Filomat 35:14 (2021), 4777–4787
https://doi.org/10.2298/FIL2114777S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Almost $\mathcal{I}$-Hurewicz Covering Property

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Abstract. In this paper, we introduce the almost $\mathcal{I}$-Hurewicz (A$\mathcal{I}$H) property which is a simultaneous generalization of the $\mathcal{I}$-Hurewicz ($\mathcal{I}$H) (see [1, 5]) and the almost Hurewicz [23] properties. It is shown that the A$\mathcal{I}$H property independent to the weakly $\mathcal{I}$-Hurewicz (W$\mathcal{I}$H) [6] property, where $\mathcal{I}$ is the proper admissible ideal of $\mathbb{N}$. It is shown that in a regular space, the A$\mathcal{I}$H property implies the $\mathcal{I}$H property hence the W$\mathcal{I}$H property, but not in Urysohn space. In a similar way, we consider almost $\mathcal{I}\gamma$-set and the almost star-$\mathcal{I}$-Hurewicz property (AS$\mathcal{I}$H) and it is shown that the image of almost $\mathcal{I}\gamma$-set under θ-continuous mapping has the A$\mathcal{I}$H property.

1. Introduction

In 1996, Scheepers [20] initiated the systematic study of selection principles in topology and their relation to game theory and Ramsey theory (see also [12]). Kočinac and Scheepers [14] studied the Hurewicz property in detail and found its relations with function spaces, game theory and Ramsey theory. Subsequently this topic became one of the most active areas of set theoretic topology. The classical selection principles have been used to define and characterize various topological properties. Maio and Kočinac [17] introduced certain types of open covers and selection principles using the ideal of asymptotic density zero of $\mathbb{N}$. Das, Chandra and Kočinac (see [1–5]) studied the open covers and selection principles using arbitrary ideals of $\mathbb{N}$ (also see [25]). Further Das et al. (see [1, 5]) defined the ideal analogues of some variants of the Hurewicz property such as the $\mathcal{I}$-Hurewicz, the star-$\mathcal{I}$-Hurewicz and the weakly $\mathcal{I}$-Hurewicz, where $\mathcal{I}$ is the proper admissible ideal of $\mathbb{N}$. Recently authors continued (see [21, 25]) the study of Hurewicz covering properties using ideal.

The purpose of this paper is to extend the use of arbitrary ideal of $\mathbb{N}$ to the covering properties. Briefly, we define the almost $\mathcal{I}$-Hurewicz (A$\mathcal{I}$H) which is a generalization of the $\mathcal{I}$-Hurewicz ($\mathcal{I}$H) property and the almost Hurewicz property and independent to the weakly $\mathcal{I}$-Hurewicz (W$\mathcal{I}$H) property. In a similar way we define almost $\mathcal{I}\gamma$-set, which is a generalization of almost γ-set [13]. We also define the almost star-$\mathcal{I}$-Hurewicz property (AS$\mathcal{I}$H) which is a further generalization of the A$\mathcal{I}$H property.

This paper is organized as follows. Section-2 contains the necessary preliminaries. In Section-3, we define the almost $\mathcal{I}$-Hurewicz property and study its relationship with the $\mathcal{I}$H, W$\mathcal{I}$H and almost Hurewicz properties by giving some examples. Several results are proved in this section for instance; (1) It is shown

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2020 *Mathematics Subject Classification*. Primary 54D20; Secondary 54B20, 54C10
Keywords. Hurewicz, almost Hurewicz, $\mathcal{I}$-Hurewicz, almost $\mathcal{I}$-Hurewicz, Ideal, Covering, Star-Covering, topological space.
Received: 18 November 2020; Accepted: 31 January 2021
Communicated by Pratulananda Das
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that in a regular space the A$\mathcal{I}$H property implies the $\mathcal{I}$H property hence the W$\mathcal{I}$H property but not in a Urysohn space, (2) The A$\mathcal{I}$H property and the W$\mathcal{I}$H are independent properties, (3) We can use regular open subsets instead of open sets in the definition of the A$\mathcal{I}$H property. In Section-4, we study the preservation properties of the A$\mathcal{I}$H under several types of mappings and subspaces. In Section-5 we define almost $\mathcal{I}\gamma$-set and it is obtained that the image of almost $\mathcal{I}\gamma$-set under θ-continuous mapping has the A$\mathcal{I}$H property. Section-6 contains the almost star-$\mathcal{I}$-Hurewicz (AS$\mathcal{I}$H) property and its relationship with the S$\mathcal{I}$H, A$\mathcal{I}$H and almost star-Hurewicz properties.

2. Preliminaries

Throughout the paper (X,τ) stands for a Hausdorff topological space. Let A be a subset of X and $\mathcal{U}$ be a cover of space X. Then St(A,$\mathcal{U}$)=$\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We usually write St($x$,$\mathcal{U}$)=St($\{x\}$,$\mathcal{U}$).

Let us recall some types of open covers which are used in this paper. An open cover $\mathcal{U}$ of X is an ω-cover [10] if for each finite subset F of X there exists a $U \in \mathcal{U}$ such that $F \subset U$ and X is not a member of $\mathcal{U}$. An open cover $\mathcal{U}$ of X is a γ-cover [10] if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. Gerlits and Nagy [10] defined the notion of γ-sets, a space X is a γ-set if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $(V_n : n \in \mathbb{N})$ is a γ-cover of X. An open cover $\mathcal{U}$ of X is an almost γ-cover [13] if it is infinite and for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin \overline{U}\}$ is finite. Kocev [13] defined the notion of almost γ-sets, a space X is an almost γ-set if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $(V_n : n \in \mathbb{N})$ is an almost γ-cover of X.

A space X is said to have the Hurewicz property [11] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

A space is called Hurewicz space if it has the Hurewicz property and so on throughout the paper.

