Filomat 35:14 (2021), 4811–4824
https://doi.org/10.2298/FIL2114811P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Kontorovich-Lebedev-Clifford Transform

Akhilesh Prasad[a], U. K. Mandal[b]

[a]*Department of Mathematics & Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand-826004, India*
[b]*Department of Mathematics, Nalanda College, Biharsharif, Patliputra University, Patna-803101, India*

Abstract. The Kontorovich-Lebedev-Clifford transform (KLC-transform) as well as its inversion are defined. The useful preliminary results like translation and convolution operators are introduced and their estimates are obtained. Continuity of translation and convolution operators on function spaces $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are discussed. Further, pseudo-differential operator (p.d.o.) associated with the KLC-transform is defined and studied its properties on some function spaces.

1. Introduction

The Kontorovich-Lebedev transform (KL-transform) is basically defined in terms of modified Bessel function of second kind (Macdonald function) of purely imaginary index given as [3, 10]

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh t}\cos(\tau t)\,dt,\ x > 0,\ \tau > 0, \tag{1}$$

and generally it is used to solve certain boundary-value problems of mathematical physics in cylindrical co-ordinate systems. The function $K_{i\tau}(x)$ satisfies the differential equation

$$(x^2 D_x^2 + xD_x - (x^2 - \tau^2))y = 0.$$

The KL-transform was first introduced in 1938 by M. I. Kontorovich and N. N. Lebedev [8, 9], and its theory has since been developed and continued by various authors like Yakubovich [11, 41, 42], Srivastava et. al. [33, 34], Prasad and Mandal [22, 23] and many more [1, 5, 15, 16, 30, 40]. Analogue theories and investigations for different types of KL-transforms, other integral transforms, pseudo-differential operators as well as wavelet transforms may also be viewed in [15, 17, 21, 23, 31, 32, 37, 39].

The Hankel transform was first introduced by a German mathematician H. Hankel by using the Bessel function of first kind $J_\nu(x)$ of order ν and then studied by many authors [14, 19, 38, 43]. After that an English mathematician W. K. Clifford, slightly modified the Bessel function by replacing x by $2\sqrt{x}$ and obtained a new function named as Bessel-Clifford function

$$C_\nu(x) = x^{-\nu/2} J_\nu(2\sqrt{x}),$$

2020 *Mathematics Subject Classification*. Primary 44A20; 35S05.
Keywords. Kontorovich-Lebedev transform; Pseudo-differential operator; Convolution.
Received: 24 November 2020; Revised: 26 June 2021; Accepted: 29 August 2021
Communicated by Hari M. Srivastava
Email addresses: apr_bhu@yahoo.com (Akhilesh Prasad), upainmandal@gmail.com (U. K. Mandal)

which is a solution of the differential equation

$$(xD_x^2 + (\nu+1)D_x + 1)y = 0.$$

It leads to another integral transform and known as the Hankel-Clifford transform, for instance see [2, 13, 20]. Subsequently, corresponding to the Mehler-Fock transform [4, 12, 35, 36], Prasad and Verma [25], constructed the Mehler-Clifford integral transform and studied various theories related to it.
In similar spirit, in this paper we have considered the Macdonald function as

$$K_{2i\sqrt{\tau}}(2\sqrt{x}) = \int_0^\infty e^{-2\sqrt{x}\cosh t}\cos(2\sqrt{\tau}t)\,dt,\ x>0,\ \tau>0, \tag{2}$$

which is a solution of the differential equation

$$(x^2D_x^2 + xD_x - (x-\tau))y = 0.$$

The asymptotic behavior of the Macdonald function $K_\nu(2\sqrt{x})$ with respect to x [11]

$$\begin{aligned} K_\nu(2\sqrt{x}) &\approx \frac{\sqrt{\pi}}{2x^{\frac{1}{4}}}e^{-2\sqrt{x}}\Big[1+O\Big(\frac{1}{\sqrt{x}}\Big)\Big],\ x\to\infty,\\ K_\nu(2\sqrt{x}) &\approx O(x^{-\Re(\nu)/2}),\ x\to 0,\\ K_0(2\sqrt{x}) &\approx O(\log x),\ x\to 0. \end{aligned}$$

Moreover, $K_{2i\sqrt{\tau}}(2\sqrt{x})$ is an eigenfunction of the operator

$$\mathcal{A}_x = x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - x, \tag{3}$$

and

$$\mathcal{A}_x K_{2i\sqrt{\tau}}(2\sqrt{x}) = -\tau K_{2i\sqrt{\tau}}(2\sqrt{x}). \tag{4}$$

Also, we can write

$$\mathcal{A}'_x = \frac{d^2}{dx^2}x^2 - \frac{d}{dx}x - x, \tag{5}$$

which represents the adjoint operator of $\mathcal{A}_x$. The series representation of $\mathcal{A}_x^q$ can be written as:

$$\mathcal{A}_x^q = \sum_{j=0}^{2q} x^j P_j^q(x) D_x^j,\ \forall\, q\in\mathbb{N}_0, \tag{6}$$

where the P_j^q are polynomials of degree $q-\frac{j}{2}$ for even j and $q-\frac{(j+1)}{2}$ for odd j respectively.
Thus $P_{2q}^q(x) = 1$ and $P_{2q-1}^q(x) = q(2q-1)$.
Now, we define here an integral transform by using the Macdonald function $K_{2i\sqrt{\tau}}(2\sqrt{x})$ on the positive half line $\mathbb{R}_+ = (0,\infty)$, provided the integral exists, as:

$$(\mathbb{K}\varphi)(\tau) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\varphi(x)x^{-1}\,dx,\ \tau\in\mathbb{R}_+, \tag{7}$$

and we named it as the Kontorovich-Lebedev-Clifford transform (KLC-transfrom). The inversion formula of (7), is given by

$$\varphi(x) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\sinh(2\pi\sqrt{\tau})(\mathbb{K}\varphi)(\tau)\,d\tau,\ x\in\mathbb{R}_+. \tag{8}$$

Besides, we define adjoint of the KLC-transform for any function ψ as:

$$(\mathbb{K}'\psi)(x) = \frac{1}{2x}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\psi(\tau)\,d\tau,\ x \in \mathbb{R}_+, \tag{9}$$

and corresponding inversion formula as:

$$\psi(\tau) = \frac{4}{\pi^2}\sinh(2\pi\sqrt{\tau})\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})(\mathbb{K}'\psi)(x)\,dx,\ \tau \in \mathbb{R}_+. \tag{10}$$

