Filomat 35:14 (2021), 4765–4776
https://doi.org/10.2298/FIL2114765A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

m-Convex Structure on b-Metric Spaces

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Abstract. We apply the concept of a m-convex b-metric space by introducing of m-convex structure on b-metric spaces. We obtain fixed point theorems in this structure. Recent results are concluded in our targets, as well. Some illustrated examples are presented to confirm our main results. As an application, we apply our main result to finding existence and uniqueness the solution of the Fredholm non-linear integral equation.

1. Introduction

Toader introduced the m-convexity in [27], as an intermediate among the general convexity and star shaped property. The concept of m-convex function play basic role in the theory of discrete convex analysis which has been used to mathematical economics.

We generalize m-convex structure on b-metric spaces. And we get some of very famous Theorems by this way. Some illustrated examples are presented to confirm our main results. As an application, we apply our main result to finding existence and uniqueness the solution of the Fredholm non-linear integral equation. For more detail refer to [1–4, 14, 16–18]

Definition 1.1 ([26]). *Let $m \in [0,1]$. Then the real number set $C \subseteq \mathbb{R}$ is said to be*

1. *convex if $tx + (1-t)y \in C$;*
2. *m-convex if $tx + (1-t)my \in C$;*

for all $x, y \in C$ and $t, m \in [0,1]$.

Definition 1.2 ([12, 23, 24, 26]). *Let $m \in [0,1]$ and $C \subseteq \mathbb{R}$. A function $f : C \subset \mathbb{R} \to \mathbb{R}$ is said to be an*

1. *convex, if C be a convex set and*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y);$$

2020 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10
Keywords. m-convex, b-metric space, fixed point theorems, convex structure
Received: 17 November 2020; Revised: 01 August 2021; Accepted: 13 October 2021
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2. *m-convex, if C be a m-convex set and*

$$f(tx + (1-t)my) \le tf(x) + (1-t)mf(y);$$

for all $x, y \in C$ *and* $t, m \in [0, 1]$.

Geometrically m-convex set C contains the line segment between the points x and my for every $x, y \in C$. Obviously see that a function$f : C \to \mathbb{R}$ is m-convex if for any $x, y \in C$,say $x \le y$, the segment between the points $(x, f(x))$ and $(my, mf(y))$ is above the graph of f in $[x, my]$ see Figure 1.

Definition 1.3 ([6]). *Under the hypotheses of Definition 1.2,*

1. f *is concave if* $-f$ *is convex;*
2. f *is m-concave if* $-f$ *is m-convex.*

Example 1.4. $f(x) = x^2$ *and* $f(x) = \|x\|$ *(on normed space) are m-convex set.* $f(x) = \sqrt{x}$ *is m-concave set.*

Example 1.5 ([24]). *Let* $f(x) = 2\ln x$ *on* $(0, \infty)$ *defines for any* $m \in [0, 1]$.
The set of $(0, \infty)$ *it isn't m-convex set for* $t = 1$ *and* $m = 0$*, and contradicts Definition* (1.2)*. If we choose* $f : [0, \infty)$ *wouldn't be defined. Therefore Example* (1.5) *of [24], isn't correct.*

Example 1.6. *Take the closed unit disk of the Euclidean space* $\mathbb{R}^2$*. It is convex set but it isn't m-convex set. Put:* $f(x, y) = -\|x - y\|^2$*. Then it is clear that* $f(x, \cdot)$ *is concave for any fixed* $x \in X$.

Remark 1.7. ([20, 24])

1. *Definition 1.2 is equivalent to* $f(m(1-t)x + ty) \le m(1-t)f(x) + tf(y)$.
2. *If* f *is a m-convex function and* $x = y = 0$ *in Definition 1.2, then* $f(0) \le 0$.
3. *From Definition 1.2 we clearly see that the* 1*-convex function is a convex function in the ordinary sense and the* 0*-convex function is the star shaped function. If we take* $m = 1$*, then we recapture the concept of convex functions. If we take* $t = 1$*, then we get* $f(my) \le mf(y)$ *for all* $x, y \in I$*, which implies that the function* f *is sub-homogeneous.*
4. *If* f *was convex function and* $m = 1$*, it would be m-convex function.*

Lemma 1.8. ([11, 20])

