



Characterizations of Twisted Spacetime

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Abstract. The object of the present paper is to characterize pseudo Ricci symmetric twisted spacetimes. At first it is shown that a pseudo Ricci symmetric twisted spacetime is a GRW spacetime. Next, we obtain a necessary and sufficient condition for a pseudo Ricci symmetric twisted spacetime to be a perfect fluid spacetime. It is also proved that a pseudo Ricci symmetric twisted spacetime is perfect fluid, provided the conformal curvature tensor is divergence free. Moreover, we investigate shear and vorticity vector fields of such type of spacetimes and pointed out its implications towards the evolution of the Early Universe. Finally, we study conformally flat $(PRS)_n$ spacetimes.

1. Introduction

Investigating conformally flat space (M, g) of class one Sen and Chaki [28] found that the covariant derivative of the Ricci tensor R_{ab} takes the form

$$R_{ab,c} = 2u_c R_{ab} + u_a R_{cb} + u_b R_{ac}, \quad (1)$$

where u_a is a non-zero covariant vector and $'$ denotes covariant differentiation with respect to the metric tensor g_{ab} . Later Chaki [6] called an n -dimensional non-flat Riemannian manifold whose Ricci tensor satisfies the condition (1), a pseudo Ricci symmetric manifold and denoted by $(PRS)_n$. Pseudo Ricci symmetric manifolds have been investigated by several authors in different aspects, see Ref.([12], [3], [27]).

In 1907, Einstein proposed a set of ten equations, known as the Einstein's field equations, defined by

$$R_{ab} - \frac{R}{2}g_{ab} + \lambda g_{ab} = \kappa T_{ab}, \quad (2)$$

where R_{ab} denotes the Ricci tensor, R , the scalar curvature, λ , the cosmological constant, κ , the gravitational constant and T_{ab} is the energy momentum tensor of the fluid spacetime corresponding to the metric tensor g_{ab} . Since then, many researchers found the solutions of the equation (2) under the certain assumptions and try to explain the behaviour of the universe. They noticed that our universe is in expanding era. A group of four scientists, namely Alexander Friedmann, Georges Lemaître, Howard P. Robertson and Arthur Geoffrey Walker proposed a metric by considering that the space is isotropic and homogeneous, and called such

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metric as the Friedmann–Lemaître–Robertson–Walker (FLRW) or Friedmann–Robertson–Walker (FRW) or Robertson–Walker (RW) or Friedmann–Lemaître (FL) metric and proved that it is an exact solution of the Einstein’s field equations of general relativity. Many researchers are considering this model as the standard model of modern cosmology.

The study of warped product manifold was initiated by Kručkovič [15] in 1957. Again in 1969 Bishop and O’Neill [5] also obtained the same notion of the warped product manifolds while they were constructing a large class of manifolds of negative curvature. Warped product are generalizations of the Cartesian product of Riemannian manifolds. Let (\bar{M}, \bar{g}) and (M^*, g^*) be two Riemannian manifolds. Let \bar{M} and M^* be covered with coordinate charts $(U; x^1, x^2, \dots, x^p)$ and $(V; y^{p+1}, y^{p+2}, \dots, y^n)$ respectively.

Then the warped product $M = \bar{M} \times_f M^*$ is the product manifold of dimension n furnished with the metric

$$g = \pi^*(\bar{g}) + (f \circ \pi)\sigma^*(g^*), \tag{3}$$

where $\pi : M \rightarrow \bar{M}$ and $\sigma : M \rightarrow M^*$ are natural projections such that the warped product manifold $\bar{M} \times_f M^*$ is covered with the coordinate chart

$$(U \times V; x^1, x^2, \dots, x^p, x^{p+1} = y^{p+1}, x^{p+2} = y^{p+2}, \dots, x^n = y^n).$$

Then the local components of the metric g with respect to this coordinate chart are given by

$$g_{ij} = \begin{cases} \bar{g}_{ij} & \text{for } i=a \text{ and } j=b, \\ f g^*_{ij} & \text{for } i = \alpha \text{ and } j = \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

Here $a, b, c, \dots \in \{1, 2, \dots, p\}$ and $\alpha, \beta, \gamma, \dots \in \{p + 1, p + 2, \dots, n\}$ and $i, j, k, \dots \in \{1, 2, \dots, n\}$. Here \bar{M} is called the base, M^* is called the fiber and f is called warping function of the warped product $M = \bar{M} \times_f M^*$.

