, $A(\mu_1'')-A(\mu'')\longrightarrow 0$, $B(\mu_1'')-B(\mu'')\longrightarrow 0$ and $C(\mu_1'')-C(\mu'')\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. Therefore, by taking $A(f;t)=t$ and $f=I$, we have

$$\Omega(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3)) \leq k\max\{\Omega(\mathcal{G}_1),\Omega(\mathcal{G}_2),\Omega(\mathcal{G}_3)\}.$$

Thus, we get

$$\omega(\Omega(\Gamma(\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3))) \leq \vartheta\Big(\omega\Big(\max\{\Omega(\mathcal{G}_1),\Omega(\mathcal{G}_2),\Omega(\mathcal{G}_3)\}\Big)\Big), \qquad (19)$$

where $\omega(x) = xe^x$ and $\vartheta(x) = kx$ for all $x > 0$ and $\vartheta(x) = lx$ for all $x \leq 0$ where $l \geq 1$ (Note that $kxe^{kx} \leq kxe^x$ for all $k \in (0,1)$.) Therefore, Corollary 3.6 infers that the operator Γ enjoys a tripled fixed point. That is, the system of functional integral equations (14) admits at least one solution in $(C([0,a]^3))^3$. ■ □

Remark 4.2. *In the above theorem, we applied $\omega(x) = xe^x$ which is an element of Ω'. This function is not an element of Ω. Also, ϑ is not a function in collection Θ introduced in [22]. So, we have larger collections of functions with respect to the classes of functions introduced in [22].*

5. Example

Example 5.1. *Let:*

$$\left\{\begin{array}{l}
\sigma_1(\mu_1,\mu_2,\mu_3) = \frac{1}{3}e^{-\sum_{i=1}^{3}\mu_i^2} + \frac{\tan^{-1}\sigma_1(\mu_1,\mu_1,\mu_3)+\sinh^{-1}\sigma_2(\mu_1,\mu_1,\mu_3)+\tan^{-1}\sigma_3(\mu_1,\mu_1,\mu_3)}{10\pi+(\sum_{i=1}^{3}\mu_i^2)^{10}} \\
+\frac{1}{10}\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{\sum_{i=1}^{3}\zeta_i(\cos\sum_{i=1}^{3}\mu_i+\sqrt{(1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3))}}{e^{\sum_{i=1}^{3}\mu_i}\sqrt{\prod i=1^3(\mu_i-\zeta_i)}(1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3))}d\zeta_1 d\zeta_2 d\zeta_3 \\
\sigma_2(\mu_1,\mu_2,\mu_3) = \frac{1}{3}e^{-\sum_{i=1}^{3}\mu_i^2} + \frac{\tan^{-1}\sigma_2(\mu_1,\mu_1,\mu_3)+\sinh^{-1}\sigma_1(\mu_1,\mu_1,\mu_3)+\tan^{-1}\sigma_2(\mu_1,\mu_1,\mu_3)}{10\pi+(\sum_{i=1}^{3}\mu_i^2)^{10}} \\
+\frac{1}{10}\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{\sum_{i=1}^{3}\zeta_i(\cos\sum_{i=1}^{3}\mu_i+\sqrt{(1+\sigma_2^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_1(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))}}{e^{\sum_{i=1}^{3}\mu_i}\sqrt{\prod i=1^3(\mu_i-\zeta_i)}(1+\sigma_2^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_1(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))}d\zeta_1 d\zeta_2 d\zeta_3 \\
\sigma_3(\mu_1,\mu_2,\mu_3) = \frac{1}{3}e^{-\sum_{i=1}^{3}\mu_i^2} + \frac{\tan^{-1}\sigma_1(\mu_1,\mu_1,\mu_3)+\sinh^{-1}\sigma_2(\mu_1,\mu_1,\mu_3)+\tan^{-1}\sigma_3(\mu_1,\mu_1,\mu_3)}{10\pi+(\sum_{i=1}^{3}\mu_i^2)^{10}} \\
+\frac{1}{10}\int_0^{\mu_3}\int_0^{\mu_2}\int_0^{\mu_1}\frac{\sum_{i=1}^{3}\zeta_i(\cos\sum_{i=1}^{3}\mu_i+\sqrt{(1+\sigma_3^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_1(\zeta_1,\zeta_2,\zeta_3))}}{e^{\sum_{i=1}^{3}\mu_i}\sqrt{\prod i=1^3(\mu_i-\zeta_i)}(1+\sigma_3^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_1(\zeta_1,\zeta_2,\zeta_3))}d\zeta_1 d\zeta_2 d\zeta_3
\end{array}\right. \tag{20}$$

