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On D-Invariant Points and Local Taylor Interpolation on Algebraic Hypersurfaces in $\mathbb{R}^N$

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Abstract. We give the definition of D-invariant points on an irreducible algebraic hypersurface V in $\mathbb{R}^N$. We show that every regular point on irreducible quadratic hypersurface in $\mathbb{R}^N$ is D-invariant. We prove that the local Taylor interpolation projector at a regular point $\mathbf{a} \in V$ is an ideal projector if and only if $\mathbf{a}$ is D-invariant.

1. Introduction

Let $\mathcal{P}^d(\mathbb{R}^N)$ be the vector space of all polynomials of degree at most d on $\mathbb{R}^N$. It is well-known that the dimension $m_d(\mathbb{R}^N)$ of $\mathcal{P}^d(\mathbb{R}^N)$ is equals to $\binom{N+d}{N}$. Let q be an irreducible polynomial on $\mathbb{R}^N$ and $V = V(q)$ the algebraic hypersurface $\{\mathbf{x} \in \mathbb{R}^N : q(\mathbf{x}) = 0\}$. The set of all regular points on V is denoted by V^0. When $V^0 \neq \emptyset$, it forms an analytic manifold of dimension $N-1$. We will also consider $\mathcal{P}^d(V) = \{p_{|V} : p \in \mathcal{P}^d(\mathbb{R}^N)\}$. Bos [5] shows that the dimension $m_d(V)$ of the vector space $\mathcal{P}^d(V)$ is given by

$$m_d(V) = m_d(\mathbb{R}^N) - m_{d-\deg q}(\mathbb{R}^N),$$

where we make a convention that $m_k(\mathbb{R}^N) = 0$ when $k < 0$.

In [6, 7], Bos and Calvi studied polynomial interpolation on algebraic hypersurfaces in $\mathbb{C}^N$. They obtained many beautiful results. The authors also showed in [6] that, with simple adaptations, everything remains true in the real variable case. Here we recall notations and results of Bos and Calvi in the real settings. Let $\mathcal{L} = (a, U, R)$ be a local parametrization of a regular point $\mathbf{a}$ on an irreducible algebraic hypersurface V in $\mathbb{R}^N$ (see Section 2 for precise definition). We consider the least space $\mathcal{P}^d_{\mathcal{L}\downarrow}$ at a induced by $\mathcal{P}^d_{\mathcal{L}} = \mathcal{P}^d(\mathbb{R}^N) \circ R$, following [6]. It is a subspace of $\mathcal{P}(\mathbb{R}^{N-1})$ generated by least terms of functions in $\mathcal{P}^d(\mathbb{R}^N) \circ R$. Using this least space, we can define a linear space of local differential operators $\mathrm{Dif}(\mathcal{L}, d)$ acting on sufficiently smooth functions in neighborhoods of $\mathbf{a}$ in V. The authors in [6] used $\mathrm{Dif}(\mathcal{L}, d)$ as interpolation conditions to define the local Taylor interpolation projector $\mathbf{T}^d_{\mathcal{L}}$. This type of interpolation inherits many properties of the ordinary Taylor polynomial (see [6] for details).

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In [7], the authors continued studying the least space and the local Taylor interpolation in $\mathbb{C}^2$. They introduced the notation of Taylorian points on algebraic curves. A point $\mathbf{a}$ on V is called to be Taylorian if, for some local parametrization $\mathcal{L} = (a, U, R)$, we have

$$\mathcal{P}^d_{\mathcal{L}\downarrow} = \operatorname{span}\Big\{1, (t-a), \dots, (t-a)^{m_d(V)-1}\Big\}. \tag{1}$$

In this case, any local parametrization gives the same set of powers $\{0, 1, \dots, m_d(V) - 1\}$. It is showed in [7], for $d \geq 1$, every point on irreducible quadratic curve in $\mathbb{C}^2$ is d-Taylorian and all but finitely many points on an irreducible algebraic curve in $\mathbb{C}^2$ are d-Taylorian. The local Taylor interpolation at a Taylorian point has many interesting properties. Theorem 3.2 in [7] points out that a regular point $\mathbf{a} \in V$ is Taylorian if and only if $\ker \mathbf{T}^d_{\mathcal{L}}$ is an ideal in the space of analytic functions at $\mathbf{a}$ on $V \subset \mathbb{C}^2$. It is an important characterization of a d-Taylorian point. Bos and Calvi also claimed in [7, p. 546] that analogous results still hold when one works with real algebraic curves in $\mathbb{R}^2$.