Next, we define the translation and convolution structure for the KLC-transform. From [3], we have

$$\frac{2}{\pi^2}\int_0^\infty K_{i\eta}(p)K_{i\eta}(q)K_{i\eta}(r)\eta\sinh(\pi\eta)\,d\eta = T(p,q,r), \tag{11}$$

where $T(p,q,r)$ is symmetric in $p,\ q,\ r$ and defined as:

$$T(p,q,r) = \frac{1}{2}\exp\Big[\frac{-(p^2q^2+q^2r^2+r^2p^2)}{2pqr}\Big],\ p,q,r \in \mathbb{R}_+. \tag{12}$$

Now, putting $p = 2\sqrt{x},\ q = 2\sqrt{y},\ r = 2\sqrt{z}$ and $\eta = 2\sqrt{\tau}$ in (11) and (12), we get

$$\frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\tau}}(2\sqrt{y})K_{2i\sqrt{\tau}}(2\sqrt{z})\sinh(2\pi\sqrt{\tau})\,d\tau = D(x,y,z), \tag{13}$$

where

$$D(x,y,z) = \frac{1}{2}\exp\Big[\frac{-(xy+yz+zx)}{\sqrt{xyz}}\Big],\ x,y,z \in \mathbb{R}_+. \tag{14}$$

Clearly from (14), $D(x,y,z)$ is symmetric in x,y,z. Since

$$\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{z}} + \sqrt{\frac{zx}{y}} \ \geq\ \sqrt{x}\Big(\sqrt{\frac{y}{z}} + \sqrt{\frac{z}{y}}\Big) \ \geq\ 2\sqrt{x}.$$

Thus

$$\exp\bigg(-\Big(\sqrt{\frac{xy}{z}} + \sqrt{\frac{yz}{z}} + \sqrt{\frac{zx}{y}}\Big)\bigg) \ \leq\ \exp(-2\sqrt{x}). \tag{15}$$

From (14) and (15)

$$|D(x,y,z)| \ \leq\ \exp(-2\sqrt{x}). \tag{16}$$

By using (7), (8) and (13), the product of Macdonald functions can be written as:

$$K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\tau}}(2\sqrt{y}) \ =\ \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{z})D(x,y,z)\,z^{-1}\,dz \ =\ \Big(\mathbb{K}\,D(x,y,z)\Big)(\tau). \tag{17}$$

The translation operator $\mathfrak{T}_x$ of function $\varphi(x)$ is defined by

$$(\mathfrak{T}_x\varphi)(y) = \frac{1}{2}\int_0^\infty D(x,y,z)\varphi(z)z^{-1}\,dz, \tag{18}$$

and the corresponding convolution operator is defined as:

$$(\varphi \sharp \psi)(x) = \frac{1}{2}\int_0^\infty \mathfrak{T}_x\varphi(y)\,\psi(y)\,y^{-1}\,dy$$
$$= \frac{1}{4}\int_0^\infty\int_0^\infty D(x,y,z)\varphi(z)\psi(y)z^{-1}y^{-1}\,dz\,dy. \quad (19)$$

The Plancherel relation for the KLC-transform can be written as:

$$\frac{1}{2}\int_0^\infty \varphi(x)\,\overline{\psi(x)}\,x^{-1}\,dx = \frac{4}{\pi^2}\int_0^\infty (\mathbb{K}\varphi)(\tau)\overline{(\mathbb{K}\psi)(\tau)}\,\sinh(2\pi\sqrt{\tau})\,d\tau,$$

and the Parseval formula is given by

$$\frac{1}{2}\int_0^\infty |\varphi(x)|^2\,x^{-1}dx = \frac{4}{\pi^2}\int_0^\infty |(\mathbb{K}\varphi)(\tau)|^2\,\sinh(2\pi\sqrt{\tau})\,d\tau.$$

The paper consists of four Sections, in first Section we introduced KLC-transform and its inversion and adjoint then translation, convolution structure associated to KLC-transform are also defined. Section 2 deals with some useful results and operational formulas are discussed and estimates for the translation and convolution operators in Lebesgue space are obtained. In Section 3, we studied the continuity of the translation and convolution operators in function spaces $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$. Section 4 is devoted to the study of pseudo-differential operator (p.d.o.) associated with the KLC-transform, where symbols are in S^m. Furthermore continuity of p.d.o. is discussed from function space $\mathcal{G}_\alpha$ into $\mathcal{F}_\alpha$. Also an integral representation of p.d.o. is given and its estimate in Lebesgue space is obtained. Finally, a special case for p.d.o. is discussed.

2. Preliminary results and estimates for convolution operators

In this section, we discuss some relevant results which will be useful in upcoming Sections.
From (3) and (5), we have a relation between the differential operators $\mathcal{A}_x$ and $\mathcal{A}'_x$ as:

$$\mathcal{A}'_x(x^{-1}\varphi(x)) = x^{-1}\mathcal{A}_x\varphi(x). \quad (20)$$

Further (20) can be extended to n times, where $n \in \mathbb{N}_0$, that is

$$(\mathcal{A}'_x)^n(x^{-1}\varphi(x)) = x^{-1}\mathcal{A}_x^n\varphi(x). \quad (21)$$

Lemma 2.1. *If φ and ψ be any real valued functions defined on $\mathbb{R}_+$, then*

$$(i)\quad (\mathbb{K}(\mathcal{A}_{(\cdot)}\,\varphi))(\tau) = -\tau(\mathbb{K}\varphi)(\tau), \quad (22)$$
$$(ii)\quad (\mathbb{K}'^{-1}(\mathcal{A}'_{(\cdot)}\,\psi))(\tau) = -\tau(\mathbb{K}'^{-1}\psi)(\tau),$$

where $\mathbb{K}$, $\mathbb{K}'$ and $\mathbb{K}'^{-1}$ are defined as (7), (9) *and* (10) *respectively.*

Proof. By using (20) and (4), we can obtain the required results. □

Lemma 2.2. *If the translation and the convolution operator defined as* (18) *and* (19) *respectively, then we have*

$$(i)\quad (\mathbb{K}(\mathfrak{T}_x\varphi))\,(\tau) = K_{2i\sqrt{\tau}}(2\sqrt{x})\,(\mathbb{K}\varphi)\,(\tau), \quad (23)$$
$$(ii)\quad (\mathbb{K}(\varphi \sharp \psi))\,(\tau) = (\mathbb{K}\varphi)(\tau)(\mathbb{K}\psi)(\tau). \quad (24)$$