For a Robertson–Walker space time, we have $I \times_{f(t)} R^n(c)$, which is a warped product of time interval I and a real space form $R^n(c)$ and warping function $f(t)$. For such a spacetime, the space slices are homothetic to a real space form $R^n(c)$. Similarly for a generalized RW spacetime $I \times_{f(t)} M^n$. In the above case, slices are always homothetic to the M^n . Thus, basically, the space slices are the same. So, in such cases, space slices are “rigid”. But this may not be true since our world may change from time to time. For this reason Chen[9] defined the notion of twisted products.

For a twisted spacetime $I \times_f M^n$, the twisting function f depends not only on time t , but also on the point in M^n . In this way, the slices are no longer rigid. As a twisted space time, the universe is twisted, in the sense that the space slices are changing from time to time.

This is the reason to define twisted products. Twisted spacetimes are much general than warped spacetimes which allow that the world is changing from time to time.

Alias et al. [1], in 1995, introduced the notion of generalized Robertson-Walker (GRW) spacetime, which is a generalization of the Robertson-Walker (RW) spacetime. A GRW spacetime of dimension n is an n -dimensional Lorentzian manifold M , that is, $M = -I \times f^2 M^*$, where I is an open interval of the real line R , M^* , a Riemannian manifold of dimension $(n - 1)$ and $f (> 0)$, a smooth warping function (or scale factor). An n -dimensional Lorentzian manifold M with the metric (in local shape)

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(dt)^2 + f(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{5}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only ($\alpha, \beta, \gamma = 2, 3, \dots, n$) and f , the warping function of t only, is known as GRW spacetime. If $g^*_{\alpha\beta}$ has dimension 3 and constant curvature, then the spacetime converts into the RW spacetime. The GRW spacetimes include the Friedmann cosmological models, the Einstein-de Sitter spacetime, the static Einstein spacetime and the de Sitter spacetime.

In [10], Chen proved that

Theorem A. A Lorentzian n -manifold with $n > 3$ is a generalized Robertson-Walker spacetime if and only if it admits a timelike concircular vector field.

Also, Mantica and Molinari[16] established the following:

Theorem B. A Lorentzian manifold of dimension $n > 3$ is a GRW spacetime if and only if it admits a unit timelike torse-forming vector, $\nabla_k u_j = \phi(u_k u_j + g_{kj})$, that is also an eigenvector of the Ricci tensor.

The idea of twisted spacetime has been introduced by B. Y. Chen [9], which is the generalization of GRW spacetime, that is, a Lorentzian manifold M of dimension n with the metric (in local shape)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(dt)^2 + f^2(x, t) g_{\mu\nu}^* dx^\mu dx^\nu, \tag{6}$$

$f > 0$ is the scale factor and g^* is the metric tensor of a Riemannian sub-manifold of dimension $(n - 1)$. If f is a function of t only, then the twisted spacetime becomes the generalized Robertson–Walker spacetime. Robertson–Walker, generalized Robertson–Walker and twisted spacetimes with torse-forming timelike unit vector have been characterized by Mantica et al. [17]. For more details about the Robertson-Walker, generalized Robertson-Walker and twisted spacetimes, we refer [1], [8], [9], [10], [11], [16] – [21], [23] and [24].

In [17] Mantica and Molinari proved that

Theorem C. A Lorentzian manifold L_n is twisted if and only if it admits a unit timelike torseforming vector field. That is,

$$u_{a,b} = \phi(g_{ab} + u_a u_b),$$

where $'$, $'$ denotes covariant differentiation, ϕ is a scalar function and $u_i u^i = -1$.

In the same paper the authors obtain necessary and sufficient condition for a twisted manifold to be a GRW spacetime. Also some interesting properties of the conformal curvature tensor and spacetimes have been addressed by the authors in ([17], [4], [33] and [23]). Also in [19] Mantica, Molinari and De established the following result:

Theorem D. Let M be a perfect fluid spacetime, i.e., a Lorentzian manifold of dimension $n > 3$ with Ricci tensor $R_{kl} = Ag_{kl} + Bu_k u_l$, where A and B are scalars, u is a unit time-like vector field $u^j u_j = -1$. If $u_{j,i} - u_{i,j} = 0$ and $C_{jkl,m}^m = 0$, then M is a generalized Robertson–Walker spacetime with Einstein fibers.