A space X is said to have the almost Hurewicz property [23] if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \overline{\bigcup \mathcal{V}_n}$ for all but finitely many n.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$, while an ideal is said to be admissible ideal or free ideal $\mathcal{I}$ of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$. If $\mathcal{I}$ is a proper ideal in Y (that is, $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I}$ such that $M = Y \setminus A\}$ is a filter in Y. Throughout the paper $\mathcal{I}$ will stand for proper admissible ideal of $\mathbb{N}$. We denote the ideals of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

A space X is said to have the $\mathcal{I}$-Hurewicz property (see [1, 5]) (in short, $\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

A space X is said to have the star-$\mathcal{I}$-Hurewicz property [6] (in short, S$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the weakly $\mathcal{I}$-Hurewicz property [6] (in short, W$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a dense set $Y \subset X$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

A space X is said to have the mildly star-$\mathcal{I}$-Hurewicz property [6] (in short, MS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of clopen covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

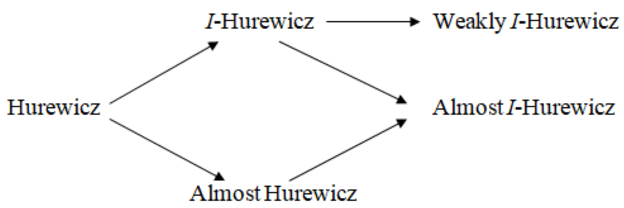
A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be an $\mathcal{I}$-γ cover [5] if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin U_n\} \in \mathcal{I}$. The set of all $\mathcal{I}$-γ covers will denoted by $\mathcal{I} - \Lambda$.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. For other terms and symbols we follow [8].

3. The almost $\mathcal{I}$-Hurewicz property

Definition 3.1. *A space X is said to have the almost $\mathcal{I}$-Hurewicz property (in short, A$\mathcal{I}$H) if for each sequence* $(\mathcal{U}_n : n \in \mathbb{N})$ *of open covers of X there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *such that for each* $n \in \mathbb{N}$, $\mathcal{V}_n$ *is a finite subset of* $\mathcal{U}_n$ *and for each* $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$.

From the definitions it is obvious that for an admissible ideal $\mathcal{I}$, the almost Hurewicz property implies the almost $\mathcal{I}$-Hurewicz property and the $\mathcal{I}$-Hurewicz property imply the almost $\mathcal{I}$-Hurewicz and the weakly $\mathcal{I}$-Hurewicz properties. Thus we have the following implications.



We give examples show that the converse implications need not be true in general.

Example 3.2. *Let X be the Euclidean plane with the deleted radius topology (see [Example 77, [22]]). Since X is not Lindelöf, X does not have the $\mathcal{I}$-Hurewicz property, since a space having the $\mathcal{I}$-Hurewicz property must be Lindelöf. Now to prove that X has the A$\mathcal{I}$H property, we will use the fact that the closure of every open set in the deleted radius topology is the same as in the usual Euclidean topology and that the Euclidean plane with the Euclidean topology is σ-compact and therefore has the A$\mathcal{I}$H property.*

Problem 3.3. *Construct an example which has the A$\mathcal{I}$H property but does not have the almost Hurewicz property.*

With $\mathcal{I}_{fin}$ ideal, the A$\mathcal{I}$H and the almost Hurewicz properties become equivalent.

Now a natural question arises that under what condition the A$\mathcal{I}$H property implies the $\mathcal{I}$H property. The following theorem gives partial answer to this question.

Theorem 3.4. *Let X be a regular space with the A$\mathcal{I}$H property. Then X has the $\mathcal{I}$H property.*

Proof. Consider $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By regularity of space X, for each $n \in \mathbb{N}$ there exists an open cover $\mathcal{V}_n$ of X such that $\mathcal{V}'_n = \{\overline{V} : V \in \mathcal{V}_n\}$ is a refinement of $\mathcal{U}_n$. Applying the A$\mathcal{I}$H property to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{W}_n}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{W}_n}\} \in \mathcal{F}(\mathcal{I})$. For every $n \in \mathbb{N}$ and for every $W \in \mathcal{W}_n$ choose $U_W \in \mathcal{U}_n$ such that $\overline{W} \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$ and hence $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{W}_n}\} \subset \{n \in \mathbb{N} : x \in \bigcup \mathcal{U}'_n\}$. Thus X has the $\mathcal{I}$H property. □

Example 3.5. *There exists an Urysohn space having the A$\mathcal{I}$H property which does not having the $\mathcal{I}$H property.*

Let $A = \{a_\alpha : \alpha < \omega_1\}$, $B = \{b_i : i \in \omega\}$ and $Y = \{\langle a_\alpha, b_i \rangle \ : \alpha < \omega_1,\ i \in \omega\}$. Let $X = Y \cup A \cup \{a\}$ where $a \notin Y \cup A$. We topologize X as follows: every point of Y is isolated, a basic neighborhood of $a_\alpha \in A$ for each $\alpha < \omega_1$ takes of the form $U_{a_\alpha}(i) = \{a_\alpha\} \cup \{\langle a_\alpha, b_j \rangle \ : i \leq j\}$ where $i \in \omega$ and a basic neighborhood of a takes the form $U_a(\alpha) = \{a\} \cup \bigcup\{\langle a_\beta, b_i \rangle \ : \beta > \alpha,\ i \in \omega\}$ where $\alpha < \omega_1$. Clearly, X is an Urysohn space. However X is not regular, since the point a can not be separated from the closed set $\{a_\alpha : \alpha < \omega_1\}$ of X. Since $\{a_\alpha : \alpha < \omega_1\}$ is an uncountable discrete closed set of X. Thus X is not Lindelöf and X does not have the $\mathcal{I}$-Hurewicz property, since a space having the $\mathcal{I}$-Hurewicz property must be Lindelöf (see [[25], Theorem 4.6]).

