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Composition Results of Stepanov (μ, ν)-Pseudo Almost Automorphic Functions

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Abstract. In this work, we give sufficient conditions ensuring that the space $S^pPAA(\mathbb{R}, X, \mu, \nu)$ of (μ, ν)-pseudo almost automorphic functions in Stepanov's sense is invariant by translation and we provide new composition theorems of (μ, ν)-pseudo almost automorphic functions in the sense of Stepanov.

1. Introduction

The notion of almost automorphy introduced by Bochner [7] is not restricted just to continuous functions. One can generalize that notion to measurable functions with some suitable conditions of integrability, namely, Stepanov almost automorphic functions, see [12]. Details can be found in [2–6, 9, 10, 12, 13]
Now, throughout this work $(\mathbb{H}, \|.\|)$ is a Banach space. The notation $C(\mathbb{R}, \mathbb{H})$ stands for the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{H}$. We denote by $BC(\mathbb{R}, \mathbb{H})$ is the space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{H}$ endowed with the supremum norm defined by

$$\|x\|_{BC(\mathbb{R},\mathbb{H})} := sup_{t\in\mathbb{R}}\{\|x(t)\|\}.$$

Furthermore, $BC(\mathbb{R} \times \mathbb{H}, \mathbb{H})$ is the space of all bounded continuous functions $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$.

Definition 1.1. *[8] Let $p \in [1; +\infty)$. The space $\mathcal{BS}^p(\mathbb{R}; \mathbb{H})$ of all bounded functions in Stepanov's sense, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{H}$ such that $\|f\|_{BS^p} := \sup_{t\in\mathbb{R}}(\int_t^{t+1} |f(s)|^p ds)^{\frac{1}{p}} < \infty$. This is a Banach space when it is equipped with the norm $\|f\|_{BS^p}$.*

Remark 1.2. *$f \in \mathcal{BS}^p(\mathbb{R}; \mathbb{H})$ iff $f^b \in L^\infty(\mathbb{R}, L^p([0, 1], \mathbb{H}))$, with f^b is is the Bochner transform of f defined by $f^b : \mathbb{R} \longrightarrow L^p([0, 1], \mathbb{H})$, $f^b(t)(s) = f(t + s), \forall(t, s) \in \mathbb{R} \times [0, 1]$. And $\|f\|_{BS^p} = \|f^b\|_\infty$.*

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2. Almost automorphic functions

Definition 2.1. *[11], [Definition 1.29] A continuous function $f : \mathbb{R} \to \mathbb{H}$ is called almost automorphic if for every sequence $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n,m\to+\infty} f(t + s_n - s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

Equivalently,

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined and

$$\lim_{n\to\infty} g(t - s_n) = f(t),$$

for all $t \in \mathbb{R}$.

Let $AA(\mathbb{R}, \mathbb{H})$ denote the collection of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{H}$

Definition 2.2. *[6] A function $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ is said to be almost automorphic in t uniformly with respect to x in $\mathbb{H}$ if the following two conditions hold:*
(i) for all $x \in \mathbb{H}$, $f(., x) \in AA(\mathbb{R}, \mathbb{H})$,
(ii) f is uniformly continuous on each compact set K in $\mathbb{H}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{H}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, \ x_2 \ \in K$, one has

$$\|x_1 - x_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \le \varepsilon.$$

Denote by $AAU(\mathbb{R} \times \mathbb{H}, \mathbb{H})$ the set of all such functions.

Definition 2.3. *[12] Let $p \in [1; +\infty)$. A function $f \in \mathcal{BS}^p(\mathbb{R}; \mathbb{H})$ is said to be S^p-almost automorphic if its Bochner transform $f^b \in AA(\mathbb{R}, L^p([0, 1], \mathbb{H}))$.*
Denote by $AA^p(\mathbb{R}, \mathbb{H})$ the set of all such functions.
The following remark is immediate.