1. *If* $f : C \to \mathbb{R}$ *is m-convex and* $0 \le n < m \le 1$*, then* f *is n-convex.*
2. *Let* $f, g : [a, b] \to \mathbb{R}$*,* $a \ge 0$*. If* f *is n-convex and* g *is m-convex, with* $n \le m$*, then* $f + g$ *and* αf*,* $\alpha \ge 0$ *a constant, are n-convex.*
3. *Let* $f : [0, a] \to \mathbb{R}$*,* $g : [0, b] \to \mathbb{R}$*, with* $\text{renge}(f) \subseteq [0, b]$*. If* f *and* g *are m-convex and* g *is increasing, then* $g \circ f$ *is m-convex on* $[0, a]$.
4. *If* $f, g : [0, a] \to \mathbb{R}$ *are both nonnegative, increasing and m-convex, then* fg *is m-convex.*

Definition 1.9 ([24]). $f : [a, b] \to \mathbb{R}$ *is said to be star shaped if*

$$f(tx) \le tf(x)$$

for all $t \in [0, 1]$ *and* $x \in [a, b]$.

For the concept of generalized convexity sets, let

$$\text{co} = \{f : f \text{ is convex}\},$$
$$\text{co}_m = \{f : f \text{ is } m-\text{convex}\}.$$

We have $\text{co}_m \subsetneq \text{co}$, since

$$f \in \text{co} \setminus \text{co}_m, \text{ where } f(x) = \begin{cases} 1 - x, & 0 \le x \le 1; \\ \frac{-x+1}{2}, & 1 \le x \le 3, \end{cases}$$

just enough we put $t = 0.2$ and $m = 0$, see Figure 2.