We show that X has the almost $\mathcal{I}$-Hurewicz property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, choose $U_n \in \mathcal{U}_n$ such that $a \in U_n$. By the definition of topology of X, there exists a $\beta_n < \omega_1$

such that $U_a(\beta_n) \subset U_n$, then $\{a_\alpha : \alpha > \beta_n\} \cup \{a\} \cup \{\langle a_\alpha, b_i\rangle\ : \alpha > \beta_n,\ i \in \omega\} \subset \overline{U_n}$. Let $\beta = sup\{\beta_n : n \in \mathbb{N}\}$. Then $\beta < \omega_1$, since for each $n \in \mathbb{N}$, β_n is countable. Then the subset $C = \bigcup_{\alpha \leq \beta}(a_\alpha \cup \{\langle a_\alpha, b_i\rangle\ : i \in \omega\})$ is countable by the definition of X. So C can be enumerate as $C = \{c_n : n \in \mathbb{N}\}$. Now $\mathcal{W}_n = \{U_n\} \cup \{U_{n,1}, U_{n,2}, ..., U_{n,n}\}$, where $U_{n,i}$ is the open set such that $c_i \in U_{n,i} \in \mathcal{U}_n$ for each $i \in \{1, 2, ..., n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$. We can easily verify that for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{W}_n}\} \in \mathcal{I}$. Thus X has the almost $\mathcal{I}$-Hurewicz property.

Recall that a space X is almost Lindelöf [26] if every open cover $\mathcal{U}$ of X there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{\overline{V} : V \in \mathcal{V}\} = X$.

Lemma 3.6. *If a space X has the A$\mathcal{I}$H property then X is almost Lindelöf.*

Proof. Let $\mathcal{U}$ be an any open cover of X. Let $\mathcal{U}_n = \mathcal{U}$ for each $n \in \mathbb{N}$. Then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Applying the A$\mathcal{I}$H property of X to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$. Let $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$. Clearly $\mathcal{V}$ is a countable subset of $\mathcal{U}$ also for each $x \in X$, $x \in \overline{\bigcup \mathcal{V}}$. Therefore X is almost Lindelöf. □

Theorem 3.7. *[26] A regular almost Lindelöf space is Lindelöf.*

Theorem 3.8. *[6] Every separable space has the W$\mathcal{I}$H property.*

We have the following corollary From Theorem 3.4.

Corollary 3.9. *Let X be a regular space with the A$\mathcal{I}$H property. Then X has the W$\mathcal{I}$H property.*

However the following example shows that the converse of above corollary need not be true, that is, if a regular space X has the W$\mathcal{I}$H property then X need not have the A$\mathcal{I}$H property.

Example 3.10. *There exists a regular space with the W$\mathcal{I}$H property which does not have the A$\mathcal{I}$H property.*

Let X be the Sorgenfrey plane. Then X is regular, since product of two regular space is regular. Also X is separable. By Theorem 3.8, X has the W$\mathcal{I}$H property. The existence of uncountable discrete closed subspace $L = \{x \times (-x) : x \in \mathbb{R}\}$ of X, shows that X is not Lindelöf. Thus by Theorem 3.7, X is not almost Lindelöf. Hence by Lemma 3.6, X does not have the A$\mathcal{I}$H property.

Problem 3.11. *Does there exist a space having the A$\mathcal{I}$H property but not having the W$\mathcal{I}$H property.*

In [6] it was shown that a paracompact Hausdorff space X has S$\mathcal{I}$H property if and only if it has $\mathcal{I}$H property. So from Theorem 3.4, we have the following theorem.

Theorem 3.12. *Let X be a paracompact space. Then the following statements are equiavlent.*

1. *X has the $\mathcal{I}$H property;*
2. *X has the S$\mathcal{I}$H property;*
3. *X has the A$\mathcal{I}$H property.*

Theorem 3.13. *[6] The $\mathcal{I}$H, S$\mathcal{I}$H and MS$\mathcal{I}$H properties are equivalent in a paracompact Hausdorff zero-dimensional space.*

From Theorem 3.12 and Theorem 3.13, we obtain the following corollary.

Corollary 3.14. *Let X be a paracompact zero-dimensional space. Then the following statements are equivalent.*

1. *X has the $\mathcal{I}$H property;*
2. *X has the S$\mathcal{I}$H property;*
3. *X has the MS$\mathcal{I}$H property;*
4. *X has the A$\mathcal{I}$H property.*

The following example of Tychonoff space has the W$\mathcal{I}$H property but does not have the A$\mathcal{I}$H property, also we use this example later in the text.

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 3.15. *There exists a Tychonoff space having the W$\mathcal{I}$H property which does not have the A$\mathcal{I}$H property.*

Let D be a discrete space of cardinality ω_1. Let $X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$ be the subspace of the product of βD and $\omega + 1$. To showing that X has the W$\mathcal{I}$H property it is enough to show that there exists a dense set in X which has the $\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since every countable set has the $\mathcal{I}$H property and thus ω has the $\mathcal{I}$H property also βD is a compact set. Since product of space having the $\mathcal{I}$H property and a compact set has the $\mathcal{I}$H property. Thus $\beta D \times \{\omega\}$ has $\mathcal{I}$H property, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in \beta D \times \{\omega\}$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$. Also since $\beta D \times \{\omega\}$ is a dense subset of X. Thus X has the W$\mathcal{I}$H property. To prove that X does not have the A$\mathcal{I}$H property, by Lemma 3.6, it is enough to show that X is not almost Lindelöf. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $U_\alpha = \{d_\alpha\} \times (\omega + 1)$. For each $n \in \omega$, let $V_n = \beta D \times \{n\}$. Let us consider the open cover $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_n : n \in \omega\}$ of X. Clearly for each countable subset $\mathcal{V}$ of $\mathcal{U}$, $\bigcup \mathcal{V} = \bigcup\{\overline{V} : V \in \mathcal{V}\}$. Consider $\mathcal{V}$ be any countable subset of $\mathcal{U}$ and let $\alpha_0 = sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \omega_1$, since $\mathcal{V}$ is countable. So for any $\alpha' > \alpha_0$, $\langle d'_\alpha, \omega\rangle \notin \bigcup\{\overline{V} : V \in \mathcal{V}\}$, since U'_α is the only element of $\mathcal{U}$ containing $\langle d'_\alpha, \omega\rangle$. Thus X is not almost Lindelöf.