Remark 2.4. *The map $B : (\mathcal{BS}^p(\mathbb{R}, \mathbb{H}), \|.\|_{\mathcal{BS}^p}) \longrightarrow L^\infty(\mathbb{R}, L^p([0, 1], \mathbb{H}))$, $f \longmapsto f^b$ is a linear isometry, in particular it is continuous.*

Definition 2.5. *[6] A function $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ is said to be S^p-almost automorphic in t uniformly with respect to x in $\mathbb{H}$ if the following two conditions hold:*
(i) for all $x \in \mathbb{H}$, $f(., x) \in AA^p(\mathbb{R}, \mathbb{H})$,
(ii) $f^b : \mathbb{R} \times \mathbb{H} \longrightarrow L^p([0, 1], \mathbb{H})$; $f^b(t, x)(s) = f(t + s, x)$ is uniformly continuous on each compact set K in $\mathbb{H}$ with respect to the second variable x, namely, for each compact set K in $\mathbb{H}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_1, \ x_2 \ \in K$, one has

$$\|x_1 - x_2\| \le \delta \Rightarrow \sup_{t\in\mathbb{R}} \Big(\int_0^1 \|f(t + s, x_1) - f(t + s, x_2)\|^p ds\Big)^{\frac{1}{p}} \le \varepsilon.$$

Denote by $AA^pU(\mathbb{R} \times \mathbb{H}, \mathbb{H})$ the set of all such functions.

3. Ergodic functions

Let $\mathcal{B}$ denote the Lebesgue σ-field of $\mathbb{R}$ and let $\mathcal{M}$ be the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \le b)$. From now on, $\mu, \nu \in \mathcal{M}$.

Definition 3.1. *[3] A function $f : \mathbb{R} \longrightarrow \mathbb{H}$ is said to be $(\mu, \nu)-$ergodic if*

$$\lim_{r\to\infty} \frac{1}{\nu([-r, r])} \int_{-r}^{r} \|f(s)\| d\mu(t) = 0.$$

We then denote the set of all such functions by $\mathcal{E}(\mathbb{R}, \mathbb{H}, \mu, \nu)$.

Definition 3.2. *[13] A function $f \in \mathcal{BS}^p(\mathbb{R},\mathbb{H})$ is said to be $\mathcal{S}^p-(\mu,\nu)-$ergodic if*

$$\lim_{r\to\infty}\frac{1}{\nu([-r,r])}\int_{-r}^{r}(\int_{t}^{t+1}\|f(s)\|^p ds)^{\frac{1}{p}}d\mu(t)=0$$

Equivalently, $f^b \in \mathcal{E}(\mathbb{R}, L^p([0,1],\mathbb{H}),\mu,\nu)$.

We then denote the collection of all such functions by $\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$.

Definition 3.3. *A $f:\mathbb{R}\times\mathbb{H}\to\mathbb{H}$ is said to be $\mathcal{S}^p$-(μ,ν)-ergodic in t uniformly with respect to $x\in\mathbb{H}$ if the following conditions are satisfied:*
(i) For all $x\in\mathbb{H}$, $f(.,x)\in\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$.
(ii) $f^b:\mathbb{R}\times\mathbb{H}\longrightarrow L^p([0,1],\mathbb{H})$; $f^b(t,x)(s)=f(t+s,x)$ is uniformly continuous on each compact set K in $\mathbb{H}$ with respect to the second variable $x\in\mathbb{H}$.

The set of such function is denoted by $\mathcal{E}^pU(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$.

4. Pseudo almost automorphic functions

Definition 4.1. *A continuous function $f:\mathbb{R}\to\mathbb{H}$ is said to be (μ,ν)-pseudo almost automorphic if it is written in the form*

$$f=g+h,$$

where $g\in AA(\mathbb{R},\mathbb{H})$ and $h\in\mathcal{E}(\mathbb{R},\mathbb{H},\mu,\nu)$. The set of such functions is denoted by $PAA(\mathbb{R},\mathbb{H},\mu,\nu)$.