Our purpose is to generalize the definition of d-Taylorian points in the higher dimension and to study the properties of generalized points. Relation (1) suggests an extension of the concept of d-Taylorian points. That is, in the multivariate case, the space $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is invariant under differentiations (D-invariant for short). Fortunately, the D-invariant property is independent of the choice of the local parametrization and depends only on $\mathbf{a}$. This enables us to define D-invariant points on algebraic hypersurfaces in $\mathbb{R}^N$. We show in Proposition 2.8 that the set of D-invariant points of order $d = 1$ on an irreducible algebraic hypersurface V in $\mathbb{R}^N$ is open and dense in V^0. On a quadratic hypersurface in $\mathbb{R}^N$, Theorem 3.1 asserts that every regular point is D-invariant of order $d \geq 1$. Moreover, we can find a basis for the least space $\mathcal{P}^d_{\mathcal{L}\downarrow}$. Using this, we show that the local approximation order of $\mathcal{P}^d_{\mathcal{L}}$ is $d + 1$ in sense of de Boor and Ron [4]. Next we extend [7, Theorem 3.2] to the case of algebraic hypersurfaces in $\mathbb{R}^N$. In Theorem 4.2 we show that $\ker \mathbf{T}^d_{\mathcal{L}}$ is an ideal if and only if $\mathbf{a}$ is a D-invariant point. We also prove that, at a D-invariant point, the local Taylor interpolation projectors obey the Leibniz property (see Proposition 4.4).

After this paper had finished, we realized that analogous problems was studied by Izumi. In [13], the author used deep tools from complex geometry to investigate the least spaces, Taylorian projectors, etc. on complex manifolds X in $\mathbb{C}^N$. He showed that the set of points which are not D-invariant of order ∞ are contained in a countable union of thin analytic subsets of X. He also pointed out that the kernel of a local Taylor interpolation projector at a point $\mathbf{a} \in X$ is an ideal if and only if $\mathbf{a}$ is D-invariant. Note that the methods used in [13] are quite abstract and different from ours. Izumi utilized results of the complex geometry to deal with the complex case, whereas we use direct methods to compute least spaces and study the local Taylor interpolation projectors in the real case. In [13, p. 5], Izumi also noted that all results remain valid also in the real analytic category. We hope that our results are of independent interests and meaningful. We finally note that D-invariant spaces and ideal interpolation have been studied in many papers [1–4]. The notations of the least spaces, their properties and applications can be found in remarkable works of de Boor and Ron.

2. Least spaces and *D*-invariant points

2.1. Least spaces

The partial derivatives in $\mathbb{R}^n$ are defined by $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The linear differential operator with constant coefficients induced by $p \in \mathcal{P}^d(\mathbb{R}^n)$ is given by

$$p(D) = \sum_\alpha c_\alpha D^\alpha \quad \text{with} \quad p(x) = \sum_\alpha c_\alpha x^\alpha.$$

Let $\mathcal{F}$ be a finite-dimensional vector space of analytic functions on neighborhoods of $a \in \mathbb{R}^n$. For $f \in \mathcal{F}$, we can uniquely expand it in a Taylor series about a,

$$f(x) = \sum_{|\alpha|=0}^{\infty} c_\alpha (x-a)^\alpha, \quad c_\alpha \in \mathbb{R}.$$

If f is nonzero, then the order of f at a, denoted by $\mathrm{ord}_a(f)$, is the lowest non-vanishing a-homogenenous polynomial of the Taylor series,

$$\mathrm{ord}_a(f) = \min\Big\{k : \sum_{|\alpha|=k} |c_\alpha| \neq 0\Big\}.$$

The least term of f at a, denoted by $f_{a,\downarrow}$, is defined by

$$f_{a,\downarrow} = \sum_{|\alpha|=\mathrm{ord}_a(f)} c_\alpha (x-a)^\alpha.$$

Hence, we can write

$$f(x) = f_{a,\downarrow}(x) + (\text{terms of higher } a\text{-order}).$$

If $f = 0$, then we set $f_{a,\downarrow} = 0$. The least space of $\mathcal{F}$ at a is the vector space spanned by least terms,

$$\mathcal{F}_{a,\downarrow} := \mathrm{span}\Big\{f_{a,\downarrow} : f \in \mathcal{F}\Big\}.$$