Theorem 3.16. *If the complement of dense set has the A$\mathcal{I}$H property then the W$\mathcal{I}$H property implies the A$\mathcal{I}$H property.*

Proof. Let X has the W$\mathcal{I}$H property and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By the W$\mathcal{I}$H property of X, there is a dense set $Y \subset X$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$. Since Y is dense in X thus $X \setminus Y$ has the A$\mathcal{I}$H property. Then for each $n \in \mathbb{N}$ there is a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in X \setminus Y$, $\{n \in \mathbb{N} : y \notin \overline{\bigcup \mathcal{V}'_n}\} \in \mathcal{I}$. Let $\mathcal{W}_n = \mathcal{V}_n \bigcup \mathcal{V}'_n$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{W}_n}\}$. Therefore X has the A$\mathcal{I}$H property. □

Recall that a subset A of a topological space X is called regular open (regular closed) if $A = Int(\overline{A})$ $(A = \overline{Int(A)})$.

The next theorem shows that we can use regular open sets instead of open sets in the definition of the A$\mathcal{I}$H property.

Theorem 3.17. *A space X has the A$\mathcal{I}$H property if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$.*

Proof. Let X has the A$\mathcal{I}$H property $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by regular open sets. Since every regular open sets is open, so by the definition of A$\mathcal{I}$H property proof follows. Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Put $\mathcal{U}'_n = \{Int(\overline{U}) : U \in \mathcal{U}_n\}$. Then for each $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of covers of X by regular open sets. Applying hypothesis, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $\overline{V} = \overline{Int(\overline{U_V})}$. Put $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. Then $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in \overline{\bigcup \mathcal{V}_n}$ for some $n \in \mathbb{N}$ as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $V \in \mathcal{V}_n$ such that $x \in \overline{V}$ also U_V is open set, $\overline{U_V}$ is a regular closed set. Thus $\overline{U_V} = \overline{Int(\overline{U_V})} = \overline{V}$, $x \in \overline{\bigcup \mathcal{W}_n}$. Therefore $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{V}_n}\} \subset \{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{W}_n}\}$. Thus X has the A$\mathcal{I}$H property. □

4. The almost $\mathcal{I}$-Hurewicz property and mappings

Recall that a mapping $f : X \to Y$ is almost continuous if for each regular open set $A \subset Y$, $f^{-1}(A)$ is an open set in X.

Theorem 4.1. *The A$\mathcal{I}$H property is preserved under almost continuous mappings.*

Proof. Let $f : X \to Y$ is almost continuous mapping and space X has the A$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of $f(X)$ by regular open sets. Put $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Since f is almost continuous surjection, then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of open covers of X. Applying the A$\mathcal{I}$H property of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$, choose $U_V \in \mathcal{U}_n$ such that $V = f^{-1}(U_V)$. Let $\mathcal{V}'_n = \{\overline{U_V} : V \in \mathcal{V}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in X$. Then $f(x) = y$ for some $y \in f(X)$. Let $x \in \overline{\bigcup \mathcal{V}_n}$ for some n. Choose $V \in \mathcal{V}_n$ such that $x \in \overline{V}$, $x \in \overline{V} = \overline{f^{-1}(U_V)} \subset f^{-1}(\overline{U_V})$. Hence $y = f(x) \in \overline{U_V}$. Therefore $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{V}_n}\} \subset \{n \in \mathbb{N} : f(x) \in \overline{\bigcup \mathcal{V}'_n}\}$. Thus Y has the A$\mathcal{I}$H property. □

Corollary 4.2. *The A$\mathcal{I}$H property is preserved under continuous mappings.*

To show that the preimage of a space with the A$\mathcal{I}$H property under a closed 2-to-1 continuous map need not have the A$\mathcal{I}$H property, we use the Alexandorff duplicate $A(X) = X \times \{0, 1\}$ of a space X. The basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X and each points $\langle x, 1\rangle \in X \times \{1\}$ are isolated points.

Example 4.3. *There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a Urysohn space having the A$\mathcal{I}$H property, but A(X) does not have the A$\mathcal{I}$H property.*

Let X be the space X of Example 3.5. Then X has the A$\mathcal{I}$H property and has an uncountable discrete closed subset $A = \{a_\alpha : \alpha < \omega_1\}$. Hence the Alexandorff duplicate A(X) of X does not have the A$\mathcal{I}$H property, since $A \times \{1\}$ is an uncountable discrete, open and closed set in A(X) and by Theorem 4.10, the A$\mathcal{I}$H property preserved under an open and closed subsets. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map.

Recall that a mapping $f : X \to Y$ is called almost open [24] if $f^{-1}(\overline{U}) \subset \overline{f^{-1}(U)}$ for each open subset U of Y and f is said to be perfect [8] if X is Hausdorff, f is closed and for each $y \in Y$, $f^{-1}\{y\}$ is a compact subset of X.

Theorem 4.4. *The A$\mathcal{I}$H property is inverse invariant under perfect almost open mappings.*