Definition 4.2. *A continuous function $f:\mathbb{R}\times\mathbb{H}\to\mathbb{H}$ is said to be (μ,ν)-pseudo almost automorphic in the first variable uniformly with respect to the second variable if is written in the form*

$$f=g+h,$$

where $g\in AAU(\mathbb{R}\times\mathbb{H},\mathbb{H})$ and $h\in\mathcal{E}U(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$. The set of such functions is denoted by $PAAU(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$.

Definition 4.3. *A function $f\in\mathcal{BS}^p(\mathbb{R}\to\mathbb{H})$ is said to be $\mathcal{S}^p-(\mu,\nu)$-pseudo almost automorphic if it can be written in the form*

$$f=g+h,$$

where $g\in AA^p(\mathbb{R},\mathbb{H},\mu)$ and $h\in\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$. The set of such functions will be denoted by $PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$.

Definition 4.4. *A function $f:\mathbb{R}\times\mathbb{H}\to\mathbb{H}$ is said to be S^p-(μ,ν)-pseudo almost automorphic in the first variable uniformly with respect to the second variable if it can be written in the form*

$$f=g+h,$$

where $g\in AA^pU(\mathbb{R}\times\mathbb{H},\mathbb{H})$ and $h\in\mathcal{E}^pU(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$. The set of such functions is denoted by $PAA^pU(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$.

We define the following conditions.

(M1):

$$\limsup_{r\to+\infty}\frac{\mu([-r,r])}{\nu([-r,r])}:=M<\infty. \qquad (1)$$

(M2): For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau:\ a\in A\})\le\beta\mu(A)\quad\text{when } A\in\mathcal{B}\text{ satisfies } A\cap I=\emptyset.$$

Theorem 4.5. *If (M2) and (M1) are satisfied, Then:*

1. $AA^p(\mathbb{R},\mathbb{H})$ *is a translation invariant closed subspace of* $\mathcal{BS}^p(\mathbb{R};\mathbb{H})$.
2. $\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ *is a translation invariant closed subspace of* $\mathcal{BS}^p(\mathbb{R};\mathbb{H})$.
3. $PAA^p(\mathbb{R},\mathbb{H},\mu,\nu) = AA^p(\mathbb{R},\mathbb{H}) \bigoplus \mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ *is a Banach space for the direct sum norm.*

Proof:

1. By [11], [Theorem 2.1.3], $AA(\mathbb{R},L^p([0,1],\mathbb{H}))$ is a translation invariant subspace of $BC(\mathbb{R},L^p([0,1],\mathbb{H}))$. Let $t \mapsto f_a(t) := f(t+a)$ define a translation of f. We have $((f_a)^b(t)(s) = f_a(t+s) = f(t+s+a) = f^b(t+a)(s) = (f^b)_a(t)(s)$. That is $(f_a)^b = (f^b)_a$ and then for $f \in AA^P(\mathbb{R},\mathbb{H})$, $f^b \in AA(\mathbb{R},L^p([0,1],\mathbb{H}))$ then $(f^b)_a = (f_a)^b \in AA(\mathbb{R},L^p([0,1],\mathbb{H}))$ that means $f_a \in AA^p(\mathbb{R},\mathbb{H}))$, then $AA^P(\mathbb{R},\mathbb{H})$ is translation invariant. By [12], Theorem 2.3 $AA^P(\mathbb{R},\mathbb{H})$ is a closed subspace of $\mathcal{BS}^p(\mathbb{R};\mathbb{H})$.