Proof. (*i*) By using (7), (18) and the Fubini's theorem, we get

$$(\mathbb{K}(\mathfrak{T}_x\varphi))(\tau) = \frac{1}{2}\int_0^\infty \Big(\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{y})D(x,z,y)y^{-1}dy\Big)\varphi(z)z^{-1}dz.$$

Thus by using (17) and then (7), we get

$$(\mathbb{K}(\mathfrak{T}_x\varphi))(\tau) = K_{2i\sqrt{\tau}}(2\sqrt{x})(\mathbb{K}\varphi)(\tau).$$

(*ii*) By using (7), (19) and Fubini's theorem, we have

$$\begin{aligned}(\mathbb{K}(\varphi \sharp \psi))(\tau) &= \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\Big(\frac{1}{2}\int_0^\infty \mathfrak{T}_x\varphi(y)\psi(y)y^{-1}dy\Big)x^{-1}\,dx\\ &= \frac{1}{2}\int_0^\infty \Big(\frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\mathfrak{T}_y\varphi(x)x^{-1}\,dx\Big)\psi(y)y^{-1}dy\\ &= \frac{1}{2}\int_0^\infty (\mathbb{K}\mathfrak{T}_y\varphi)(\tau)\psi(y)y^{-1}dy.\end{aligned}$$

Thus by using (23), we get

$$(\mathbb{K}(\varphi \sharp \psi))(\tau) = (\mathbb{K}\varphi)(\tau)\,\frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{y})\psi(y)y^{-1}dy = (\mathbb{K}\varphi)(\tau)(\mathbb{K}\psi)(\tau).$$

Hence proof is complete. □

We now discuss some operational formulas associated with the differential operator, translation and convolution operator and function $D(x,y,z)$ defined by (13) and (14).
(*i*) By using (4) and (13), we have quite obvious result between the differential operator (3) and $D(x,y,z)$ defined by (13)

$$\mathcal{A}_x^n D(x,y,z) = \mathcal{A}_y^n D(x,y,z) = \mathcal{A}_z^n D(x,y,z), \tag{25}$$

where $n \in \mathbb{N}_0$.
(*ii*) If the differential operator $\mathcal{A}_x$ and the convolution operator is defined as (3) and (19) respectively, then we have

$$\mathcal{A}_x(\varphi \sharp \psi) = \mathcal{A}_x\varphi \sharp \psi = \varphi \sharp \mathcal{A}_x\psi. \tag{26}$$

Proof. By using (7), (22) and (24), we get

$$(\mathbb{K}\mathcal{A}_x(\varphi \sharp \psi))(\tau) = -\tau(\mathbb{K}\varphi)(\tau)(\mathbb{K}\psi)(\tau) = (\mathbb{K}\mathcal{A}_x\varphi)(\tau)(\mathbb{K}\psi)(\tau) = (\mathbb{K}\,((\mathcal{A}_x\varphi) \sharp \psi))(\tau).$$

Similarly

$$\mathcal{A}_x(\varphi \sharp \psi) = \mathcal{A}_x\varphi \sharp \psi,\ \mathcal{A}_x(\varphi \sharp \psi) = \varphi \sharp \mathcal{A}_x\psi.$$

Hence the proof is complete. □

(*iii*) If the translation and the convolution operator is defined as (18) and (19) respectively, then

$$\mathfrak{T}_x(\varphi \sharp \psi) = \mathfrak{T}_x\varphi \sharp \psi = \varphi \sharp \mathfrak{T}_x\psi. \tag{27}$$

Proof. By using (23) and (24) and proceeding as the proof of (26), (27) can be easily obtained. □

Next, we find out some estimates and preliminary results that are quite interesting and useful for our purpose in this paper.

$$(i) \quad K_0(2\sqrt{x+y}) \ \le \ K_0(2\sqrt{y}) \ \text{ or } \ K_0(2\sqrt{x}), \tag{28}$$

$$(ii) \quad \int_0^\infty D(x,y,z)z^{-1}dz \ = \ 2\,K_0(2\sqrt{x+y}), \tag{29}$$

$$(iii) \quad |\,K_{2i\sqrt{\tau}}(2\sqrt{x})\,| \ \le \ C'(b)\,x^{\frac{-b}{4}}[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}},\ 0<b<1/2, \tag{30}$$

$$(iv) \quad |\mathcal{A}_x^n D(x,y,z)| \ \le \ C\,(xyz)^{\frac{-b}{4}},\ 0<b<1/2,\ n\in\mathbb{N}_0, \tag{31}$$

$$(v) \quad |D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x})| \le C'x^{-r},\ \forall\ r\in\mathbb{N}, \tag{32}$$

where $C'(b)$ and C are some positive constants.

Proof. (i) From [42, p. 212], we have an integral representation for Macdonald function of index zero as

$$K_0(x) = \frac{1}{2}\int_0^\infty e^{-t-\frac{x^2}{4t}}\ t^{-1}dt. \tag{33}$$

Hence from (33), the inequality is obvious.

(ii) From [26, p. 344], we have

$$\int_0^\infty z^{\alpha-1}e^{-pz-\frac{q}{z}}dz = 2\Big(\frac{q}{p}\Big)^{\frac{\alpha}{2}}K_\alpha(2\sqrt{pq}), \quad \mathrm{Re}(p),\ \mathrm{Re}(q)>0,\ \alpha\in\mathbb{R}. \tag{34}$$

Thus by putting $\alpha = 0,\ p = \frac{1}{2}\frac{x+y}{\sqrt{xy}}$ and $q = 2\sqrt{xy}$ in (34) and keeping in view (14), we get the required result.

(iii) To prove the inequality (30), we use another representation of Macdonald function as in [3, p. 97(69)]

$$[K_{2i\sqrt{\tau}}(2\sqrt{x})]^2\sinh(2\pi\sqrt{\tau}) \ = \ \pi\int_0^\infty J_0(4\sqrt{x}\sinh(t))\sin(4\sqrt{\tau}t)dt. \tag{35}$$

Also from [15, p. 32], we have

$$|J_0(x)|x^b \ \le \ C(b),\ \ 0<b<1/2,\ \forall\ x\in\mathbb{R}_+. \tag{36}$$

Thus from (35) and (36), we get

$$\begin{aligned} |[K_{2i\sqrt{\tau}}(2\sqrt{x})]^2\sinh(2\pi\sqrt{\tau})| \ &\le \ C(b)\pi\int_0^\infty[4\sqrt{x}\sinh(t)]^{-b}dt \ \le \ C(b)\pi(4\sqrt{x})^{-b}\int_0^\infty[\sinh(t)]^{-b}dt \\ &\le \ C(b)\pi(4\sqrt{x})^{-b}\int_0^\infty\frac{u^{-b}}{(1+u^2)^{\frac{1}{2}}}du, \end{aligned}$$

for $0<b<1/2$, the integral is convergent. Thus

$$|\,K_{2i\sqrt{\tau}}(2\sqrt{x})\,| \ \ \le \ \ C'(b)\,x^{\frac{-b}{4}}[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}.$$