Proof. Let $f : X \to Y$ be a perfect almost open mapping and let Y be a space having the A$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Then for each $y \in Y$ and for each $n \in \mathbb{N}$, there is a finite subset $\mathcal{U}_{n_y}$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subset \bigcup \mathcal{U}_{n_y}$. Let $U_{n,y} = \bigcup \mathcal{U}_{n_y}$. Then $V_{n,y} = Y \setminus f(X \setminus U_{n,y})$ is a neighborhood of y, since f is closed. Now for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ is an open cover of Y. Applying the A$\mathcal{I}$H property of Y to $(\mathcal{V}_n : n \in \mathbb{N})$, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ such that and for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \overline{\bigcup \mathcal{V}'_n}\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : y \in \overline{\bigcup \mathcal{V}'_n}\} \in \mathcal{F}(\mathcal{I})$. Suppose that $\mathcal{V}'_n = \{V_{n,y_i} : i \leq n'\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \bigcup_{i \leq n'} \mathcal{U}_{n,y_i}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Let $y \in Y$. Then $y = f(x)$ for some $x \in X$. Let $y \in \bigcup \overline{\mathcal{V}'_n}$ for some n. Choose $V_{n,y_i} \in \mathcal{V}'_n$ such that $x \in f^{-1}(\overline{V_{n,y_i}})$ for $i \leq n'$. Since f is almost open, then $x \in f^{-1}(\overline{V_{n,y_i}}) \subset \overline{f^{-1}(V_{n,y_i})} \subset \overline{U_{n,y_i}} = \overline{\bigcup \mathcal{U}_{n,y_i}}$ for $i \leq n'$. Thus $x \in \bigcup \overline{\mathcal{U}'_n}$ for some n. Therefore $\{n \in \mathbb{N} : y \in \overline{\bigcup \mathcal{V}'_n}\} \subset \{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{U}'_n}\}$. Hence X has the A$\mathcal{I}$H property. □

Since every open mapping is almost open, we have following by Theorem 4.4.

Corollary 4.5. *If X has the A$\mathcal{I}$H property and Y is compact then $X \times Y$ has the A$\mathcal{I}$H property.*

Theorem 4.6. *The $A\mathcal{I}H$ property is closed under countable unions.*

Proof. The proof is easy and thus omitted. □

We have the following from Corollary 4.5 and Theorem 4.6.

Corollary 4.7. *If X has the $A\mathcal{I}H$ property and Y is σ-compact then, $X \times Y$ has the $A\mathcal{I}H$ property.*

Remark 4.8. *In Example 3.5, the set $A = \{a_\alpha : \alpha < \omega_1\}$ is an uncountable discrete closed set of X, thus A is not almost Lindelöf and by Lemma 3.6, the set A does not have the $A\mathcal{I}H$ property. This shows that the $A\mathcal{I}H$ property does not preserved under closed subsets.*

The following example shows that the A$\mathcal{I}$H property is not preserved under an regular closed subsets.

Example 4.9. *There exist an Urysohn space the $A\mathcal{I}H$ property but their regular closed subset not having the $A\mathcal{I}H$ property.*

Let S_1 be the same space X of Example 3.5. Then S_1 has the A$\mathcal{I}$H property. Let S_2 be the same space X of Example 3.15. Then S_2 does not have the A$\mathcal{I}$H property. We assume that $S_1 \cap S_2 = \emptyset$. Since $|D| = \omega_1$, we can enumerate D as $\{d_\alpha : \alpha < \omega_1\}$. Let $\psi : D \times \{\omega\} \to A$ be a bijection defined by $\psi(\langle d_\alpha, \omega\rangle) = a_\alpha$ for each $\alpha < \omega_1$. Let X be the quotient space obtained from discrete sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, \omega\rangle$ with $\psi(\langle d_\alpha, \omega\rangle)$ for each $\alpha < \omega_1$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map and $Y = \pi(S_2)$. Then Y is a regular closed subset of X by the construction of X. Since Y is homeomorphic to S_2, thus Y does not have the A$\mathcal{I}$H property.

Now we show that X has the A$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since $\pi(S_1)$ have the A$\mathcal{I}$H property, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and for each $x_1 \in \pi(S_1)$, $\{n \in \mathbb{N} : x_1 \notin \overline{\bigcup \mathcal{V}'_n}\} \in \mathcal{I}$. On the other hand, since every countable set has the A$\mathcal{I}$H property and thus ω has the A$\mathcal{I}$H property also βD is a compact set. Then by Corollary 4.5, $\pi(\beta D \times \{\omega\})$ has the A$\mathcal{I}$H property, there exists a sequence $(\mathcal{V}''_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}''_n$ is a finite subset of $\mathcal{U}_n$ and for each $x_2 \in \pi(\beta D \times \{\omega\})$, $\{n \in \mathbb{N} : x_2 \notin \overline{\bigcup \mathcal{V}''_n}\} \in \mathcal{I}$. Let $\mathcal{V}_n = \mathcal{V}'_n \bigcup \mathcal{V}''_n$. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{V}_n}\} \in \mathcal{I}$.

Theorem 4.10. *The $A\mathcal{I}H$ property is preserved clopen subsets.*

Proof. Let X be a space having the A$\mathcal{I}$H property and Y be a clopen subset of X. Consider a sequence of open covers $(\mathcal{U}_n : n \in \mathbb{N})$ of Y. For each $n \in \mathbb{N}$, $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus Y\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Applying the A$\mathcal{I}$H property of X there exists $(\mathcal{H}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{V}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{H}_n}\} \in \mathcal{I}$. Since Y is open thus $\overline{X \setminus Y} = X \setminus Y$. Let $\mathcal{W}_n = \mathcal{H}_n \setminus \{X \setminus Y\}$. Then $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and $\overline{\bigcup \mathcal{W}_n} = \overline{\bigcup \mathcal{H}_n} \setminus \{X \setminus Y\}) \subset \overline{\bigcup \mathcal{H}_n}$. Therefore for each $y \in Y$, $\{n \in \mathbb{N} : y \notin \overline{\bigcup \mathcal{W}_n}\} \subset \{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{H}_n}\}$. □

Recall that a mapping $f : X \to Y$ is θ-continuous [9] (strongly θ-continuous [16]) if for each $x \in X$, and each open set V in Y containing $f(x)$ there is an open set U in X containing x such that $f(\overline{U}) \subset \overline{V}$ ($f(\overline{U}) \subset V$).

It is clear that each strongly θ-continuous mapping is θ-continuous.