2. It is immediate to prove $\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ is a subspace of $\mathcal{BS}^p(\mathbb{R};\mathbb{H})$.
   Now take $f \in \mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ and f_a its translate
$$\begin{aligned}\frac{1}{\nu([-r,r])}\int_{-r}^{r}(\int_0^1 |f_a(t+s)|^p ds)^{\frac{1}{p}} d\mu(t) &= \frac{1}{\nu([-r,r])}\int_{-r}^{r}(\int_0^1 |f(t+a+s)|^p ds)^{\frac{1}{p}} d\mu(t)\\ &= \frac{1}{\nu([-r,r])}\int_{-r+a}^{r+a}(\int_0^1 |f(y+s)|^p ds)^{\frac{1}{p}} d\mu(y-a)\\ &\le \beta\frac{\nu(Q_r)}{\nu([-r,r])}\frac{1}{\nu(Q_r)}\int_{-r-|a|}^{r+|a|}(\int_0^1 |f(y+s)|^p ds)^{\frac{1}{p}} d\mu(y)\end{aligned}$$
   where $Q_r = [-r-|a|, r+|a|]$. The factor $\beta\frac{\nu([-r-|a|,r+|a|])}{\nu([-r,r])}$ is bounded and
$$\lim_{r\to\infty}\frac{1}{\nu([-r-|a|,r+|a|])}\int_{-r-|a|}^{r+|a|}(\int_0^1 |f(y+s)|^p ds)^{\frac{1}{p}} d\mu(y) = 0,$$
   then $f_a \in \mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$.
   For the closedness of $(\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ take a sequence (f_n) in it. Assume that it converges in $\mathcal{BS}^p$ to f. By Remark 2.4, the $((f_n)^b)$ of $\mathcal{E}(\mathbb{R},L^p([0,1],\mathbb{H}))$ converges to f^b in $L^\infty(\mathbb{R},L^p([0,1],\mathbb{H}))$. According to [1], Theorem 3 $(\mathcal{E}(\mathbb{R},L^p([0,1],\mathbb{H}),\mu,\nu),\|.\|_\infty)$ is closed, then $f^b \in (\mathcal{E}(\mathbb{R},L^p([0,1],\mathbb{H}),\mu,\nu)$ that is $f \in (\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu),\|.\|_{\mathcal{BS}^p})$.
3. It is enough to show that $AA^p(\mathbb{R},\mathbb{H}) \cap \mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu) = \{0\}$. Let $f \in AA^p(\mathbb{R},\mathbb{H}) \cap \mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$ then $f^b \in AA(\mathbb{R},L^p([0,1],\mathbb{H})) \cap \mathcal{E}(\mathbb{R},L^p([0,1],\mathbb{H}),\mu,\nu)$. According to [1], Theorem 5, $f^b = 0$ then $f = 0$, by the injectivity of B in Remark 2.4.
   Let $(f_n) \in PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$ that converges in $\mathcal{BS}^p$ to f then $(f_n)^b \in PAA(\mathbb{R},L^p([0,1],\mathbb{H}),\mu,\nu)$ and converges in $L^\infty(\mathbb{R},L^p([0,1],\mathbb{H}))$ to f^b. According to [1], Theorem 6, $f^b \in PAA(\mathbb{R},L^p([0,1],\mathbb{H}),\mu,\nu)$ that is $f \in PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$.

Remark 4.6. *In the space* $PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$*, the direct sum norm and the* $\|.\|_{\mathcal{BS}^p}$ *are equivalent.*

Theorem 4.7. *Let* $G \in AA^pU(\mathbb{R}\times\mathbb{H},\mathbb{H})$ *and* $h \in AA^p(\mathbb{R},\mathbb{H})$ *satisfying the following:*

1. **(A0):** *There exists a nonnegative function* $L \in \mathcal{BS}^p(\mathbb{R})$ *such that*
$$\forall x,y \in \mathbb{H}, t \in \mathbb{R}\|G(t,x) - G(t,y)\| \le L(t)\|x-y\|.$$
   And there exists $\xi > 0$ *such that for all* $t \in \mathbb{R}, f \in \mathcal{BS}^p(\mathbb{R},\mathbb{H})$*, we have:*
$$\Big(\int_0^1 L^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}} \le \xi\Big(\int_0^1 \|f(s)\|^p ds\Big)^{\frac{1}{p}},$$

2. $K = \overline{\{h(t), t \in \mathbb{R}\}}$ *is compact.*

Then $[t \longmapsto G(t, h(t))] \in AA^p(\mathbb{R}, \mathbb{H})$.