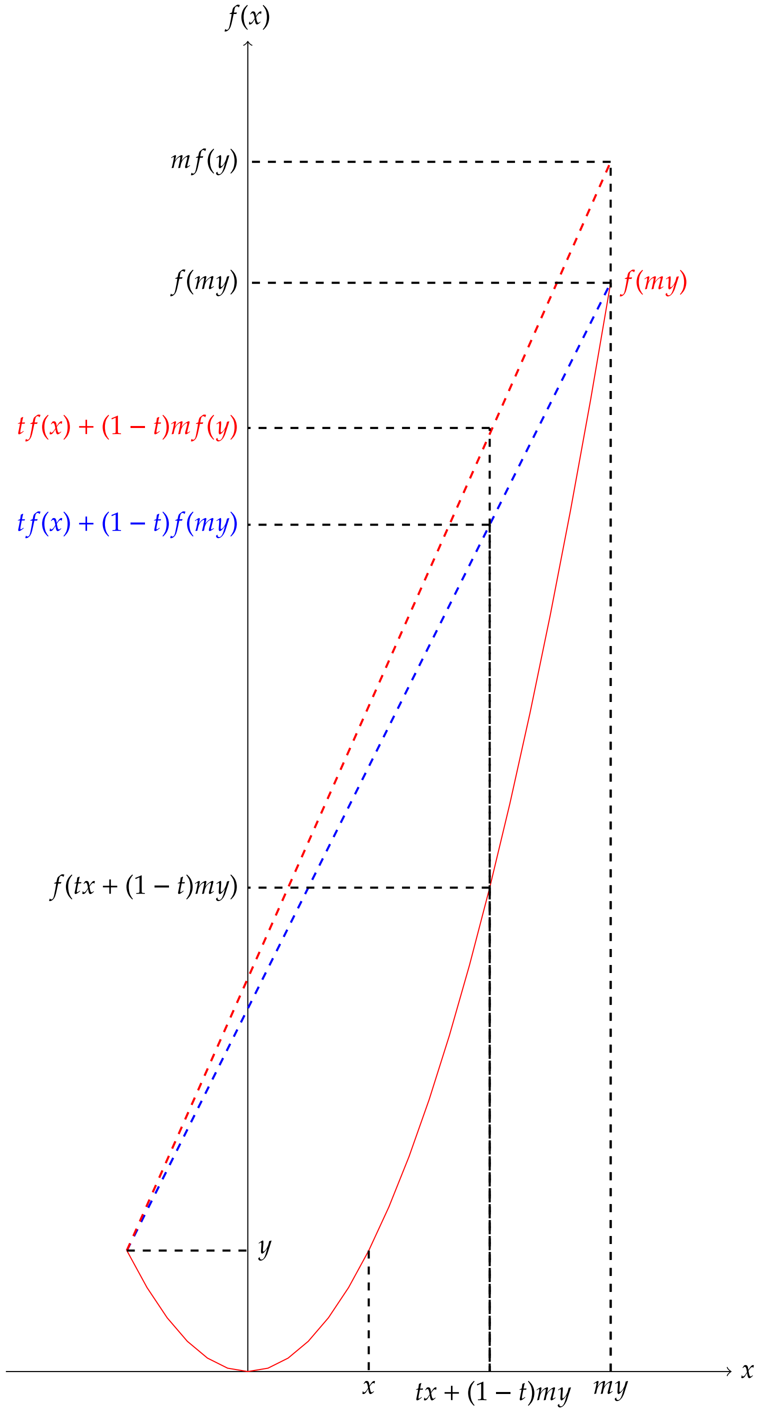

Figure 1: Illustration for Definition 1.2

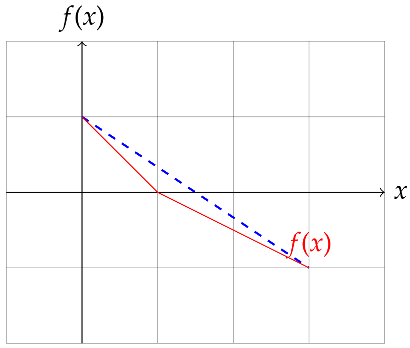

Figure 2: Illustration for Section 3:"Concept of Generalized Convexity Sets"

Example 1.10. *Let* $B_1 = \{\lambda = (\lambda_1, \ldots, \lambda_n, \ldots) \in l^1; \|\lambda\|_1 \leq 1\}$.
$L = \{\lambda = (\lambda_1, \ldots, \lambda_n, \ldots) \in l^1 : \lambda_i \geq 0,\ \forall\, i \in \mathbb{N} \text{ and } \|\lambda\|_1 = \sum_{i=1}^{\infty} \lambda_i = 1\}$.
Obviously $\partial B_1 = L$ *is convex set but it isn't m-convex set. It is easily seen that* B_1 *is m-convex set.*

Now we present some notations and definitions which will be used in sequel. For more details refer to [5, 7, 8, 10, 13, 15, 21, 22].

Definition 1.11. *Let* X *be a non-empty set and* $d_b : X \times X \to [0, \infty)$ *be a mapping such that for all* $x, y, z \in X$ *and for some real number* $s \geq 1$,

(i) $d_b(x, y) = 0 \iff x = y$;

(ii) $d_b(x, y) = d_b(y, x)$;

(iii) $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$.

Then (X, d_b) *is called a b-metric space with parameter* $s \geq 1$.

When $s = 1$ the definition of metric space is attained.

Definition 1.12. *Let* (X, d_b) *be a b-Branciari metric space and* $\{x_n\}$ *be a sequence in* X *and* $x \in X$.

(a) A sequence $\{x_n\}$ *in* X *is said to converge to* $x \in X$ *if*

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } d_b(x_n, x) < \epsilon, \quad \forall n > N.$$

Show

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \text{ as } n \to \infty.$$

(b) A sequence $\{x_n\}$ *in* X *is said to be a Cauchy sequence if*

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } d_b(x_n, x_{n+p}) < \epsilon \quad \forall n > N, p > 0$$

or equivalently, if $\lim_{n\to\infty} d_b(x_n, x_{n+p}) = 0$ *for all* $p > 0$.

(c) (X, d_b) *is complete if and only if every Cauchy sequence in* X *converges to some element in* X.

It should be noted some warnings about b-Branciari metric spaces:

(1) The limit of the sequence in a b-Branciari metric spaces is not necessarily unique.

(2) A convergent sequence in a b-Branciari metric spaces may not be a Cauchy sequence.

(3) A Branciari b-metric may not be continuous.

Definition 1.13. *Let $X \neq \emptyset$ and $I = [0,1]$. Define the mapping $d_b : X \times X \to [0,\infty)$ and a continuous function $w : X \times X \times J \times I \to X$. Then w is said to be the m-convex structure on X if the following holds:*

$$d_b(z, w(x, y; t, m)) \leq t d_b(z, x) + (1-t) m d_b(z, y) \tag{1}$$

for each $z \in X$ and $(x, y; t, m) \in X \times X \times J \times I$, where $J \subseteq I$.

We note that $d_b(z, w(x, y; 0, m)) \leq m d_b(z, y)$ for every $x, y, z \in X$ and $m \in [0,1]$.

2. Main Results

In this section, we begin with the definition of a m-convex b-metric space.