Theorem 4.11. *The $A\mathcal{I}H$ property is preserved under θ-continuous mappings.*

Proof. Let $f : X \to Y$ is θ-continuous mapping and space X has the A$\mathcal{I}$H property. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of $f(X)$. For each $n \in \mathbb{N}$ and for each $x \in X$ there is a open set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. Since f is θ-continuous there is an open set $U_{x,n}$ containing x such that $f(\overline{U_{x,n}}) \subset \overline{V_{x,n}}$. Therefore for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{x,n} : x \in X\}$ is an open cover of X. Applying the A$\mathcal{I}$H property of X to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{H}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{H}_n}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{H}_n}\} \in \mathcal{F}(\mathcal{I})$. For each $H \in \mathcal{H}_n$ there is $W_H \in \mathcal{V}_n$ such that $f(\overline{H}) \subset \overline{W_H}$. Let $\mathcal{W}_n = \{W_H : H \in \mathcal{H}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $x \in X$ then $f(x) = y$ for some $y \in f(X)$. Let $x \in \overline{\bigcup \mathcal{H}_n}$ for some $n \in \mathbb{N}$ as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $H \in \mathcal{H}_n$ such that $x \in \overline{H}$ then $y = f(x) \in f(\overline{H}) \subset \overline{W_H}$. Hence the result follows from the fact that $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{H}_n}\} \subset \{n \in \mathbb{N} : f(x) \in \overline{\bigcup \mathcal{W}_n}\}$. □

Corollary 4.12. *If $f : X \to Y$ is strongly θ-continuous mapping and X has the $A\mathcal{I}H$ property then f(X) has the $\mathcal{I}H$ property.*

A mapping $f : X \to Y$ is called contra-continuous [7] if the preimage of each open set in Y is closed in X. f is said to be precontinuous [18] if $f^{-1}(V) \subset Int(\overline{f^{-1}(V)})$ whenever V is a open in Y.

Theorem 4.13. *Let $f : X \to Y$ be contra-continuous and precontinuous mapping. If X has the $A\mathcal{I}H$ property then f(X) has the $\mathcal{I}H$ property.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of $f(X)$. Since f is a contra-continuous, for each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$ the set $f^{-1}(V)$ is a closed in X. Also f is precontinuous $f^{-1}(V) \subset Int(\overline{f^{-1}(V)})$, so that $f^{-1}(V) \subset Int(f^{-1}(V))$, that is $f^{-1}(V) = Int(f^{-1}(V))$. Therefore for each $n \in \mathbb{N}$, the set $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a open cover of X. By the A$\mathcal{I}$H property of X there is a sequence $(\mathcal{G}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{\bigcup \mathcal{G}_n}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{G}_n}\} \in \mathcal{F}(\mathcal{I})$. Put $\mathcal{H}_n = \{f(G) : G \in \mathcal{G}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{V}_n$. Let $x \in X$. Then $f(x) = y$ for some $y \in f(X)$. Let $x \in \overline{\bigcup \mathcal{G}_n}$ for some n. Choose $G \in \mathcal{G}_n$ such that $x \in \overline{G} = G$, $y = f(x) = f(G)$. Therefore $\{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{G}_n}\} \subset \{n \in \mathbb{N} : f(x) \in \bigcup \mathcal{H}_n\}$. Thus $f(X)$ has the $\mathcal{I}$H property. □

Recall that a mapping $f : X \to Y$ is called weakly continuous [15] if for each $x \in X$ and each neighborhood V of $f(X)$ there is neighborhood U of x such that $f(U) \subset \overline{V}$.

Theorem 4.14. *If $f : X \to Y$ be weakly continuous and X has the $\mathcal{I}H$ property, then f(X) has the $A\mathcal{I}H$ property.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of $f(X)$. Let $x \in X$. Then for each $n \in \mathbb{N}$ there is a $V \in \mathcal{V}_n$ such that $f(x) \in V$. Since f is weakly continuous there is a open set $U_{x,n}$ in X such that $x \in U_{x,n}$ and $f(U_{x,n}) \subset \overline{V}$. The set $\mathcal{U}_n = \{U_{x,n} : x \in X\}$ is a open cover of X. Applying the $\mathcal{I}$H property of X there exist a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{F}_n\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \bigcup \mathcal{F}_n\} \in \mathcal{F}(\mathcal{I})$. For each $U \in \mathcal{F}_n$ assign a set $V_U \in \mathcal{V}_n$ such that $f(U) \subset \overline{V_U}$. Put $\mathcal{W}_n = \{V_U : U \in \mathcal{F}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $x \in X$. Then $f(x) = y$ for some $y \in f(X)$. Let $x \in \bigcup \mathcal{F}_n$ for some n. Choose $U \in \mathcal{F}_n$ such that $x \in U$, $y = f(x) = f(U) \subset \overline{V_U}$. Therefore $\{n \in \mathbb{N} : x \in \bigcup \mathcal{F}_n\} \subset \{n \in \mathbb{N} : f(x) \in \overline{\bigcup \mathcal{W}_n}\}$. Thus $f(X)$ has the A$\mathcal{I}$H property. □

By Theorem 3.4, Theorem 4.13 and Theorem 4.14, we conclude the following.

Corollary 4.15. *(1) If Y is a regular space then the $A\mathcal{I}H$ property is preserved by strongly θ-continuous mappings.*
(2) If X is a regular space then the $\mathcal{I}H$ property is preserved by contra-continuous and precontinuous mappings.
(3) If Y is a regular space then the $A\mathcal{I}H$ property is preserved by contra-continuous and precontinuous mappings.
(4) If X is a regular space then the $A\mathcal{I}H$ property is preserved by weakly continuous mappings.
(5) If Y is a regular space then the $\mathcal{I}H$ property is preserved by weakly continuous mappings.

By Theorem 3.5, Theorem 4.4, Theorem 4.13 and Theorem 4.14, we conclude the following.

Corollary 4.16. *Let $f : X \to Y$ be an almost open, perfect, contra-continuous, precontinuous and X and Y both regular spaces. Then*
(1) X has the $\mathcal{I}H$ property if and only if f(X) has the $\mathcal{I}H$ property.
(2) X has the $A\mathcal{I}H$ property if and only if f(X) has the $A\mathcal{I}H$ property.