(iv) By using (13), (4) and then (30), we get

$$|\mathcal{A}_x^n D(x,y,z)| \;\leq\; \frac{4}{\pi^2}[C'(b)]^3(xyz)^{\frac{-b}{4}}\int_0^\infty \tau^n[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}d\tau \;\leq\; C\,(xyz)^{\frac{-b}{4}}.$$

Hence the estimate (31) is true.
(v) By using (2), we have

$$D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x}) \;=\; \sum_{j=1}^{r} C_{j,r}(-1)^n x^{\frac{j-2r}{2}}\int_0^\infty (\cosh(t))^j e^{-2\sqrt{x}\cosh(t)}\cos(2\sqrt{\tau}t)dt,$$

where $C_{j,r}>0$ is some constant. Thus

$$\begin{aligned} |D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x})| \;&\leq\; \sum_{j=1}^{r} C_{j,r}\,x^{\frac{j-2r}{2}}\int_0^\infty e^{tj}\,e^{-\sqrt{x}e^t}dt \;=\; \sum_{j=1}^{r} C_{j,r}\,x^{\frac{j-2r}{2}}\int_0^\infty u^{j-1}\,e^{-\sqrt{x}u}du \\ &=\; \sum_{j=1}^{r} C_{j,r}\,x^{\frac{j-2r}{2}}\,\frac{(j-1)!}{x^{\frac{j}{2}}} \;\leq\; C'\,x^{-r}, \end{aligned}$$

where $C'>0$ is some constant. □

Remark 2.3. *From* (25) *and* (31), *we can conclude that*

$$\mathcal{A}_x^n D(x,y,z)=\mathcal{A}_y^n D(x,y,z)=\mathcal{A}_z^n D(x,y,z) \quad\leq\quad C\,(xyz)^{\frac{-b}{4}}.$$

Theorem 2.4. *If $\varphi,\ \psi\in L^1(\mathbb{R}_+;x^{-1}dx)$ then the translation and the convolution operator defined as* (18) *and* (19) *respectively, exist and belong to $L^1(\mathbb{R}_+;dx)$. Further, we have the following estimates*

$$(i)\quad \|\mathfrak{T}_x\varphi\|_{L^1(\mathbb{R}_+;\,dx)}\leq\frac{1}{4}\,\|\varphi\|_{L^1(\mathbb{R}_+;\,x^{-1}dx)},$$

$$(ii)\quad \|\varphi\,\sharp\,\psi\|_{L^1(\mathbb{R}_+;\,dx)}\leq\;\frac{1}{8}\,\|\varphi\|_{L^1(\mathbb{R}_+;\,x^{-1}dx)}\,\|\psi\|_{L^1(\mathbb{R}_+;x^{-1}dx)}.$$

Proof. (i) From (18) and (16), we have

$$|\,\mathfrak{T}_x\varphi(y)\,|\leq\frac{1}{2}e^{-2\sqrt{y}}\|\varphi\|_{L^1(\mathbb{R}_+;\,x^{-1}dx)}.$$

Thus

$$\|\mathfrak{T}_x\varphi\|_{L^1(\mathbb{R}_+;\,dx)}\;\leq\;\frac{1}{4}\|\varphi\|_{L^1(\mathbb{R}_+;\,x^{-1}dx)}.$$

(ii) Now from (19) and (16), we get

$$|\,(\varphi\,\sharp\,\psi)(x)\,|\quad\leq\quad\frac{1}{4}e^{-2\sqrt{x}}\Big(\int_0^\infty\int_0^\infty|\varphi(z)|\,|\psi(y)|z^{-1}y^{-1}\,dz\,dy\Big).$$

Hence

$$\|\varphi\,\sharp\,\psi\|_{L^1(\mathbb{R}_+;\,dx)}\leq\;\frac{1}{8}\,\|\varphi\|_{L^1(\mathbb{R}_+;\,x^{-1}dx)}\,\|\psi\|_{L^1(\mathbb{R}_+;x^{-1}dx)}.$$

Hence proof is complete. □

Theorem 2.5. *If $\varphi \in L^p(\mathbb{R}_+; x^{-1}dx)$, $1 < p < \infty$ and $\psi \in L^1(\mathbb{R}_+; x^{-1}dx)$, then the convolution operator* (19) *exists and belongs to $L^p(\mathbb{R}_+;\ dx)$. Moreover*

$$\|\varphi \sharp \psi\|_{L^p(\mathbb{R}_+;dx)} \leq C\,\|\varphi\|_{L^p(\mathbb{R}_+;\ x^{-1}dx)}\,\|\psi\|_{L^1(\mathbb{R}_+;\ x^{-1}dx)}, \tag{37}$$

where $C > 0$ is certain constant.

Proof. By using (19) and Hölder's inequality, we have

$$\begin{aligned} |(\varphi \sharp \psi)(x)| &\leq \frac{1}{4}\Big(\int_0^\infty\int_0^\infty D(x,y,z)\,|\varphi(z)|^p\,|\psi(y)|\,y^{-1}z^{-1}\,dy\,dz\Big)^{\frac{1}{p}} \\ &\times \Big(\int_0^\infty\int_0^\infty D(x,y,z)\,|\psi(y)|\,y^{-1}z^{-1}\,dy\,dz\Big)^{\frac{1}{q}}. \end{aligned}$$

Thus from (16), we get

$$\begin{aligned} |(\varphi \sharp \psi)(x)|^p &\leq \Big(\frac{1}{4}\Big)^p e^{-2\sqrt{x}}\Big(\int_0^\infty\int_0^\infty |\varphi(z)|^p\,|\psi(y)|\,y^{-1}z^{-1}\,dy\,dz\Big) \\ &\times \Big(\int_0^\infty\Big(\int_0^\infty D(x,y,z)z^{-1}\,dz\Big)|\psi(y)|\,y^{-1}\,dy\Big)^{\frac{p}{q}}. \end{aligned}$$

From (28) and (29), we have

$$\begin{aligned} \|\varphi \sharp \psi\|_{L^p(\mathbb{R}_+;dx)} &\leq \frac{1}{4}\Big(\int_0^\infty e^{-2\sqrt{x}}2^{\frac{p}{q}}K_0^{\frac{p}{q}}(2\sqrt{x})dx\Big)^{\frac{1}{p}}\|\varphi\|_{L^p(\mathbb{R}_+;\frac{dx}{x})}\|\psi\|_{L^1(\mathbb{R}_+;\frac{dx}{x})} \\ &\leq C\,\|\varphi\|_{L^p(\mathbb{R}_+;x^{-1}dx)}\|\psi\|_{L^1(\mathbb{R}_+;x^{-1}dx)}, \end{aligned}$$

where $C > 0$ is certain constant. Hence the proof is complete. □

3. Function spaces $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ and continuity of translation and convolution operator

In this Section we consider the function spaces $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ analogous to [6, 7] and discuss the continuity of translation and convolution operators.