Motivated by the works in twisted spacetimes of Mantica and Molinari[17], in the present paper we mainly characterize pseudo Ricci symmetric twisted spacetimes. Precisely we prove the following theorems:

Theorem 1.1. A pseudo Ricci symmetric twisted spacetime is a GRW spacetime.

Theorem 1.2. A pseudo Ricci symmetric twisted spacetime is a perfect fluid spacetime if and only if $C_{abcd} u^a u^d = 0$, where u_a is the velocity vector field and $u^a = g^{ab} u_b$.

Theorem 1.3. A $(PRS)_n$ twisted spacetime with $\text{div}C = 0$ has vanishing vorticity and vanishing shear.

Theorem 1.4. In a perfect fluid pseudo Ricci symmetric spacetime, the state equation is $p + \mu = 0$ which leads to rapid expansion of the spacetime, now termed as inflation.

Finally, using the Theorem D of Mantica et. al[19] we prove

Theorem 1.5. A conformally flat pseudo Ricci symmetric spacetime is a GRW spacetime.

2. Preliminaries

It is observed that in a $(PRS)_n$, Chaki[6] obtained the following theorems:

Theorem 2.1. [6] If the scalar curvature R of a $(PRS)_n$ is constant, then $R = 0$, while if $R \neq 0$ then the associated vector u_a of the manifold is irrotational.

Theorem 2.2. [6] In a conformally flat $(PRS)_n$, the scalar curvature R is necessarily different from zero.

Thus in a $(PRS)_n$, u_a is irrotational. Also in [6] the author prove that $u^a R_{ab} = 0$. Throughout the present paper we assume that the associated vector u_a is a unit timelike vector field.

A Lorentzian manifold L_n is said to be a perfect fluid spacetime if the Ricci tensor R_{ab} is of the form:

$$R_{ab} = Ag_{ab} + Bu_a u_b,$$

where A, B are scalars and u_a is non-zero 1-form such that $u_a u^a = -1$.

In perfect fluid spacetime, the energy momentum tensor T_{ab} of type $(0, 2)$ is of the form ([24]):

$$T_{ab} = pg_{ab} + (\sigma + p)u_a u_b, \tag{7}$$

where σ and p are the energy density and the isotropic pressure respectively and u_a is a unit timelike vector. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [14].

Definition 2.3. A vector field V is called torse-forming if $V_{i;j} = \omega_j V_i + \phi g_{ij}$, being ϕ is a scalar function and ω_j is a covariant vector.

Torse-forming vector fields in Riemannian spaces were studied by Yano([31], [32]). Sinyukov[29], Mikeš and Rachunek[22] studied torse-forming vector fields in pseudo-Riemannian spaces. It can be easily prove that for a unit torse-forming vector u_i , we get

$$u_{i;j} = \phi(g_{ij} + u_i u_j),$$

ϕ is a scalar. If ω_j is locally the gradient of a scalar, then torse-forming vectors are called concircular by Yano. Also Fialkow[13] defined concircular vector field in a different way as follows:

Definition 2.4. A vector field V is called concircular vector field if $V_{i;j} = f g_{ij}$, f is a scalar function.

In 2014, Chen[10] characterized GRW spacetime in the presence of a timelike vector field given by Theorem A. On the other hand, Mantica and Molinari[23] proved that in a twisted spacetime the vector field u_a is Weyl compatible:

$$(u_a C_{bcde} + u_b C_{cade} + u_c C_{abde})u^e = 0, \tag{8}$$

where C denotes the conformal curvature tensor and u_a is unique.

3. Proof of the main Theorem

Proof of Theorem 1.1:

Let us assume that the pseudo Ricci symmetric spacetime is a twisted spacetime. Hence we have from Theorem C that

$$u_{a;b} = \phi(g_{ab} + u_a u_b), \tag{9}$$

where ϕ is a scalar and $u_a u^a = -1$.

Taking covariant differentiation of (9) with respect to x^c and using (9) we have

$$u_{a,bc} = \phi_c(g_{ab} + u_a u_b) + \phi^2(g_{ac}u_b + g_{bc}u_a + 2u_a u_b u_c).$$

From the foregoing equation we reveal that

$$u_{a,bc} - u_{a,cb} = \phi_c(g_{ab} + u_a u_b) - \phi_b(g_{ac} + u_a u_c) + \phi^2(g_{ac}u_b - g_{ab}u_c),$$

where $\phi_c = \phi, c$.