A theorem of de Boor and Ron [2, p. 291] asserts that

$$\dim \mathcal{F}_{a,\downarrow} = \dim \mathcal{F}. \tag{2}$$

Definition 2.1. *A subspace Q of $\mathcal{P}(\mathbb{R}^n)$ is said to be D-invariant if it is invariant under differentiations. Equivalently, for every differential operator $p(D)$, we have*

$$p(D)(Q) \subset Q.$$

More generally, a vector space $\mathcal{F}$ of analytic functions in neighborhoods of $0 \in \mathbb{R}^n$ is called D-invariant if

$$p(D)(\mathcal{F}) \subset \mathcal{F}, \quad \forall p \in \mathcal{P}(\mathbb{R}^n).$$

Obviously, $\mathcal{F}$ is D-invariant if and only if $\frac{\partial f}{\partial x_i} \in \mathcal{F}$ for all $i = 1, \dots, n$ and $f \in \mathcal{F}$. The following result is showed in [2].

Theorem 2.2. *If $\mathcal{F}$ is D-invariant, then so is $\mathcal{F}_{0,\downarrow}$.*

2.2. D-invariant points on algebraic hypersurfaces

In this subsection, we always assume that q is an irreducible polynomial on $\mathbb{R}^N$ and $V = V(q)$ with $V^0 \neq \emptyset$.

Definition 2.3. *A local parametrization of V (and of V^0) at $\mathbf{a} \in V^0$ is a 3-tuple $\mathcal{L} = (a, U, R)$, where $a \in \mathbb{R}^{N-1}$, U is a domain in $\mathbb{R}^{N-1}$ containing a and $R : U \to \mathbb{R}^N$ an analytic mapping such that $R(a) = \mathbf{a}$, $R(U) \subset V^0$ and $R = (R_1, \dots, R_N)$ is a diffeomorphism from U onto $R(U)$.*

Note that R_i is analytic on U for every $i = 1, \dots, N$. Next, we consider the finite-dimensional space of analytic functions in neighborhoods of a,

$$\mathcal{P}^d_{\mathcal{L}} := \mathcal{P}^d(\mathbb{R}^N) \circ R = \Big\{p \circ R : p \in \mathcal{P}^d(\mathbb{R}^N)\Big\}.$$

Therefore, we get the corresponding least space at a,

$$\mathcal{P}^d_{\mathcal{L}\downarrow} := \Big(\mathcal{P}^d_{\mathcal{L}}\Big)_{a,\downarrow} \subset \mathcal{P}(\mathbb{R}^{N-1}).$$

From (2) and [6, Lemma 3.2], we have

$$\dim \mathcal{P}^d_{\mathcal{L}\downarrow} = \dim \mathcal{P}^d_{\mathcal{L}} = m_d(V).$$

Proposition 2.4. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two local parametrizations at* $\mathbf{a} \in V^0$. *If $\mathcal{P}^d_{\mathcal{L}_2\downarrow}$ is D-invariant then so is $\mathcal{P}^d_{\mathcal{L}_1\downarrow}$. Moreover, if*

$$\mathcal{P}^k(\mathbb{R}^{N-1}) \subseteq \mathcal{P}^d_{\mathcal{L}_2\downarrow},$$

then $\mathcal{P}^d_{\mathcal{L}_1\downarrow}$ also has the same property.

Proof. By Lemma 2.2 in [6], there exists an affine automorphism $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ such that

$$\mathcal{P}^d_{\mathcal{L}_1\downarrow} = \mathcal{P}^d_{\mathcal{L}_2\downarrow} \circ \Phi. \tag{3}$$

We write $\Phi = (\phi_1, \ldots, \phi_{N-1})$ where $\phi_i : \mathbb{R}^{N-1} \to \mathbb{R}$ are affine functions, $i = 1, \ldots, N-1$. For $P \in \mathcal{P}^d_{\mathcal{L}_1\downarrow}$ with $P = Q \circ \Phi$, $Q \in \mathcal{P}^d_{\mathcal{L}_2\downarrow}$, we have

$$\frac{\partial P}{\partial x_k} = \frac{\partial}{\partial x_k}(Q \circ \Phi) = \sum_{i=1}^{N-1} \Big(\frac{\partial Q}{\partial x_i} \circ \Phi\Big) \frac{\partial \phi_i}{\partial x_k}, \quad 1 \le k \le N-1. \tag{4}$$