Definition 3.1. *The space $\mathcal{F}_\alpha$ consists of all complex-valued infinitely differentiable functions φ defined on $\mathbb{R}_+$ with the topology generated by the collection of seminorms*

$$\gamma_{n,\alpha}(\varphi) = \sup_{x\in\mathbb{R}_+} |\lambda_\alpha^-(x)x^{-1}\mathcal{A}_x^n\varphi(x)| < \infty, \tag{38}$$

where $\alpha > 0$, $n \in \mathbb{N}_0$, $\mathcal{A}_x$ is the differential operator as (3) *and $\lambda_\alpha^-(x)$ denotes the continuous function*

$$\lambda_\alpha^-(x) = \begin{cases} e^{\frac{-\alpha}{x}} & \text{if } 0 < x \leq 1, \\ e^{-\alpha x} & \text{if } 1 \leq x < \infty. \end{cases} \tag{39}$$

Remark 3.2. *The differential operator $\mathcal{A}_x$ is continuous linear mapping from the space $\mathcal{F}_\alpha$ into itself.*

Definition 3.3. *The space $\mathcal{G}_\alpha$ consists of all complex-valued infinitely differentiable functions φ defined on $\mathbb{R}_+$ with the topology generated by the collection of seminorms*

$$\Gamma_{n,\alpha}(\varphi) = \sup_{x\in\mathbb{R}_+} |\lambda_\alpha^+(x) x^{-1} \mathcal{A}_x^n \varphi(x)| < \infty, \tag{40}$$

where $\alpha > 0$, $n \in \mathbb{N}_0$, $\mathcal{A}_x$ is the differential operator (3) *and $\lambda_\alpha^+(x)$ denotes the continuous function*

$$\lambda_\alpha^+(x) = \begin{cases} e^{\frac{\alpha}{x}} & \text{if } 0 < x \le 1, \\ e^{\alpha x} & \text{if } 1 \le x < \infty. \end{cases} \tag{41}$$

Remark 3.4. *The differential operator $\mathcal{A}_x$ is continuous linear mapping from the space $\mathcal{G}_\alpha$ into itself.*

Remark 3.5. *The differential operator $\mathcal{A}_x$ is continuous linear mapping from the space $\mathcal{G}_\alpha$ into the space $\mathcal{F}_\alpha$.*

Theorem 3.6. *The translation operator defined as* (18)*, is a continuous linear mapping from $\mathcal{G}_\alpha$ into $\mathcal{F}_\alpha$.*

Proof. Let $\varphi \in \mathcal{G}_\alpha$ then from (18), (25) and (21), we have

$$\mathcal{A}_y^n(\mathfrak{T}_x\varphi(y)) = \frac{1}{2}\int_0^\infty (\mathcal{A}_z')^n (z^{-1}D(x,y,z))\, \varphi(z)\, dz = \frac{1}{2}\int_0^\infty z^{-1}D(x,y,z)\, \mathcal{A}_z^n\varphi(z)\, dz.$$

Now by using (38), (39) and (16), we get

$$\gamma_{n,\alpha}(\mathfrak{T}_x\varphi(y)) \le \Gamma_{n,\alpha}(\varphi) \sup_{y\in\mathbb{R}_+} |\lambda_\alpha^-(y) y^{-1}| \int_0^\infty e^{-2\sqrt{z}} [\lambda_\alpha^+(z)]^{-1} dz,$$

from (41), it is clear that the integral is convergent. Thus

$$\gamma_{n,\alpha}(\mathfrak{T}_x\varphi(y)) \le C\, \Gamma_{n,\alpha}(\varphi),$$

where $C > 0$ is certain constant. Hence our proof is complete. □

Theorem 3.7. *The convolution operator defined as* (19)*, is a continuous linear mapping from $\mathcal{G}_\alpha$ into $\mathcal{F}_\alpha$.*

Proof. Let $\varphi,\ \psi \in \mathcal{G}_\alpha$ then by using (19), we have

$$\mathcal{A}_x^n(\varphi \sharp \psi)(x) = \frac{1}{2}\int_0^\infty \mathcal{A}_x^n \mathfrak{T}_y \varphi(x)\, \psi(y)\, y^{-1}\, dy.$$

Now by using (38), (40) and Theorem 3.6, we get

$$\gamma_{n,\alpha}(\varphi \sharp \psi) \le \frac{1}{2}\gamma_{n,\alpha}(\mathfrak{T}_y\varphi(x))\, \Gamma_{0,\alpha}(\psi) \int_0^\infty [\lambda_\alpha^+(y)]^{-1} dy,$$

from (41), it is clear that the integral is convergent. Thus

$$\gamma_{n,\alpha}(\varphi \sharp \psi) \le C\, \gamma_{n,\alpha}(\mathfrak{T}_y\varphi(x))\, \Gamma_{0,\alpha}(\psi),$$

where $C > 0$ is some constant. Thus the proof is complete. □

4. Pseudo-differential operator

Definition 4.1. *Let us consider the the symbol class* S^m *as collection of functions* $a(x,\tau) : C^\infty(\mathbb{R}^2_+) \to \mathbb{C}$ *which is exponentially decreasing in variable* τ *and its derivative satisfies*

$$(1+x)^{-l} \mid D_x^p D_\tau^q a(x,\tau) \mid \leq C\, e^{-m\tau}, \tag{42}$$

for all $l, p, q \in \mathbb{N}_0,\ m > 0$ *and constant* $C = C_{l,m,p,q} > 0$.

Definition 4.2. *Let the symbol* $a(x,\tau) \in C^\infty(\mathbb{R}^2_+) \to \mathbb{C}$ *satisfies* (42). *Then the p.d.o. involving KLC-transform is denoted by* $\mathbb{P}_a$ *and defined by*

$$\mathbb{P}_a\varphi(x) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\sinh(2\pi\sqrt{\tau})a(x,\tau)(\mathbb{K}\varphi)(\tau)\,d\tau, \tag{43}$$

where $\mathbb{K}$ *is as* (7).