Corollary 4.17. *Let $f : X \to Y$ be an almost open, perfect, weakly continuous and X and Y both regular spaces. Then*
(1) X has the $\mathcal{I}H$ property if and only if f(X) has the $\mathcal{I}H$ property.
(2) X has the $A\mathcal{I}H$ property if and only if f(X) has the $A\mathcal{I}H$ property.

5. Almost $\mathcal{I}\gamma$-set

In this section we consider almost $\mathcal{I}\gamma$-set which is a generalized form of γ-set [10] and almost γ-set [13]. First we define almost $\mathcal{I}$-γ-cover which is a generalization of $\mathcal{I}$-γ-cover [5].

A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be an almost $\mathcal{I}$-γ-cover if for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{U_n}\} \in \mathcal{I}$.

Definition 5.1. *A space X is said to be an almost $\mathcal{I}\gamma$-set if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $(U_n : n \in \mathbb{N})$ is an almost $\mathcal{I}$-γ-cover of X.*

Theorem 5.2. *A space X is almost $\mathcal{I}\gamma$-set if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-covers of X by regular open sets, there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $(U_n : n \in \mathbb{N})$ is an almost $\mathcal{I}$-γ-cover of X.*

Proof. The necessary part follows from the fact that every regular open set is open. Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of X. Let $\mathcal{U}'_n = \{Int(\overline{U}) : U \in \mathcal{U}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}'_n$ is a sequence of ω-covers of X by regular open sets. By the hypothesis, there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{U_n}\} \in \mathcal{I}$. Follow the same argument as in the proof of Theorem 3.17. Hence X is almost $\mathcal{I}\gamma$-set. □

Lemma 5.3. *Let $f : X \to Y$ be almost continuous surjective mapping and $\mathcal{U}$ be an ω-cover of Y by regular open sets. Then $\mathcal{U}' = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an ω-cover of X.*

Proof. Let F is a finite subset of X then $f(F)$ is a finite subset of Y. Since $\mathcal{U}$ is an ω-cover of Y by regular open sets, there exists $U \in \mathcal{U}$ such that $f(F) \subset U$. Then $F \subset f^{-1}(U)$. Since f is almost continuous, $f^{-1}(U)$ is an open set for every $U \in \mathcal{U}$. So $\mathcal{U}'$ is an ω-cover of X. □

Theorem 5.4. *An almost $\mathcal{I}\gamma$-set preserved under almost continuous surjective mappings.*

Proof. Let $f : X \to Y$ is almost continuous surjective mapping and space X be an almost $\mathcal{I}\gamma$-set. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of Y by regular open sets. Let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. By Lemma 5.3, for each $n \in \mathbb{N}$, $\mathcal{U}'_n$ is an ω-cover of X. Since X is an almost $\mathcal{I}\gamma$-set there exists a sequence $(V_n : n \in \mathbb{N})$ and $U_n \in \mathcal{U}_n$ such that for each $n \in \mathbb{N}$, $V_n = f^{-1}(U_n)$ and $(V_n : n \in \mathbb{N})$ is an almost $\mathcal{I}$-γ-cover of X. That is for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{V_n}\} \in \mathcal{I}$. So $\{n \in \mathbb{N} : x \in \overline{V_n}\} \in \mathcal{F}(\mathcal{I})$. Consider a sequence $(U_n : n \in \mathbb{N})$. Let $x \in X$. Then $y = f(x)$ for some $y \in Y$. Let $x \in \overline{V_n}$ for some n, thus $x \in \overline{V_n} = \overline{f^{-1}(U_n)} \subset f^{-1}(\overline{U_n})$ which implies $y = f(x) \in \overline{U_n}$. Therefore $\{n \in \mathbb{N} : x \in \overline{V_n}\} \subset \{n \in \mathbb{N} : y \in \overline{U_n}\}$. This completes the proof. □

Theorem 5.5. *If $f : X \to Y$ is θ-continuous mapping and X is an almost $\mathcal{I}\gamma$-set then f(X) has the A$\mathcal{I}$H property.*

Proof. Consider $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of $f(X)$. Let $x \in X$. For each $n \in \mathbb{N}$ there is a open set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. Since f is θ-continuous there is an open set $U_{x,n}$ containing x such that $f(\overline{U_{x,n}}) \subset \overline{V_{x,n}}$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the set of all finite unions of sets $U_{x,n}, x \in X$. Then each $\mathcal{U}_n$ is an ω-cover of X. As X is an almost $\mathcal{I}\gamma$-set there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{U_n}\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in \overline{U_n}\} \in \mathcal{F}(\mathcal{I})$. Let $U_n = U_{x_1,n} \cup U_{x_2,n} \cup ... \cup U_{x_{i_n},n}$. For each $U_{x_j,n}, j \leq i_n$, assign a set $V_{x_j,n} \in \mathcal{V}_n$ with $f(\overline{U_{x_j,n}}) \subset \overline{V_{x_j,n}}$. Put $\mathcal{W}_n = \{V_{x_j,n} : j = 1, 2, ..., i_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $x \in X$. Then $y = f(x)$ for some $y \in f(X)$. Let $x \in \overline{\bigcup \mathcal{U}_n}$ for some n as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Then $x \in \overline{U_{x_p,n}}$ for some $1 \leq p \leq i_n$, $y = f(x) \in f(\overline{U_{x_p,n}}) \subset \overline{V_{x_p,n}}$. Therefore $\{n \in \mathbb{N} : x \in \overline{U_n}\} \subset \{n \in \mathbb{N} : y \in \overline{\bigcup \mathcal{W}_n}\}$. Hence $f(X)$ has the A$\mathcal{I}$H property. □

6. The almost star-$\mathcal{I}$-Hurewicz property

Definition 6.1. *A space X is said to have the almost star-$\mathcal{I}$-Hurewicz property (in short, AS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n)}\} \in \mathcal{I}$.*

Theorem 6.2. *A space X has the AS$\mathcal{I}$H property if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by regular open sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}_n)}\} \in \mathcal{I}$.*