The above system is a particular case of the system (14) with

$$A(t) = B(t) = C(t) = t, \qquad \alpha = \frac{1}{2}, \qquad t \in [0,1],$$

$$h(\mu_1,\mu_2,\mu_3,\sigma_1(\mu_1,\mu_2,\mu_3),\sigma_2(\mu_1,\mu_2,\mu_3),\sigma_3(\mu_1,\mu_2,\mu_3),p) \tag{21}$$

$$= \frac{1}{3}e^{-\sum_{i=1}^{3}\mu_i^2} + \frac{\tan^{-1}\sigma_1(\mu_1,\mu_2,\mu_3)+\sinh^{-1}\sigma_2(\mu_1,\mu_2,\mu_3)+\tan^{-1}\sigma_3(\mu_1,\mu_2,\mu_3)}{10\pi+(\sum_{i=1}^{3}\mu_i)^{10}} + \frac{p}{10} \tag{22}$$

and

$$g(\mu_1,\mu_2,\mu_3,\zeta_1,\zeta_2,\zeta_3,\sigma_1(\zeta_1,\zeta_2,\zeta_3),\sigma_2(\zeta_1,\zeta_2,\zeta_3),\sigma_3(\zeta_1,\zeta_2,\zeta_3)) \tag{23}$$

$$= \frac{\sum_{i=1}^{3}\zeta_i(\cos\sum_{i=1}^{3}\mu_i+\sqrt{(1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3))}}{e^{\sum_{i=1}^{3}\mu_i}(1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3))(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3))}. \tag{24}$$

We need to investigate the suppositions (i)-(iv) of Theorem 4.1 to show that the above system has a solution.

Example 5.2. *Supposition (i) is explicit. We define $\omega(s) = se^s$ and $k = \frac{1}{10}$.*
Now,

$$\begin{aligned}
&\omega\Big(\Big|h(\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3,m) - h(\mu_1,\mu_2,\mu_3,\sigma_1',\sigma_2',\sigma_3',n)\Big|\Big) \\
&\le \omega\Big(\frac{|\tan^{-1}\sigma_1-\tan^{-1}\sigma_1'|+|\sinh^{-1}\sigma_2-\sinh^{-1}\sigma_2'|+|\tan^{-1}\sigma_3-\tan^{-1}\sigma_3'|}{10\pi+(\sum_{i=1}^{3}\mu_i^2)^{10}} + \frac{|m-n|}{10}\Big) \\
&\le \omega\Big(\frac{|\sigma_1-\sigma_1'|+|\sigma_2-\sigma_2'|+|\sigma_3-\sigma_3'|}{10\pi} + \frac{|m-n|}{10}\Big) \\
&\le \omega\Big(\frac{\max_i\{|\sigma_i-\sigma_i'|\}+|m-n|}{10}\Big) \\
&\le \vartheta\Big(\omega\Big(\max_i\{|\sigma_i-\sigma_i'|\}+|m-n|\Big)\Big).
\end{aligned}$$

Hence, h fulfills supposition (ii) of Theorem 4.1. Also,

$$M = \sup\{|h(\mu_1, \mu_2, \mu_3, 0, 0, 0, 0)| : \mu_1, \mu_2, \mu_3 \in [0, 1]\} = \sup\{\frac{1}{3}e^{-\sum_{i=1}^{3} \mu_i^2} : \mu_1, \mu_2, \mu_3 \in [0, 1]\} \simeq 0.33$$

Explicitly, supposition (iii) of Theorem 4.1 is accurate, that is, g is continuous on $[0, a]^3 \times [0, a]^3 \times \mathbb{R}^3$*, and*

$$K = \sup\left\{\left|\frac{\sum_{i=1}^{3} \zeta_i(\cos\sum_{i=1}^{3} \mu_i + \sqrt{1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3)))}}{e^{\sum_{i=1}^{3}\mu_i}(1+\sigma_1^2(\zeta_1,\zeta_2,\zeta_3)^2)(1+\sin^2\sigma_2(\zeta_1,\zeta_2,\zeta_3))(1+\cos^2\sigma_3(\zeta_1,\zeta_2,\zeta_3))}\right|\right.$$
$$\left. : \mu_1, \mu_2, \mu_3 \in [0, 1], \zeta_1, \zeta_2, \zeta_3 \in [0, 1], \sigma_i \in [-r, r]\right\}$$
$$\leq \sup \frac{t^2}{e^t} \simeq 0.55.$$

As well as, every $r \geq 0.3744$ *satisfies the condition (iv), i.e.,*

$$kr + kK\frac{[A(a)-A(0)]^\alpha}{\alpha}\frac{[B(a)-B(0)]^\alpha}{\alpha}\frac{[C(a)-C(0)]^\alpha}{\alpha} + M + 0.33$$
$$= \frac{1}{10}r + \frac{1}{10}\frac{55}{100}\frac{1}{8} + 0.33 \leq r.$$

As a result, the suppositions of Theorem 4.1 are valid and hence, the above system of integral equations enjoys at least one solution in $\{C([0, a]^3)\}^3$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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