For p.d.o's involving Kontorovich-Lebedev transform, Fourier Jacobi transform, Fourier transform, Hankel-Clifford transform, fractional Fourier transform, linear cannonical transform we refer to [15, 17, 22–24, 27, 28, 32, 37, 39] respectively.

Theorem 4.3. *Let* $a(x,\tau) \in \mathrm{S}^m$ *then the p.d.o. is continuous linear mapping from* $\mathcal{G}_\alpha$ *into* $\mathcal{F}_\alpha$.

Proof. By using (43) and (6), we have

$$\begin{aligned}\mathcal{A}_x^q\mathbb{P}_a\varphi(x) &= \frac{4}{\pi^2}\sum_{j=0}^{2q} x^j P_j^q(x)\int_0^\infty D_x^j K_{2i\sqrt{\tau}}(2\sqrt{x})a(x,\tau)\sinh(2\pi\sqrt{\tau})(\mathbb{K}\varphi)(\tau)d\tau\\ &= \frac{4}{\pi^2}\sum_{j=0}^{2q} x^j P_j^q(x)\int_0^\infty \sum_{r=0}^{j}\binom{j}{r}D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x})D_x^{j-r}a(x,\tau)\sinh(2\pi\sqrt{\tau})\\ &\times\ (1+\tau)^{-s}\sum_{n=0}^{s}\binom{s}{n}(-1)^n(-\tau)^n(\mathbb{K}\varphi)(\tau)d\tau.\end{aligned} \tag{44}$$

Now by using (4) and (21), we get

$$(-\tau)^n(\mathbb{K}\varphi)(\tau) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{t})t^{-1}\mathcal{A}_t^n\varphi(t)dt.$$

Thus from (30) and (40), we obtain

$$\begin{aligned}\mid(-\tau)^n(\mathbb{K}\varphi)(\tau)\mid &\leq \frac{1}{2}C'(b)[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\int_0^\infty t^{\frac{-b}{4}}t^{-1}\mathcal{A}_t^n\varphi(t)dt\\ &\leq \frac{1}{2}C'(b)[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\Gamma_{n,\alpha}(\varphi)\int_0^\infty \frac{t^{\frac{-b}{4}}}{\lambda_\alpha^+(t)}dt,\end{aligned}$$

where $0 < b < 1/2$. From (41), clearly the integral is convergent, hence

$$\mid(-\tau)^n(\mathbb{K}\varphi)(\tau)\mid \leq C''[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\Gamma_{n,\alpha}(\varphi), \tag{45}$$

where $C'' > 0$ is a constant. Now from (44) and (45), we get

$$\begin{aligned}
|\mathcal{A}_x^q \mathbb{P}_a\varphi(x)| &\leq \frac{4}{\pi^2}\sum_{j=0}^{2q} x^j P_j^q(x)\int_0^\infty \sum_{r=0}^{j}\binom{j}{r}|D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x})|\,|D_x^{j-r}a(x,\tau)|\\
&\times \sinh(2\pi\sqrt{\tau})(1+\tau)^{-s}\sum_{n=0}^{s}\binom{s}{n}C''[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\Gamma_{n,\alpha}(\varphi)d\tau\\
&= \frac{4}{\pi^2}\int_0^\infty |K_{2i\sqrt{\tau}}(2\sqrt{x})|\,|a(x,\tau)|\sinh(2\pi\sqrt{\tau})\\
&\times (1+\tau)^{-s}\sum_{n=0}^{s}\binom{s}{n}C''[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\Gamma_{n,\alpha}(\varphi)d\tau\\
&+ \frac{4}{\pi^2}\sum_{j=1}^{2q} x^j P_j^q(x)\int_0^\infty \sum_{r=1}^{j}\binom{j}{r}|D_x^r K_{2i\sqrt{\tau}}(2\sqrt{x})|\,|D_x^{j-r}a(x,\tau)|\\
&\times \sinh(2\pi\sqrt{\tau})(1+\tau)^{-s}\sum_{n=0}^{s}\binom{s}{n}C''[\sinh(2\pi\sqrt{\tau})]^{\frac{-1}{2}}\Gamma_{n,\alpha}(\varphi)d\tau.
\end{aligned}$$

By using (30), (42) and (32), we get

$$\begin{aligned}
|\mathcal{A}_x^q \mathbb{P}_a\varphi(x)| &\leq \frac{4}{\pi^2}\sum_{n=0}^{s}\binom{s}{n}\Gamma_{n,\alpha}(\varphi)C''C'(b)x^{\frac{-b}{4}}C(1+x)^{-l}\int_0^\infty e^{-m\tau}(1+\tau)^{-s}d\tau\\
&+ \frac{4}{\pi^2}\sum_{j=1}^{2q} x^j P_j^q(x)\sum_{r=1}^{j}\binom{j}{r}\sum_{n=0}^{s}\binom{s}{n}\Gamma_{n,\alpha}(\varphi)C''C'\, x^{-r}\\
&\times C(1+x)^{-l}\int_0^\infty e^{-m\tau}(1+\tau)^{-s}[\sinh(2\pi\sqrt{\tau})]^{\frac{1}{2}}d\tau,
\end{aligned}$$

for large value of $m(>0)$ the integrals are convergent. Thus from (38)

$$\begin{aligned}
\gamma_{q,\alpha}(\mathbb{P}_a\varphi) &\leq C_1\sum_{n=0}^{s}\binom{s}{n}\Gamma_{n,\alpha}(\varphi)\sup_{x\in\mathbb{R}_+}|\lambda_\alpha^-(x)x^{-1}x^{\frac{-b}{4}}(1+x)^{-l}|\\
&+ C_2\sum_{n=0}^{s}\binom{s}{n}\Gamma_{n,\alpha}(\varphi)\sup_{x\in\mathbb{R}_+}|\lambda_\alpha^-(x)x^{-1}(1+x)^{-l}\sum_{j=1}^{2q}x^jP_j^q(x)\sum_{r=1}^{j}\binom{j}{r}x^{-r}|.
\end{aligned}$$

Using (39), we have

$$\gamma_{q,\alpha}(\mathbb{P}_a\varphi) \leq C_3\,\Gamma_{n,\alpha}(\varphi),$$

where $C_3 > 0$ is a constant, from which the theorem follows. □

An integral representation of p.d.o.