By the hypothesis, $\frac{\partial Q}{\partial x_i} \in \mathcal{P}^d_{\mathcal{L}_2\downarrow}$, and hence $\frac{\partial Q}{\partial x_i} \circ \Phi \in \mathcal{P}^d_{\mathcal{L}_1\downarrow}$. On the other hand, $\frac{\partial \phi_i}{\partial x_k}$ is a real number. It follows that the polynomials in (4) belongs to $\mathcal{P}^d_{\mathcal{L}_1\downarrow}$. This completes the proof of the first part. The second assertion follows directly from (3) since Φ is an affine automorphism. □

Definition 2.5. *A point* $\mathbf{a} \in V^0$ *is called D-invariant of order d if, for any (or some) local parametrization $\mathcal{L}$ of V at* $\mathbf{a}$, *$\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant.*

The following result points out that D-invariant points are invariant with respect to affine automorphisms.

Lemma 2.6. *Let* $\mathbf{a} \in V^0$ *be D-invariant of order d and $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ an affine automorphism. Then* $\Phi(\mathbf{a})$ *is a D-invariant point on $\Phi(V)$ of order d.*

Proof. The proof follows directly from the computation in [6, p. 42]. Indeed, let $\mathcal{L} = (a, U, R)$ be a local parametrization of V at $\mathbf{a}$. Then $\mathcal{L}_\Phi = (a, U, \Phi \circ R)$ is a local parametrization of $\Phi(V) = \{q \circ \Phi^{-1} = 0\}$ at $\Phi(\mathbf{a})$. Since Φ is an affine automorphism of $\mathbb{R}^N$, we have $\mathcal{P}^d_{\mathcal{L}} = \mathcal{P}^d_{\mathcal{L}_\Phi}$. Hence

$$\mathcal{P}^d_{\mathcal{L}\downarrow} = \mathcal{P}^d_{\mathcal{L}_\Phi\downarrow}. \tag{5}$$

The proof is complete. □

Example 2.7. *1) It is easy to show that a finite-dimensional subspace Q of $\mathcal{P}(\mathbb{R})$ is D-invariant if and only if there exists k such that*

$$Q = \operatorname{span}\big\{1, t, \ldots, t^k\big\}.$$

Hence, on an algebraic curve in $\mathbb{R}^2$, a point is D-invariant if and only if it is Taylorian.
2) Example 3.4 in [6] points out that all points on hyperplanes in $\mathbb{R}^N$ are D-invariant of order d for $d \ge 1$.

Proposition 2.8. *Let $V(q)$ be an irreducible algebraic hypersurface of degree $d \ge 2$ in $\mathbb{R}^N$ with $V^0(q) \neq \emptyset$. Then the set of D-invariant points of order 1 on $V(q)$ is open and dense in $V^0(q)$.*

Proof. Let $\mathbf{a} \in V^0$, $V = V(q)$. Without loss of generality, we assume that $\dfrac{\partial q}{\partial x_N}(\mathbf{a}) \neq 0$. By the implicit function theorem, there exists a local parametrization $R : U \to \mathbb{R}^N$, $R(x) = \big(x, \rho(x)\big)$ with $\mathbf{x} = (x, x_N)$, $\mathbf{a} = (a, a_N)$ and

$R(a) = \mathbf{a}$. Remark that ρ is real analytic on U and $\mathrm{D}^2\rho$ is not identical to 0 on U since $\deg q \geq 2$, where $\mathrm{D}^2\rho(x)$ is the second-order total derivative of ρ at $x \in U$. It follows that

$$U_0 = \{b \in U : \mathrm{D}^2\rho(b) \neq 0\} = \left\{ b \in U : \left[\frac{\partial^2 \rho(b)}{\partial x_i \partial x_j}\right]_{1\leq i,j\leq N-1} \neq 0 \right\}$$

is open and dense in U, and hence $R(U_0)$ is open and dense in $R(U)$. It suffices to show that

$$U_0 = \{b \in U : \mathcal{I}_b \text{ is } D\text{-invariant}\},$$

where $\mathcal{I}_b$ is the least space of order 1,

$$\mathcal{I}_b = \operatorname{span}\Big\{\big(p(x, \rho(x))\big)_{b,\downarrow} : p \in \mathcal{P}^1(\mathbb{R}^N)\Big\}.$$