Proof: Let (x_n) be a sequence such that $\lim_{n,m\to\infty} \|h(t + x_n - x_m + .) - h(t + .)\|_{L^p[0,1]} = 0$.
Take $\varepsilon > 0$ and $K \subset \bigcup_{1\le i\le r} B(y_i, \varepsilon)$, for some $y_i \in K$.
For $t \in \mathbb{R}$, let $E_1 := \{s \in [0, 1] : \ h(t + s) \in B(y_1, \varepsilon)\}$ and for $2 \le i \le r$, we define $E_i := \{s \in \Big([0, 1] \setminus \bigcup_{1\le j\le i-1} E_j\Big) :$ $h(t + s) \in B(y_i, \varepsilon)\}$.
Here $\{E_i, 1 \le i \le r\}$ is a partition of $[0, 1]$ and the sum of Lebesgue measures: $\sum_i \lambda(E_i) = 1$.

$$
\begin{aligned}
I : &= (\int_0^1 |G(t + s + x_n - x_m, h(t + s + x_n - x_m)) - G(t + s, h(t + s))|^p ds)^{\frac{1}{p}} \\
&\le (\int_0^1 |G(t + s + x_n - x_m, h(t + s + x_n - x_m)) - G(t + s + x_n - x_m, h(t + s))|^p ds)^{\frac{1}{p}} \\
&+ (\int_0^1 |G(t + s + x_n - x_m, h(t + s)) - G(t + s, h(t + s))|^p ds)^{\frac{1}{p}}.
\end{aligned}
$$

Denote by I_1 and I_2, respectively, the first and the second term of the previous sum.
By **(A0)**, $I_1 \le (\int_0^1 (L(t + s + x_n - x_m)|h(t + s + x_n - x_m) - h(t + s)|)^p ds)^{\frac{1}{p}}$
$\le \xi(\int_0^1 (\||h(t + s + x_n - x_m) - h(t + s)|)^p ds)^{\frac{1}{p}} \le \varepsilon\xi$, for $n, m \ge N_0$, since $h \in AA^p(\mathbb{R}, \mathbb{H})$.
For I_2:

$$
I_2 = (\sum_1^r \int_{E_i} |G(t + s + x_n - x_m, h(t + s)) - G(t + s, h(t + s))|^p ds)^{\frac{1}{p}}.
$$

Let

$$
\begin{aligned}
G(t + s + x_n - x_m, h(t + s)) - G(t + s, h(t + s)) &= (G(t + s + x_n - x_m, h(t + s)) - G(t + s + x_n - x_m, y_i)) \\
&+ (G(t + s + x_n - x_m, y_i) - G(t + s, y_i)) \\
&+ (G(t + s, y_i) - G(t + s, h(t + s))) \\
&= f_{1,i}(s) + f_{2,i}(s) + f_{3,i}(s)
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= (\sum_1^r \int_{E_i} |f_{1,i}(s) + f_{2,i}(s) + f_{3,i}(s)|^p ds)^{\frac{1}{p}} \\
&\le \Big(\sum_1^r \Big[(\int_{E_i} |f_{1,i}(s)|^p ds)^{\frac{1}{p}} + (\int_{E_i} |f_{2,i}(s)|^p ds)^{\frac{1}{p}} + (\int_{E_i} |f_{3,i}(s)|^p)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_1^r \int_{E_i} |f_{1,i}(s)|^p ds\Big)^{\frac{1}{p}} + \Big(\sum_1^r \int_{E_i} |f_{2,i}(s)|^p ds\Big)^{\frac{1}{p}} + \Big(\sum_1^r \int_{E_i} |f_{3,i}(s)|^p ds\Big)^{\frac{1}{p}} \\
&:= S_1 + S_2 + S_3.
\end{aligned}
$$