Proof. Suppose X has AS$\mathcal{I}$H property. Since every regular open set is open. Thus result follows. Conversely, let the $(\mathcal{U}_n : n \in \mathbb{N})$ be the sequence of covers of X by regular open sets. Consider $\mathcal{U}'_n = \{Int(\overline{U}) : U \in \mathcal{U}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}'_n$ is cover of X by regular open sets. Applying the hypothesis to the sequence $(\mathcal{U}'_n : n \in \mathbb{N})$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \in \mathcal{F}(\mathcal{I})$. For every $V \in \mathcal{V}_n$, choose $U_V \in \mathcal{U}_n$ such that $V = Int(\overline{U_V})$. Put $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}$ for some n. Then for every neighborhood U_x of x, $U_x \cap St(\bigcup \mathcal{V}_n, \mathcal{U}'_n) \neq \emptyset$. Then there exists $U \in \mathcal{U}_n$ such that $U_x \cap \{Int(\overline{U}) : Int(\overline{U}) \cap \bigcup \mathcal{V}_n \neq \emptyset\} \neq \emptyset$ which implies $U_x \cap \{U : U \cap \bigcup \mathcal{V}_n \neq \emptyset\} \neq \emptyset$. Now using the fact that $St(U, \mathcal{U}_n) = St(Int(\overline{U}), \mathcal{U}_n)$ for each $U \in \mathcal{U}_n$, we have $U_x \cap \{U : U \cap \bigcup \mathcal{W}_n \neq \emptyset\} \neq \emptyset$. Thus $x \in \overline{St(\bigcup \mathcal{W}_n, \mathcal{U}_n)}$ for some n. Therefore $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \subset \{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{W}_n, \mathcal{U}_n)}\}$. Hence X has the AS$\mathcal{I}$H property. □

Theorem 6.3. *The AS$\mathcal{I}$H property is preserved under almost continuous surjection mappings.*

Proof. Let $f : X \to Y$ is almost continuous surjection mapping and space X has AS$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of Y by regular open sets. Put $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then for each $n \in \mathbb{N}$, $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of open covers of X, since f is almost continuous surjection. Applying the AS$\mathcal{I}$H property of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \in \mathcal{F}(\mathcal{I})$. Put $\mathcal{V}'_n = \{U : f^{-1}(U) \in \mathcal{V}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}_n$ and $f^{-1}(\bigcup \mathcal{V}'_n) = \bigcup \mathcal{V}_n$. Let $x \in X$ then $f(x) = y$ for some $y \in f(X)$. Let $x \in \overline{St(f^{-1}(\bigcup \mathcal{V}'_n), \mathcal{U}'_n)}$ for some n. Then $y = f(x) \in f(\overline{St(f^{-1}(\bigcup \mathcal{V}'_n), \mathcal{U}'_n)}) \subset \overline{f(St(f^{-1}(\bigcup \mathcal{V}'_n), \mathcal{U}'_n))} \subset \overline{St(\bigcup \mathcal{V}'_n, \mathcal{U}_n)}$. For proving last inclusion, suppose that $f^{-1}(\bigcup \mathcal{V}'_n) \cap f^{-1}(U) \neq \emptyset$. Then $f(f^{-1}(\bigcup \mathcal{V}'_n)) \cap f(f^{-1}(U)) \neq \emptyset$ which implies the last inclusion $\bigcup \mathcal{V}_n \cap U \neq \emptyset$. Therefore $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{V}_n, \mathcal{U}'_n)}\} \subset \{n \in \mathbb{N} : y \in \overline{St(\bigcup \mathcal{V}'_n, \mathcal{U}_n)}\}$. Hence Y has the AS$\mathcal{I}$H property. □

The following lemma is immediate from the definitions.

Lemma 6.4. *For a topological space X and an admissible ideal $\mathcal{I}$, A$\mathcal{I}$H $\Rightarrow$ AS$\mathcal{I}$H and S$\mathcal{I}$H $\Rightarrow$ AS$\mathcal{I}$H.*

Theorem 6.5. *If a paracompact space has the AS$\mathcal{I}$H property then it has the A$\mathcal{I}$H property.*

Proof. Let X be a paracompact space having the AS$\mathcal{I}$H property and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. By the Stone characterization of paracompactness [8] for each $n \in \mathbb{N}$, $\mathcal{U}_n$ has an open star-refinement, say $\mathcal{V}_n$. Applying the AS$\mathcal{I}$H property of X to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that such that for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and each $x \in X$, $\{n \in \mathbb{N} : x \notin \overline{St(\bigcup \mathcal{W}_n, \mathcal{V}_n)}\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{W}_n, \mathcal{V}_n)}\} \in \mathcal{F}(\mathcal{I})$. For each $W \in \mathcal{W}_n$, let $U_W \in \mathcal{U}_n$ such that $St(W, \mathcal{V}_n) \subset U_W$. Let $\mathcal{H}_n = \{U_W : W \in \mathcal{W}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$. Let $x \in \overline{St(\cup \mathcal{W}_n, \mathcal{V}_n)}$ for some n. Then for every neighborhood U_x of x, $U_x \cap St(\cup \mathcal{W}_n, \mathcal{V}_n) \neq \emptyset$. Choose $W \in \mathcal{W}_n$ such that $U_x \cap St(W, \mathcal{V}_n) \neq \emptyset$ and $St(W, \mathcal{V}_n) \subset U_W$ which implies $U_x \cap U_W \neq \emptyset$. Thus $x \in \overline{U_W} \subset \bigcup \{\overline{U_W} : U_W \in \mathcal{H}_n\}$ for some $n \in \mathbb{N}$. Hence $\{n \in \mathbb{N} : x \in \overline{St(\bigcup \mathcal{W}_n, \mathcal{V}_n)}\} \subset \{n \in \mathbb{N} : x \in \overline{\bigcup \mathcal{H}_n}\}$. Therefore X has the A$\mathcal{I}$H property. □

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