Definition 2.1. *Let the mapping $w : X \times X \times J \times I \to X$ be a m-convex structure on a b-metric space (X, d_b) with constant $s \geq 1$ and $J \subseteq I = [0,1]$. Then (X, d_b, w) is said to be a m-convex b-metric space.*

Proposition 2.2. *[19] Let $\{x_n\}$ be a Cauchy sequence in a Branciari metric space (X, d_b) such that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X$. Then $\lim_{n\to\infty} d_b(x_n, y) = d_b(x, y)$, for all $y \in X$. In particular, the sequence $\{x_n\}$ dose not converge to y if $y \neq x$.*

Remark 2.3. *If we replace Branciari metric by b-Branciari metric in proposition (2.2) , the proposition is still valid.*

Example 2.4. *Let $X := \{1, 2, 3\}$. Define d_b by*

$$\begin{aligned} d_b(x,x) &= 0 \\ d_b(1,2) &= d_b(2,1) = 1 \\ d_b(1,3) &= d_b(3,1) = 0 \\ d_b(3,2) &= d_b(2,3) = 6. \end{aligned}$$

d_b is a b-metric with $s = 6$.

$$\begin{aligned} &1 = d_b(1,2) \leq 6(d_b(1,3) + d_b(3,2)) = 6(0+6) \\ &0 = d_b(1,3) \leq 6(d_b(1,2) + d_b(2,3)) = 6(1+6) \\ &6 = d_b(2,3) \leq 6(d_b(2,1) + d_b(1,3)) = 6(1+0). \end{aligned}$$

But it isn't ordinary metric, because

$$6 = d_b(3,2) > d_b(3,1) + d_b(1,2) = 0 + 1.$$

b-metric d_b has $\frac{1}{3}$-convex structure. Also, if we put $x \in X$ and $\frac{y}{3} \in X$ which means $y = 3$, then

$$d_b(x, my) = d_b\Big(x, \frac{y}{3}\Big) \leq \frac{1}{3} d_b(x, y).$$

Also, we take $t = \frac{1}{2}$

$$\begin{aligned} w(x, y; \frac{1}{2}, \frac{1}{3}) &:= tx + (1-t)my = \frac{3x+y}{3} \\ d_b(z, w(x, y; \frac{1}{2}, \frac{1}{3})) &= d_b(z, tx + (1-t)my) \leq t d_b(z, x) + (1-t) m d_b(z, y) \\ d_b(z, \frac{3x+y}{3}) &\leq \frac{1}{2} d_b(z, x) + \frac{1}{6} d_b(z, y), \end{aligned}$$

$$0 = d_b(1,1) = d_b(1, \frac{3\times 1+3}{3}) \le \frac{1}{2}d_b(1,1) + \frac{1}{6}d_b(1,3) = 0,$$
$$1 = d_b(2,1) = d_b(2, \frac{3\times 1+3}{3}) \le \frac{1}{2}d_b(2,1) + \frac{1}{6}d_b(2,3) = 1,$$
$$0 = d_b(3,1) = d_b(1, \frac{3\times 1+3}{3}) \le \frac{1}{2}d_b(3,1) + \frac{1}{6}d_b(3,3) = 0,$$

Example 2.5. *Let* $d_b : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ *by*

$$d_b(x,y) = |x-y|^p \quad p > 1$$

and

$$w(x,y;\frac{1}{2},1) = \frac{x+y}{2}.$$

Then, $(\mathbb{R}, d_b, w)$ *is a* 1*-convex b-metric space with* $s = 2^{p-1}$*. However,* $(\mathbb{R}, d_b, w)$ *is not a metric space in the usual sense.*

Now we shall prove Banach's contraction principle for complete m-convex b-metric spaces by means of Mann's iteration algorithm.

Theorem 2.6. *Let* (X, d_b, w) *be a complete m-convex b-metric space with constant* $s > 1$*. If T be a self mapping on X satisfying in the following conditions:*

- *there exists* $k \in [0,1)$ *such that*

$$d_b(Tx,Ty) \le kd_b(x,y), \quad \forall x,y \in X. \tag{2}$$

- $x_0 \in X$ *with* $d_b(x_0, Tx_0) = M < \infty$
- $x_n := w(x_{n-1}, Tx_{n-1}; t_{n-1}, m)$*, where* $0 \le t_{n-1} < 1$ *and* $n \in \mathbb{N}$*.*
- $kms^4 < 1$ *and* $0 < t_{n-1} < \frac{\frac{1}{s^4}-km}{1-km}$ *for each* $n \in \mathbb{N}$*.*

Then, T has a unique fixed point in X.