By **(A0)**,

$$\begin{aligned}
S_1 &= \Big(\sum_1^r \int_{E_i} |G(t+s+x_n-x_m, h(t+s)) - G(t+s+x_n-x_m, y_i)|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \int_{E_i} (L(t+s+x_n-x_m)|h(t+s)-y_i|)^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \int_{E_i} (L(t+s+x_n-x_m)\varepsilon)^p ds\Big)^{\frac{1}{p}} \\
&= \varepsilon\Big(\sum_1^r \int_{E_i} (L(t+s+x_n-x_m))^p ds\Big)^{\frac{1}{p}} \\
&= \varepsilon\Big(\sum_1^r \int_0^1 (\chi_{E_i}(s)L(t+s+x_n-x_m))^p ds\Big)^{\frac{1}{p}} \\
&= \varepsilon\Big(\sum_1^r [(\int_0^1 (\chi_{E_i}(s)L(t+s+x_n-x_m))^p ds)^{\frac{1}{p}}]^p\Big)^{\frac{1}{p}} \\
&\leq \varepsilon\Big(\sum_1^r [\xi(\int_0^1 (\chi_{E_i}(s))^p ds)^{\frac{1}{p}}]^p\Big)^{\frac{1}{p}} \\
&= \xi\varepsilon\Big(\sum_1^r \lambda(E_i)\Big)^{\frac{1}{p}} \\
&= \xi\varepsilon.
\end{aligned}$$

In the same way $S_3 \leq \varepsilon\xi$.

For S_2 :

$S_2 = \Big(\sum_1^r \int_{E_i} |G(t+s+x_n-x_m, y_i) - G(t+s, y_i)|^p ds\Big)^{\frac{1}{p}}$.

$G(., y_1) \in AA^p(\mathbb{R}, \mathbb{H})$, then there exists a subsequence $(\sigma_{1n}) \subseteq (x_n)$ and $N_1 \in \mathbb{N}$ such that

$$n, m \geq N_1 \Rightarrow (\int_0^1 |G(t+s+\sigma_{1n}-\sigma_{1m}, y_1) - G(t+s, y_1)|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

$G(., y_2) \in AA^p(\mathbb{R}, \mathbb{H})$, then there exists a subsequence $(\sigma_{2n}) \subseteq (\sigma_{1n})$ and $N_2 \geq N_1$ such that

$$n, m \geq N_2 \Rightarrow (\int_0^1 |G(t+s+\sigma_{2n}-\sigma_{2m}, y_2) - G(t+s, y_2)|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

Since $G(., y_j) \in AA^p(\mathbb{R}, \mathbb{H})$, then there exists a subsequence $(\sigma_{jn}) \subseteq (\sigma_{(j-1)n})$ and $N_j \geq N_{j-1}$ such that

$$n, m \geq N_j \Rightarrow (\int_0^1 |G(t+s+\sigma_{jn}-\sigma_{jm}, y_j) - G(t+s, y_j)|^p ds\Big)^{\frac{1}{p}} \leq \frac{\varepsilon}{r^{\frac{1}{p}}}.$$

Preserving the same notation of S_2, for $N = \max_{1\leq i\leq r}\{N_i\}$, $n, m \geq N$, we have

$$\begin{aligned}
S_2 &= \Big(\sum_1^r \int_{E_i} |G(t+s+\sigma_{rn}-\sigma_{rm}, y_j) - G(t+s, y_j)|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \int_0^1 |G(t+s+\sigma_{rn}-\sigma_{rm}, y_j) - G(t+s, y_j)|^p ds\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_1^r \Big(\frac{\varepsilon}{r^{\frac{1}{p}}}\Big)^p\Big)^{\frac{1}{p}} = \varepsilon.
\end{aligned}$$

And then, for $n, m \geq \max\{N, N_0\}, I \leq \varepsilon(1+3\xi)$. This completes the proof.