Now applying Ricci identity we get

$$u_e R_{abc}^e = \phi_c(g_{ab} + u_a u_b) - \phi_b(g_{ac} + u_a u_c) + \phi^2(g_{ac} u_b - g_{ab} u_c). \tag{10}$$

Multiplying the above equation with g^{ab} entails that

$$u^e R_{ec} = \phi^2(1 - n)u_c + (n - 2)\phi_c - \phi_a u^a u_c. \tag{11}$$

It is known[6] that $u^e R_{ec} = 0$ in a $(PRS)_n$.

Using the above result from (11) we infer that

$\phi_c = f u_c$, where $f = \frac{1}{n-2}[-\phi^2(1 - n) + \phi_a u^a]$ and f is a scalar.

Then $(\phi u_a), b - (\phi u_b), a = \phi_b u_a + \phi u_{a,b} - \phi_a u_b - \phi u_{b,a}$.

Using $\phi_c = f u_c$ and remembering that u_a is irrotational, we finally obtain $(\phi u_a), b - (\phi u_b), a = 0$, that is, ϕu_a is locally a gradient. Hence $\phi u_a = \sigma_{,a}$, σ is a scalar.

Now defining $X_a = u_a e^{-\sigma}$, we have $X_a X^a = u_a u^a e^{-\sigma} e^\sigma = -1 < 0$ and $X_{a,b} = u_{a,b} e^{-\sigma} + u_a e^{-\sigma} (-\sigma_b) = \phi(g_{ab} + u_a u_b)(-u_a e^\sigma (-\phi u_b)) = (\phi e^{-\sigma}) g_{ab}$, using (9) and $\phi u_a = \sigma_{,a}$.

Therefore by Chen’s Theorem A the spacetime is GRW spacetime. This completes the proof.

Proof of Theorem 1.2:

In an n -dimensional Lorentzian manifold L_n the $(1, 3)$ conformal curvature tensor is defined by

$$C_{bcd}^a = R_{bcd}^a - \frac{1}{n-2}[\delta_d^a R_{bc} - \delta_c^a R_{bd} + R_d^a g_{bc} - R_c^a g_{bd}] + \frac{R}{(n-1)(n-2)}[\delta_d^a g_{bc} - \delta_c^a g_{bd}], \tag{12}$$

where R_d^a is the $(1, 1)$ Ricci tensor and R denotes the scalar curvature.

Transvecting (12) by u_a and using (10) it follows

$$u_a C_{bcd}^a = \phi_d(g_{bc} + u_b u_c) - \phi_c(g_{bd} + u_b u_d) + \phi^2(u_c g_{bd} - u_d g_{bc}) - \frac{1}{(n-2)}[u_d R_{bc} - u_c R_{bd}] + \frac{R}{(n-1)(n-2)}[u_d g_{bc} - u_c g_{bd}], \tag{13}$$

since $u^a R_{ab} = 0$ in a $(PRS)_n$.

Again transvecting with u^d and using $u^a R_{ab} = 0$ we infer

$$R_{bc} = [\frac{R}{(n-1)} - (n-2)(\psi + \phi^2)]g_{bc} + [\frac{R}{(n-1)} - (n-2)(\psi + \phi^2)]u_b u_c - C_{abcd} u^a u^d, \tag{14}$$

where $\psi = \phi_a u^a$.

The above equation implies that a pseudo Ricci symmetric twisted spacetime is a perfect fluid spacetime of the form

$$R_{bc} = \alpha g_{bc} + \beta u_b u_c, \text{ if and only if } C_{abcd} u^a u^d = 0,$$

$\alpha = [\frac{R}{(n-1)} - (n-2)(\psi + \phi^2)]$ and $\beta = [\frac{R}{(n-1)} - (n-2)(\psi + \phi^2)]$. This finishes the proof.

From (8) we have

$$(u_a C_{bcde} + u_b C_{cade} + u_c C_{abde})u^e = 0.$$

Now transvecting with u^b and using $C_{abcd}u^a u^d = 0$ we obtain $C_{cade}u^e = 0$. But it is known[23]

$$C^a_{bcd,a} = 0 \Leftrightarrow u_a C^a_{bcd} = 0.$$

Therefore $C_{abcd}u^a u^d = 0$ implies $C^a_{bcd,a} = 0$, that is, the conformal curvature tensor is divergence free. Thus we can state the following Corollary:

Corollary 3.1. *A (PRS)_n twisted spacetime is perfect fluid if the conformal curvature tensor is divergence free.*

Remark. In a 4–dimensional pseudo Ricci symmetric twisted spacetime, $C_{abde}u^e = 0$ which implies either $C_{abde} = 0$ or the vector field u^e is a null vector. Since u^e is time-like, therefore the spacetime is conformally flat. Thus such a spacetime is of Petrov classification O[25].