Proof.

$$\begin{aligned} d_b(x_n, x_{n+1}) &= d_b(x_n, w(x_n, Tx_n; t_n, m)) \\ &\le (1-t_n)md_b(x_n, Tx_n) \end{aligned}$$

and

$$\begin{aligned} d_b(x_n, Tx_n) &\le sd_b(x_n, Tx_{n-1}) + sd_b(Tx_{n-1}, Tx_n) \\ &\le sd_b(w(x_{n-1}, Tx_{n-1}; t_{n-1}, m), Tx_{n-1}) + skd_b(x_{n-1}, x_n) \\ &\le st_{n-1}d_b(x_{n-1}, Tx_{n-1}) + skd_b(x_{n-1}, w(x_{n-1}, Tx_{n-1}; t_{n-1}, m)) \\ &\le st_{n-1}d_b(x_{n-1}, Tx_{n-1}) + sk(1-t_{n-1})md_b(x_{n-1}, Tx_{n-1}) \\ &\le s(t_{n-1} + k(1-t_{n-1})m)d_b(x_{n-1}, Tx_{n-1}). \end{aligned}$$

Put $\mu_{n-1} = s(t_{n-1} + k(1-t_{n-1})m)$. By $kms^4 < 1$ and $0 < t_{n-1} < \frac{\frac{1}{s^4}-km}{1-km}$ we have

$$d_b(x_n, Tx_n) \le \mu_{n-1}d_b(x_{n-1}, Tx_{n-1}) \le \frac{1}{s^3}d_b(x_{n-1}, Tx_{n-1}). \tag{3}$$

So $\{d_b(x_n, Tx_n)\}$ is convergent to β. By takeing limit from (3) we find $\beta = 0$, because

$$\beta \le \frac{1}{s^3}\beta < \beta.$$

Also $\lim_{n\to\infty} d_b(x_n, x_{n+1}) = 0$ since

$$d_b(x_n, x_{n+1}) \le (1 - t_n)md_b(x_n, Tx_n).$$

It is time to show that $\{x_n\}$ is a Cauchy sequence. If not, let n_l be the smallest natural index with that $n_l > m_l > k$,

$$\exists\epsilon_0 > 0\ \exists\{x_{n_l}\}, \{x_{m_l}\} \subseteq \{x_n\} \quad d_b(x_{n_l}, x_{m_l}) \ge \epsilon_0,\ d_b(x_{n_l-1}, x_{m_l}) < \epsilon_0.$$

So

$$\begin{aligned}
\epsilon_0 &\le d_b(x_{n_l}, x_{m_l}) \le s(d_b(x_{n_l}, x_{m_l+1}) + d_b(x_{m_l+1}, x_{m_l})) \\
\Rightarrow \frac{\epsilon_0}{s} &\le \limsup_{l\to\infty} d_b(x_{n_l}, x_{m_l+1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
& d_b(x_{n_l}, x_{m_l+1}) = d_b(w(x_{n_l-1}, Tx_{n_l-1}; t_{n_l-1}, m), x_{m_l+1}) \\
&\le t_{n_l-1}d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1})md_b(Tx_{n_l-1}, x_{m_l+1}) \\
&\le t_{n_l-1}d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1})ms(d_b(Tx_{n_l-1}, Tx_{m_l+1}) + d_b(Tx_{m_l+1}, x_{m_l+1})) \\
&\le t_{n_l-1}d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1})ms(kd_b(x_{n_l-1}, x_{m_l+1}) + d_b(Tx_{m_l+1}, x_{m_l+1})) \\
&\le (t_{n_l-1} + (1 - t_{n_l-1})msk)d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1})msd_b(Tx_{m_l+1}, x_{m_l+1}) \\
&\le (t_{n_l-1} + (1 - t_{n_l-1})msk)s(d_b(x_{n_l-1}, x_{m_l}) + d_b(x_{m_l}, x_{m_l+1})) + (1 - t_{n_l-1})msd_b(Tx_{m_l+1}, x_{m_l+1}) \\
&\le (t_{n_l-1}s + (1 - t_{n_l-1})msk)s(d_b(x_{n_l-1}, x_{m_l}) + d_b(x_{m_l}, x_{m_l+1})) + (1 - t_{n_l-1})msd_b(Tx_{m_l+1}, x_{m_l+1}) \\
&\le \frac{1}{s^2}(d_b(x_{n_l-1}, x_{m_l}) + d_b(x_{m_l}, x_{m_l+1})) + (1 - t_{n_l-1})msd_b(Tx_{m_l+1}, x_{m_l+1})
\end{aligned}$$

therefore

$$\frac{\epsilon_0}{s} \le \limsup_{l\to\infty} d_b(x_{n_l}, x_{m_l+1}) \le \frac{1}{s^2}\epsilon_0 < \epsilon_0,$$

which means $\{x_n\}$ is a Cauchy sequence in X. Thus $x_n \to x^*$ for some $x^* \in X$.