Theorem 4.8. *Assume μ, ν satisfy* (**M1**). *Let $G \in \mathcal{E}^p U(\mathbb{R}\times\mathbb{H}, \mathbb{H}, \mu, \nu)$ and $h : \mathbb{R} \longrightarrow \mathbb{H}$ satisfying:*

1. **(A0):** *There exists a nonnegative function $L \in \mathcal{BS}^p(\mathbb{R})$ such that*

$$\forall x, y \in \mathbb{H}, t \in \mathbb{R}, \|G(t,x) - G(t,y)\| \leq L(t)\|x-y\|.$$

And there exists $\xi > 0$ such that for all $t \in \mathbb{R}, f \in \mathcal{BS}^p(\mathbb{R}, \mathbb{H})$, we have:

$$\Big(\int_0^1 L^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}} \leq \xi\Big(\int_0^1 \|f(s)\|^p ds\Big)^{\frac{1}{p}},$$

2. $K = \overline{\{h(t), t \in \mathbb{R}\}}$ *is compact.*

Then $[t \longmapsto G(t, h(t))] \in \mathcal{E}^p(\mathbb{R}, \mathbb{H}, \mu, \nu)$.

Proof: Take $\varepsilon > 0$. Let $\eta = \frac{\varepsilon}{2(M+1)\xi}$. Assume the compact $K \subset \bigcup_{1\leq i\leq m} B(y_i, \eta)$.

For $t \in \mathbb{R}$, let $E_1 := \{s \in [0,1]/h(t+s) \in B(y_1, \eta)\}$ and for $2 \leq i \leq m$, we define $E_i := \{s \in ([0,1] \setminus \bigcup_{1\leq j\leq i-1} E_j)\ /h(t+s) \in B(y_i, \eta)\}$.

Here $\{E_i, 1 \leq i \leq m\}$ is a partition of $[0,1]$ and the sum of Lebesgue measures: $\sum_{i=1}^m \lambda(E_i) = 1$.
We aim to find $R > 0$ such that:

$$r \geq R \Longrightarrow I := \frac{1}{\nu([-r,r])}\int_{[-r,r]}(\int_0^1 |G(t+s, h(t+s))|^p ds)^{\frac{1}{p}} d\mu(t) \leq \varepsilon.$$

$$\Big(\int_0^1 |G(t+s,h(t+s))|^p ds\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^m \int_{E_i} |G(t+s,h(t+s)) - G(t+s,y_i) + G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}}$$
$$\leq \Big(\sum_{i=1}^m \Big[(\int_{E_i} |G(t+s,h(t+s)) - G(t+s,y_i)|^p ds)^{\frac{1}{p}} + (\int_{E_i} |G(t+s,y_i)|^p ds)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}}$$
$$\leq \Big(\sum_{i=1}^m \int_{E_i} |G(t+s,h(t+s)) - G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^m \int_{E_i} |G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}}.$$

Denote by $S_1(t)$ the first sum and by $S_2(t)$ the second sum of the previous expression. Then

$$\begin{aligned} S_1(t) &\leq \Big(\sum_{i=1}^m \int_{E_i} L^p(t+s)\eta^p ds\Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^m \lambda(E_i)\xi^p\eta^p ds\Big)^{\frac{1}{p}} \\ &\leq \eta\xi, \end{aligned}$$

and

$$\begin{aligned} S_2(t) &= \Big(\sum_{i=1}^m \int_0^1 |\chi_{E_i}(s)G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^1 |\sum_{i=1}^m \chi_{E_i}(s)G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^m \Big(\int_0^1 |\chi_{E_i}(s)G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^m \Big(\int_0^1 |G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}}. \end{aligned}$$

Now, we return to I:

$$\begin{aligned} I &\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]} S_1(t)+S_2(t)d\mu(t) \\ &\leq \frac{1}{\nu([-r,r])}\int_{[-r,r]} \eta\|L\|_{\mathcal{BS}^p(\mathbb{R})}d\mu(t)+\sum_{i=1}^{m}\frac{1}{\nu([-r,r])}\int_{[-r,r]}\Big(\int_0^1 |G(t+s,y_i)|^p ds\Big)^{\frac{1}{p}}d\mu(t) \end{aligned}$$

Since each $G(.,y_i)$ is in $\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$, there exists $R_i>0$ such that

$$r\geq R_i \Longrightarrow \frac{1}{\nu([-r,r])}\int_{[-r,r]}(\int_0^1 |G(t+s,y_i)|^p ds)^{\frac{1}{p}}d\mu(t)\leq \frac{\varepsilon}{2m}.$$