Let us consider a function $\mathsf{a}_x(y)$ associated with the symbol $a(x,\tau)$ by

$$\mathsf{a}_x(y) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\tau}}(2\sqrt{y})\,a(x,\tau)\,\sinh(2\pi\sqrt{\tau})\,d\tau, \tag{46}$$

then from (43), (7) and by using Fubini's theorem, we have

$$(\mathbb{P}_a\varphi)(x) = \frac{1}{2}\int_0^\infty \left(\frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})K_{2i\sqrt{\tau}}(2\sqrt{y})a(x,\tau)\sinh(2\pi\sqrt{\tau})\,d\tau\right)\varphi(y)\,y^{-1}\,dy. \tag{47}$$

On using (46), (47) reduces to

$$(\mathbb{P}_a\varphi)(x) = \frac{1}{2}\int_0^\infty \mathsf{a}_x(y)\,\varphi(y)\,y^{-1}\,dy. \tag{48}$$

Theorem 4.4. *If* $\mathsf{a}_x(y)$ *is defined as* (46)*, then we have the following estimate*

$$|\mathsf{a}_x(y)| \leq C\,(xy)^{\frac{-b}{4}}(1+x)^{-l}, \tag{49}$$

where $0 < b < 1/2$, $l \in \mathbb{N}_0$ *and* $C > 0$ *is certain constant.*

Proof. By using (46), (30) and (42), we get

$$|\mathsf{a}_x(y)| \leq \frac{4}{\pi^2}(C'(b))^2(xy)^{\frac{-b}{4}}(1+x)^{-l}\int_0^\infty e^{-m\tau}d\tau,$$

where $0 < b < 1/2$, and $l \in \mathbb{N}_0$. Clearly the integral is convergent as $m > 0$, thus

$$|\mathsf{a}_x(y)| \leq C\,(xy)^{\frac{-b}{4}}(1+x)^{-l},$$

where $C > 0$ is a constant. Hence the proof is complete. □

Now we find out an estimate for p.d.o. defined as (48). From (48) and (49), we have

$$\begin{aligned}|(\mathbb{P}_a\varphi)(x)| &\leq \frac{C}{2}\int_0^\infty (xy)^{\frac{-b}{4}}(1+x)^{-l}\varphi(y)y^{-1}\,dy\\ &\leq \frac{C}{2}\,(x)^{\frac{-b}{4}}(1+x)^{-l}\,\|(\cdot)^{\frac{-b}{4}}\,\varphi\|_{L^1(\mathbb{R}_+;x^{-1}\,dx)}.\end{aligned}$$

Thus, for $l > 1 - \frac{b}{4}$, $0 < b < 1/2$

$$\|\mathbb{P}_a\varphi\|_{L^1(\mathbb{R}_+;dx)} \leq C'\,\|(\cdot)^{\frac{-b}{4}}\,\varphi\|_{L^1(\mathbb{R}_+;x^{-1}\,dx)}.$$

Special Case:
If we consider symbols $a(x,\tau)$ which can be explicitly represented as $a(x,\tau) = f(x)g(\tau)$, then p.d.o. defined as (43), we have

$$(\mathbb{P}_a\varphi)(x) = \frac{4}{\pi^2}\int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x})\sinh(2\pi\sqrt{\tau})f(x)g(\tau)(\mathbb{K}\varphi)(\tau)d\tau. \tag{50}$$

By using (8), (50) reduces to

$$\left[\mathbb{K}\left(\frac{\mathbb{P}_a\varphi}{f}\right)\right](\tau) = g(\tau)(\mathbb{K}\varphi)(\tau),\ f(x) \neq 0. \tag{51}$$

Further, if $g(\tau) = (\mathbb{K}\psi)(\tau)$, then by using (24), (51) can be represented as

$$\left[\mathbb{K}\left(\frac{\mathbb{P}_a\varphi}{f}\right)\right](\tau) = [\mathbb{K}(\psi \sharp \varphi)](\tau).$$

Or

$$(\mathbb{P}_a\varphi)(x) = f(x)\,(\psi \sharp \varphi)(x).$$

Moreover, by using Hölder's inequality, we have

$$\|(\mathbb{P}_a\varphi)(x)\|_{L^1(\mathbb{R}_+;dx)} \leq \|\psi \sharp \varphi\|_{L^p(\mathbb{R}_+;dx)}\|f\|_{L^q(\mathbb{R}_+;dx)}, \tag{52}$$

from (37) and (52), we obtain

$$\|(\mathbb{P}_a\varphi)(x)\|_{L^1(\mathbb{R}_+;dx)} \leq \|\psi\|_{L^p(\mathbb{R}_+;dx)}\|\varphi\|_{L^1(\mathbb{R}_+;dx)}\|f\|_{L^q(\mathbb{R}_+;dx)},$$

where $p,\ q > 1$.

Remark 4.5. *If in* (50), *we consider* $f(x) = 1$ *then the p.d.o. belongs to a special class of operator similar to potential operator as investigated for Fourier, Jacobi and Bessel transform [18, 29, 39].*

Acknowledgments: The authors express their gratitude to the anonymous reviewer for his/her constructive and valuable suggestions.
The first author is financially supported by Science and Engineering Research Board, Govt. of India, under grant no. EMR/2016/005141.