$$\begin{aligned}
d_b(x^*, Tx^*) &\le s(d_b(x^*, x_n) + d_b(x_n, Tx^*)) \\
&\le sd_b(x^*, x_n) + s^2d_b(x_n, Tx_n) + s^2d_b(Tx_n, Tx^*) \\
&\le sd_b(x^*, x_n) + s^2d_b(x_n, Tx_n) + s^2kd_b(x_n, x^*)
\end{aligned}$$

which implies that $x^* = Tx^*$. Uniqueness is clear. □

As a corollary, [9, Theorem 1] is one of our results.

Example 2.7. *Let all hypothesis of Example 2.4 are hold. define $T : X \to X$ by*

$$T(1) = 3, T(2) = 1 \text{ and } T(3) = 3,$$

so T satisfies in (2) *for every $k < \frac{3}{6^4}$. We notice that 3 is a unique fixed point of T in X.*

Next theorem is the Kannan type fixed point theorem for a complete m-convex b-metric space.

Theorem 2.8. *Let (X, d_b, w) be a complete m-convex b-metric space with constant $s > 1$ and $T : X \to X$ be a contraction mapping; that is, there exists $k \in [0, 1)$ such that*

$$\exists k \in [0, \frac{1}{2}) \quad d_b(Tx, Ty) \le k[d_b(x, Tx) + d_b(y, Ty)], \quad \forall x, y \in X. \tag{4}$$

Choose $x_0 \in X$ with $d_b(x_0, Tx_0) = M < \infty$ and define $x_n := w(x_{n-1}, Tx_{n-1}; t_{n-1}, m)$, where $0 \le t_{n-1} < 1$ and $n \in \mathbb{N}$. If $0 \le k \le \frac{1}{4s^2+(1-m)}$ and $0 < t_{n-1} \le \frac{1}{4s^2+(1-m)}$ for each $n \in \mathbb{N}$. Then, T has a unique fixed point in X.

Proof.

$$d_b(x_n, x_{n+1}) = d_b(x_n, w(x_n, Tx_n; t_n, m)) \leq (1 - t_n)md_b(x_n, Tx_n) \tag{5}$$

and

$$\begin{aligned} & d_b(x_n, Tx_n) = d_b(w(x_{n-1}, Tx_{n-1}; t_{n-1}, m), Tx_n) \\ \leq \; & t_{n-1}d_b(x_{n-1}, Tx_n) + (1 - t_{n-1})md_b(Tx_{n-1}, Tx_n) \\ \leq \; & t_{n-1}s(d_b(x_{n-1}, Tx_{n-1}) + d_b(Tx_{n-1}, Tx_n)) + (1 - t_{n-1})md_b(Tx_{n-1}, Tx_n) \\ \leq \; & t_{n-1}sd_b(x_{n-1}, Tx_{n-1}) + (t_{n-1}s + (1 - t_{n-1})m)d_b(Tx_{n-1}, Tx_n) \\ \leq \; & t_{n-1}sd_b(x_{n-1}, Tx_{n-1}) + (t_{n-1}s + (1 - t_{n-1})m)k(d_b(x_{n-1}, Tx_{n-1}) + d_b(x_n, Tx_n)) \end{aligned}$$

so

$$(1 - (t_{n-1}s + (1 - t_{n-1})m)k)d_b(x_n, Tx_n) \leq (t_{n-1}s + (t_{n-1}s + (1 - t_{n-1})m)k)d_b(x_{n-1}, Tx_{n-1}) \tag{6}$$

hence

$$\begin{aligned} L &:= (t_{n-1}s + (1 - t_{n-1})m)k \\ &= (t_{n-1}(s - m) + m)k \\ &< (\frac{s - m}{4s^2 + (1 - m)} + 1)\frac{1}{4s^2 + (1 - m)} \\ &< (\frac{s}{4s^2} + 1)\frac{1}{4s^2} \\ &< (\frac{1}{4} + 1)\frac{1}{4} \\ &< \frac{5}{16} < 1. \end{aligned}$$