Then for $r\geq R:=\sup_{1\leq i\leq m} R_i, I\leq \dfrac{\mu([-r,r])}{\nu([-r,r])}\eta\|L\|_{\mathcal{BS}^p(\mathbb{R})}+\dfrac{\varepsilon}{2}\leq\varepsilon.$

Theorem 4.9. *Let μ and ν satisfy* (**M1**). *Assuming that $G=G_1+G_2\in PAA^pU(\mathbb{R}\times\mathbb{H},\mathbb{H},\mu,\nu)$ and $h=h_1+h_2\in PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$. Assume that the following conditions hold:*

1. *G_1,G_2 satisfy* **(A0):** *There exists a nonnegative function $L_i\in\mathcal{BS}^p(\mathbb{R})$ such that*

$$\forall x,y\in\mathbb{H}, t\in\mathbb{R}: \|G_i(t,x)-G_i(t,y)\|\leq L_i(t)\|x-y\|,$$

for $i=1,2$. And there exists $\xi>0$ such that for all $t\in\mathbb{R}, f\in\mathcal{BS}^p(\mathbb{R})$

$$\Big(\int_0^1 L_i^p(t+s)\|f(s)\|^p ds\Big)^{\frac{1}{p}}\leq\xi\Big(\int_0^1\|f(s)\|^p ds\Big)^{\frac{1}{p}}.$$

2. *$K_i=\overline{\{h_i(t),t\in\mathbb{R}\}}$ is compact, for $i=1,2$.*

Then $t\longmapsto G(t,h(t))\in PAA^p(\mathbb{R},\mathbb{H},\mu,\nu)$.

Proof: Put $G(t,h(t))=\widetilde{G_1}(t)+\widetilde{G_2}(t)$. Where $\widetilde{G_1}(t):=G_1(t,h_1(t))$ and $\widetilde{G_2}(t):=(G(t,h(t))-G(t,h_1(t)))+G_2(t,h_1(t))$. By Theorem 4.7 , we have $t\longmapsto G_1(t,h_1(t))\in AA^p(\mathbb{R},\mathbb{H})$ that is $\widetilde{G_1}\in AA^p(\mathbb{R},\mathbb{H})$. For $\widetilde{G_2}$:
$t\longmapsto G_2(t,h_1(t))\in\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$, by Theorem 4.8.

For $t\in\mathbb{R}$, we have

$$\begin{aligned} \Big(\int_t^{t+1}\|G(s,h(s))-G(s,h_1(s))\|^p ds\Big)^{\frac{1}{p}} &\leq \Big(\int_t^{t+1}\|G_1(s,h(s))-G_1(s,h_1(s))\|^p ds\Big)^{\frac{1}{p}} \\ &+ \Big(\int_t^{t+1}\|G_2(s,h(s))-G_2(s,h_1(s))\|^p ds\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_0^1 L_1^p(t+s)\|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}}+\Big(\int_0^1 L_2^p(t+s)\|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}} \\ &\leq 2\xi\Big(\int_0^1\|h_2(t+s)\|^p ds\Big)^{\frac{1}{p}}, \text{ since } h_2(t+.)\in\mathcal{BS}^p(\mathbb{R}). \end{aligned}$$

Then

$$\frac{1}{\nu([-r,r])}\int_{[-r,r]}\Big(\int_t^{t+1}\|G(s,h(s))-G(s,h_1(s))\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\leq\frac{2\xi}{\nu([-r,r])}\int_{[-r,r]}\Big(\int_t^{t+1}\|h_2(s)\|^p ds\Big)^{\frac{1}{p}}d\mu(t)\longrightarrow 0$$

as $r\longrightarrow+\infty$. This implies that $t\longmapsto G(t,h(t))-G(t,h_1(t))\in\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$. Therefore, $\widetilde{G_2}\in\mathcal{E}^p(\mathbb{R},\mathbb{H},\mu,\nu)$.

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