References

[1] Banerji PK, Loonker D, Kalla SL. *Kontorovich-Lebedev transform for Boehmians*. Integral Transforms Spec. Funct. 2009;**20**(12):905-913.
[2] Betancor JJ., *Two complex variants of Hankel type transformation of generalized functions*. Port. Math. 1989;**46**(3):229-243.
[3] Erde'lyi A, Magnus W, Oberhettinger F, Tricomi FG. *Tables of integral transforms*. Vol. 2 McGraw-Hill Book Co., New York, 1953.
[4] Fock VA.*On the representation of an arbitrary function by an integrals involving the Legendre's functions with a complex index*. Dokl. Akad. Nauk SSSR. 1943;**39**(7):279-283. Russian.
[5] Glaeske HJ, Heβ A. *A convolution connected with the Kontorovich-Lebedev transform*. Math. Z. 1986;**193**(1):67-78.
[6] Gutiérrez-Tovar YE, Méndez-Pérez JMR. *A convolution for the Hankel-Kontorovich-Lebedev transformation*. Math. Nachr. 2008;**281**(1):1566-1581.
[7] Gutiérrez-Tovar YE, Méndez-Pérez JMR. *The Kontorovich-Lebedev integral transformation with a Hankel function kernel in a space of generalized functions of doubly exponential descent*. J. Math. Anal. Appl. 2007;**328**(1):359-369.
[8] Kontorovich MI, Lebedev NN. *On the application of inversion formulae to the solution of some electrodynamics problems*. J. Exp. Theor. Phys. 1939;**9**(6):729-742 (In Russian).
[9] Kontorovich MI, Lebedev NN. *On the one method of solution for some problems in diffraction theory and related problems*. J. Exp. Theor. Phys. 1938;**8**(10-11):1192-1206 (In Russian).
[10] Lebedev NN. *Special Functions and Their Applications*. Prentice-Hall, Englewood Cliffs, N. J., 1965.
[11] Loureiro AF, Yakubovich SB. *Central factorials under the Kontorovich-Lebedev transforms of polynomials*. Integral Transforms Spec. Funct. 2013; **24**(3):217-238.
[12] Mehler FG. *Ueber eine mit den kugel-und cylinderfunctionen verwandte function und ihre anwendung in der theorie der elektricitatsvertheilung*. Math. Anal. 1881;**18**:161-194.
[13] Méndez Pérez JMR, Socas Robayna MM. *A pair of generalized Hankel-Clifford transformations and their applications*. J. Math. Anal. Appl. 1991;**154**:543-557.
[14] Pathak RS. *Integral transform of Generalized Functions and Their Applications*. Gordon Breach Science Publishers, Amsterdem, 1997.
[15] Pathak RS. *Pseudo-differential operator associated with the Kontorovich-Lebedev transform*, Invest. Math. Sci. 2015;**5**(1):29-46.
[16] Pathak RS. Pandey JN. *The Kontorovich-Lebedev transformation of distributions*. Math. Z. 1979;**165**(1):29-51.
[17] Pathak RS, Pandey PK. *A class of pseudo-differential operator associated with Bessel operators*. J. Math. Anal. Appl. 1995;**196**:736-747.
[18] Pathak RS, Pandey PK. *Sobolev type spaces associated with Bessel operator*. J. Math. Anal. Appl. 1997; **215**(1):95-111.
[19] Prasad A, Kumar M. *Continuity of pseudo-differential operator h_μ, a involving Hankel translation and Hankel convolution on some Gevrey spaces*. Integral Transforms Spec. Funct. 2010;**21**(6):465-477.
[20] Prasad A, Kumar S. *Hankel-Clifford transformations on some ultradifferentiable function spaces and pseudo-differential operators*. J. Pseudo-Differ. Oper. Appl. 2013;**4**(4):551-567.
[21] Prasad A, Mandal UK. *Wavelet transforms associated with the Kontorovich-Lebedev transform*. Int. J. Wavelets Multiresolut. Inf. Process. 2017;**15**(02):1750011(17 pp.).
[22] Prasad A, Mandal UK. *Boundedness of pseudo-differential operators involving Kontorovich-Lebedev transform*. Integral Transforms Spec. Funct. 2017; **28**(4):300-314.
[23] Prasad A, Mandal UK. *Two versions of pseudo-differential operators involving the Kontorovich-Lebedev transform in $L^2(\mathbb{R}_+; x^{-1}dx)$*. Forum Math. 2018;**30**(1):31-42.
[24] Prasad A, Singh VK. *Pseudo-differential operators associated to a pair of Hankel-Clifford transformations on certain Beurling type function spaces*. Asian-Eur. J. of Math. 2013; **6**(3):1350039(22 pp.).

[25] Prasad A, Verma SK. *The Mehler-Fock-Clifford transform and pseudo-differential operator on function spaces*. Filomat. 2019;33(8):2457-2469.
[26] Prudnikov AP, Brychkov YA, Marichev OI. *Integrals and series, Vol.I, Elementary functions*. Gordon and Breach Science Publisher, Amsterdam, 1986.
[27] Rodino L. *Linear partial differential operators in Gevrey spaces*. World Scientific, Singapore, 1993.
[28] Salem NB, Dachraoui A. *Pseudo-differential operator associated with the Jacobi differential operator*. J. Math. Anal. Appl. 1998;**220**(1):365-381.
[29] Salem NB, Dachraoui A. *Sobolev type spaces associated with Jacobi differential operator*. Integral Transforms Spec. Funct. 2000;**9**(3):163-184.
[30] Sneddon IN. *The use of integral transforms*. New York: McGraw-Hill Book Co. 1972.
[31] Srivastava HM, Singh A, Rawat A, Singh S. *A family of Mexican hat wavelet transforms associated with an isometry in the heat equation*. Math. Methods Appl. Sci. 2021;**44**(14):11340-11349.
[32] Srivastava HM, Kumar M, Pradhan T. *A framework of linear canonical Hankel transform pairs in distribution spaces and their applications*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM. 2021;**115**(4):Article ID:171(18 pp).
[33] Srivastava HM, González B, Nergín E. *New L^p-boundedness properties for the Kontorovich-Lebedev and Mehler-Fock transforms*. Integral Transforms Spec. Funct. 2016;**27**(10):835-845.
[34] Srivastava HM, Chauhan MS, Upadhyay SK. *L^p_α-boundedness of the pseudo-differential operators associated with the Kontorovich-Lebedev transform*, Rev. Real Acad. Cienc. Exactas Fí. Natur. Ser. A. Mat. 2020; **114**(2):Article ID:80(18 pp.).
[35] Srivastava HM, González BJ, Negrín ER. *A new class of Abelian theorems for the Mehler-Fock transforms*. Russ. J. Math. Phys. 2017;**24**(1):124-126.
[36] Srivastava HM, González BJ, Negrín ER. *Erratum to "A new class of Abelian theorems for the Mehler-Fock transforms"*. Russ. J. Math. Phys. 2017;**24**(2):278-278.
[37] Srivastava HM, Upadhyay SK, Khatterwani K. *A family of pseudo-differential operators on the Schwartz space associated with the fractional Fourier transform*. Russ. J. Math. Phy. 2017;**24**(4):534-543.
[38] Watson GN. *A Treatise on the Theory of Bessel Functions*. Second Edition, Cambridge University Press, 1995.
[39] Wong MW. *An introduction to pseudo-differential operators*, 3rd edn. World Scientific, Singapore, 2014.
[40] Yakubovich SB. *Index transforms (with foreword by H. M. Srivastava).* World Scientific Publishing Company, Singapore, 1996.
[41] Yakubovich SB *On the least values of L^p-norms for the Kontorovich-Lebedev transform and its convolution*. J. Approx. Theory. 2004;**131**(2):231-242.
[42] Yakubovich SB. *The Kontorovich-Lebedev transformation on Sobolev type spaces*. Sarajevo J. Math. 2005;**1**(14):211-234.
[43] Zemanian AH. *Generalized Integral Transformations*. Interscience Publishers, New York, 1968.