Put

$$\begin{aligned} \mu_{n-1} &:= \frac{t_{n-1}s + (t_{n-1}s + (1 - t_{n-1})m)k}{(1 - (t_{n-1}s + (1 - t_{n-1})m)k)} \\ &= \frac{t_{n-1}s + L}{1 - L} < 1, \end{aligned}$$

since

$$t_{n-1}s + L < 1 - L \iff t_{n-1}s + 2L < 1$$

so

$$\begin{aligned} t_{n-1}s + 2L &\leq \frac{s}{4s^2 + 1 - m} + 2\frac{5}{16} \\ &< \frac{1}{4} + 2\frac{5}{16} = \frac{14}{16} < 1. \end{aligned}$$

We deduce that

$$d_b(x_n, Tx_n) \leq \mu_{n-1}d_b(x_{n-1}, Tx_{n-1}). \tag{7}$$

So $\{d_b(x_n, Tx_n)\}$ is convergent to β. By (7) we get $\beta = 0$. Also by (5) $d_b(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ does not be a Cauchy sequence, then, choose n_l be the smallest natural index that $n_l > m_l > k$,

$$\exists\epsilon_0 > 0 \; \exists\{x_{n_l}\}, \{x_{m_l}\} \subseteq \{x_n\} \quad d_b(x_{n_l}, x_{m_l}) \geq \epsilon_0, \; d_b(x_{n_l-1}, x_{m_l}) < \epsilon_0.$$

So

$$\begin{aligned} \epsilon_0 &\leq d_b(x_{n_l}, x_{m_l}) \leq s(d_b(x_{n_l}, x_{m_l+1}) + d_b(x_{m_l+1}, x_{m_l})) \\ \Rightarrow \frac{\epsilon_0}{s} &\leq \limsup_{l\to\infty} d_b(x_{n_l}, x_{m_l+1}). \end{aligned}$$

On the other hand

$$\begin{aligned} &d_b(x_{n_l}, x_{m_l+1}) = d_b(w(x_{n_l-1}, Tx_{n_l-1}; t_{n_l-1}, m), x_{m_l+1}) \\ &\leq t_{n_l-1} d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1}) m d_b(Tx_{n_l-1}, x_{m_l+1}) \\ &\leq t_{n_l-1} d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1}) ms(d_b(Tx_{n_l-1}, Tx_{m_l+1}) + d_b(Tx_{m_l+1}, x_{m_l+1})) \\ &\leq t_{n_l-1} d_b(x_{n_l-1}, x_{m_l+1}) + (1 - t_{n_l-1}) ms(k d_b(x_{n_l-1}, Tx_{n_l-1}) + (k+1) d_b(Tx_{m_l+1}, x_{m_l+1})) \\ &\leq t_{n_l-1} s(d_b(x_{n_l-1}, x_{m_l}) + d_b(x_{m_l}, x_{m_l+1})) \\ &+ (1 - t_{n_l-1}) ms(k d_b(x_{n_l-1}, Tx_{n_l-1}) + (k+1) d_b(Tx_{m_l+1}, x_{m_l+1})) \end{aligned}$$

we obtain

$$\limsup_{l\to\infty} d_b(x_{n_l}, x_{m_l+1}) \leq \frac{1}{4s^2 + 1 - m} s\epsilon_0 < \frac{\epsilon_0}{4s} < \frac{\epsilon_0}{s}.$$

So $x_n \to x^*$ for some $x^* \in X$. We shall show that u^* is a fixed point of T.

$$\begin{aligned} d_b(x^*, Tx^*) &\leq s(d_b(x^*, x_n) + d_b(x_n, Tx^*)) \\ &\leq s d_b(x^*, x_n) + s^2(d_b(x_n, Tx_n) + d_b(Tx_n, Tx^*)) \\ &\leq s d_b(x^*, x_n) + s^2 d_b(x_n, Tx_n) + s^2 k(d_b(x_n, Tx_n) + d_b(x^*, Tx^*)) \end{aligned}$$

we conclude

$$(1 - s^2 k) d_b(x^*, Tx^*) \leq s d_b(x^*, x_n) + (s^2 + s^2 k) d_b(x_n, Tx_n).$$

The uniqueness of the fixed point is clear.