Proof of Theorem 1.3:

A pseudo Ricci symmetric spacetime with $divC = 0$ is perfect fluid. For a twisted spacetime we have

$$u_{i,j} = \phi(g_{ij} + u_i u_j). \tag{15}$$

It is known[6] in a pseudo Ricci symmetric spacetime if the scalar curvature $R \neq 0$, then u_a is irrotational. The Roy Choudhury equation[26] for the fluid in Lorentzian manifold L_n can be written as

$$v_{a,b} = \omega_{ab} + \tau_{ab} + \phi(g_{ab} + u_a u_b), \tag{16}$$

where ω_{ab} and τ_{ab} are respectively vorticity tensor and shear tensor.

From (15) and (16) we get

$$\omega_{ab} + \tau_{ab} = 0. \tag{17}$$

Since u_a is irrotational, hence the vorticity of the fluid vanishes. Therefore $\omega_{ab} = 0$ and hence from (17) we have $\tau_{ab} = 0$. Hence the Theorem 1.3.

Proof of Theorem 1.4:

Consider a perfect fluid spacetime with energy momentum tensor

$$T_{ab} = (p + \mu)g_{ab} + \mu u_a u_b, \tag{18}$$

where u_a is the velocity vector field, p is the isotropic pressure and μ is the energy density. Einstein’s field equations are given by

$$R_{ab} - \frac{R}{2}g_{ab} = \kappa T_{ab}. \tag{19}$$

Multiplying (18) and (19) by g^{ab} we have

$$T = -(p + \mu) + np = (n - 1)p - \mu, \tag{20}$$

since $g^{ab}u_a u_b = -1$ and

$$\frac{2 - n}{2}R = \kappa T, \tag{21}$$

where $T = g^{ab}T_{ab}$.

Equations (20) and (21) together implies that

$$\frac{R}{2} = \frac{\kappa(n - 1)}{(2 - n)}p - \frac{\kappa\mu}{(2 - n)}. \tag{22}$$

From (18) and (19) we obtain

$$R_{ab} - \frac{R}{2}g_{ab} = \kappa(p + \mu)u_a u_b + \kappa p g_{ab}. \tag{23}$$

Now using (22) in (23) reveals that

$$R_{ab} = \kappa(p + \mu)u_a u_b + \kappa \frac{p - \mu}{2 - n} g_{ab}.$$

Comparison with the form (14) satisfying the condition $C_{abcd}u^a u^d = 0$, we get

$$\kappa(p + \mu) = \frac{R}{n - 1} - (n - 2)(f + \phi^2)$$

and

$$\kappa \frac{p - \mu}{2 - n} = \frac{R}{n - 1} - (n - 2)(f + \phi^2)$$

Hence $\kappa(p + \mu) = \kappa \frac{p - \mu}{2 - n}$, which implies $\mu + p = 0$.

Now $\mu + p = 0$ causes the fluid motions as a cosmological constant[30]. This is also termed as phantom barrier[7]. In cosmology, we know such a choice $\mu = -p$ leads to rapid expansion of the spacetime which is now termed as inflation[2]. This completes the proof.

Proof of Theorem 1.5:

We now consider conformally flat $(PRS)_n$ spacetime. Chaki[6] obtained an expression of the curvature tensor in a conformally flat $(PRS)_n$, which is given by

$$R_{abcd} = \frac{R}{(n - 1)(n - 2)} [g_{bc}g_{ad} - g_{ac}g_{bd}] - u_a u_c g_{bd} + u_b u_c g_{ad} - u_b u_d g_{ac} + u_a u_d g_{bc}. \tag{24}$$

Multiplying (24) by g^{bc} gives

$$R_{ad} = \left(\frac{R}{n - 2} - 1\right)g_{ad} + (n - 2)u_a u_d. \tag{25}$$

Hence a conformally flat $(PRS)_n$ is a perfect fluid. Since a conformally flat manifold implies the conformal curvature tensor is divergence free, hence by the Theorem D of Mantica et. al[19] the proof of the theorem follows.

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