If put $m = 1$. Then as a corollary, [9, Theorem 2] is especial case of Theorem 2.8. □

3. Applications

In order to show the existence and uniqueness of the solution integral equation, consider:

$$u(t) = f(t) + \lambda \int_a^b K(t, u(\tau)) d\tau, \tag{8}$$

on $X := C[a, b]$ with sup norm, and define d_b as follows:

$$d_p(u, v) = \|u - v\|^2 = \max_{t\in[a,b]} |u(t) - v(t)|^2.$$

in this case d_b is a b-metric with $s = 2$. Let

$$Tu(t) = f(t) + \lambda \int_a^b K(t, u(\tau)) d\tau \tag{9}$$

and for $m \in [0, 1]$

$$M_m := \left\{(f, g) \in X \times X | f = \frac{2m}{m+1} g\right\}.$$

We note $(0, 0), (f, f), (\frac{2m}{m+1}, 1) \in M_m \neq \emptyset$. When $(f, g) \in M_m$, then

$$|f(t) - mg(t)| = m|f(t) - g(t)|,$$

for $t \in [a, b]$.

Therefore d_b has a m-convex structure if we consider

$$w : X \times X \times \{\frac{1}{2}\} \times [0, 1] \to X,$$

$$w(f, g; \frac{1}{2}, m) = \frac{1}{2} f + (1 - \frac{1}{2}) mg.$$

Because

$$\begin{aligned}
d_b(h, w(f, g; \frac{1}{2}, m)) &= d_b(h, \frac{1}{2} f + (1 - \frac{1}{2}) mg) \\
&= \|h - (\frac{1}{2} f + (1 - \frac{1}{2}) mg\|^2 \\
&= \|\frac{1}{2} h + (1 - \frac{1}{2}) h - (\frac{1}{2} f + (1 - \frac{1}{2}) mg\|^2 \\
&\leq 2(\frac{1}{4} \|h - f\|^2 + (1 - \frac{1}{2})^2 \|h - mg\|^2) \\
&\leq \frac{1}{2} \|h - f\|^2 + \frac{1}{2} m^2 \|h - g\|^2 \\
&\leq \frac{1}{2} \|h - f\|^2 + \frac{1}{2} m \|h - g\|^2 \\
&\leq \frac{1}{2} d_b(h, f) + (1 - \frac{1}{2}) m d_b(h, g)
\end{aligned}$$

Theorem 3.1. *Consider the integral Equation* (9) *with*

1. *$K(t, s)$ is a continuous function;*
2. *$|K(t, s_1) - K(t, s_2)| \leq L(t)|s_1 - s_2|$;*
3. *$f \in C[a, b]$;*
4. *$N := \max_{t \in [a,b]} L(t)$;*
5. *$\lambda^2 (b - a)^3 N^2 < 1$.*

Then the linear integral Equation (9) *has a unique solution on the interval* $[a, b]$.

Proof.

$$\begin{aligned}
d_b(Tu, Tv) &= \|Tu - Tv\|^2 \\
&= \max_{t \in [a,b]} |Tu(t) - Tv(t)|^2 \\
&= \max_{t \in [a,b]} \left| \lambda \int_a^b K(t, u(\tau)) d\tau - \lambda \int_a^b K(t, v(\tau)) d\tau \right|^2 \\
&= \max_{t \in [a,b]} \lambda^2 \left| \int_a^b (K(t, u(\tau)) - K(t, v(\tau))) d\tau \right|^2 \\
&\leq \max_{t \in [a,b]} \lambda^2 \left(\int_a^b d\tau \right)^2 \left(\int_a^b |K(t, u(\tau)) - K(t, v(\tau))| d\tau \right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{t\in[a,b]} \lambda^2(b-a)^2N^2\left(\int_a^b |u(\tau)-v(\tau)|d\tau\right)^2\\
&\leq \max_{t\in[a,b]} \lambda^2(b-a)^3N^2\left(|u(t)-v(t)|\right)^2\\
&\leq \max_{t\in[a,b]} \lambda^2(b-a)^3N^2\|u(t)-v(t)\|^2\\
&\leq \lambda^2(b-a)^3N^2d_b(u,v)\\
&\leq \mu d_b(u,v),
\end{aligned}
$$

where $\mu = \lambda^2(b-a)^3N^2 < 1$. Meanwhile, by Theorem 2.6, T has a unique fixed point $u \in X$. □

Conclusion

In this paper, by applying the concept of a m-convex b-metric space and introducing of m-convex structure on b-metric spaces, we obtain fixed point theorems in this structure. Recent recognized results are obtained as our corollaries, as well. Example 1.4 states that the example [**?**] i.e., the set of $(0,\infty)$ it isn't m-convex set. Many illustrated examples and an application are presented. Our result applied to finding existence and uniqueness the solution of the Fredholm **non-linear** integral equation.

Acknowledgements

The authors express their deep gratitude to the referees for their valuable comments and suggestions.

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