Evidently, $1 \in \mathcal{I}_b$. Since $m_1(V) = N + 1$, we need to find N non-constant independent elements in $\mathcal{I}_b$. Note that R is also a parametrization of $\mathbf{b} \in R(U)$ with $\mathbf{b} = R(b), b \in U$. For $b = (b_1, \ldots, b_{N-1}) \in U$, let $p(\mathbf{x}) = \sum_{i=1}^{N-1} c_i(x_i - b_i) + c_N(x_N - \rho(b))$. We have

$$p(x, \rho(x)) = \sum_{i=1}^{N-1} c_i(x_i - b_i) + c_N\big(\rho(x) - \rho(b)\big).$$

Therefore, if we take $c_N = 0$, then $\big(p(x, \rho(x))\big)_{b,\downarrow} = \sum_{i=1}^{N-1} c_i(x_i - b_i)$. It follows that $N-1$ polynomials $x_1, \ldots, x_{N-1}$ belong to $\mathcal{I}_b$. Let us choose $c_N = 1$ and $c_i = -\frac{\partial \rho(b)}{\partial x_i}$ for $i = 1, \ldots, N-1$. Then, the Taylor series of ρ about b gives

$$p(x, \rho(x)) = \sum_{k=2}^{\infty} \frac{1}{k!}\mathrm{D}^k\rho(b)(x - b)^k,$$

where $\mathrm{D}^k\rho(b)$ stands for the kth total derivative of ρ at b and $\mathrm{D}^k\rho(b)(x - b)^k$ means $\mathrm{D}^k\rho(b)(x - b, \ldots, x - b)$. It follows that

$$\big(p(x, \rho(x))\big)_{b,\downarrow} = \frac{1}{k!}\mathrm{D}^k\rho(b)(x - b)^k,$$

where $k \geq 2$ is the smallest integer such that $\mathrm{D}^k\rho(b)(x - b)^k$ does not vanish. Since $\dim \mathcal{I}_b = m_1(V) = N + 1$ we get

$$\mathcal{I}_b = \operatorname{span}\Big\{1, x_1, \ldots, x_{N-1}, \mathrm{D}^k\rho(b)(x - b)^k\Big\}.$$

Evidently, $\mathcal{I}_b$ is D-invariant if and only if $k = 2$. Equivalently, $\mathrm{D}^2\rho(b) \neq 0$, i.e., $b \in U_0$. The proof is complete. □

Let q be an irreducible polynomial of degree at least 2 on $\mathbb{R}^N$ with $V^0(q) \neq \emptyset$. We conjecture that the set of D-invariant points of order d on $V(q)$ is open and dense in $V^0(q)$ for any $d \geq 1$. Izumi [13] gave the affirmative answer for the case of complex manifolds in $\mathbb{C}^N$. In the next section, we will confirm the conjecture in the special case when $\deg q = 2$.

3. *D*-invariant points on quadratic hypersurfaces

In this section, we study the set of D-invariant points on irreducible quadratic hypersurfaces in $\mathbb{R}^N$ with $N \geq 3$. The main result of the section is the following.

Theorem 3.1. *Let q be an irreducible quadratic hypersurface on $\mathbb{R}^N$ with $V(q) \neq \emptyset$. Then every regular point on $V(q)$ is D-invariant of order d for $d \geq 1$.*

The idea of the proof is as follows. By applying a coordinate transformation in $\mathbb{R}^N$ if necessary, we may assume that that the equation $q(\mathbf{x}) = 0$ takes one of the following canonical forms:

$$(I) \quad x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2 - 1 = 0, \quad 1 \leq r \leq s \leq N;$$

$$(II) \quad x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2 = 0, \quad 1 \leq r < s \leq N, \quad s \geq 3;$$

$$(III) \quad x_N - \sum_{i=1}^{s} \epsilon_i x_i^2 = 0, \quad 1 \leq s \leq N-1, \quad \epsilon_i = \pm 1, \quad i = 1, \ldots, s.$$

Moreover, $\mathbf{a}$ is assumed to be a special point on $V^0(q)$ so that we can compute the least space directly. Hence we reduce the proof of Theorem 3.1 into the proofs of three special cases which are presented in the following lemmas.

Lemma 3.2. *Let $N \geq 3$, $1 \leq m \leq N-1$ and $d \geq 1$. Let $\varepsilon_i = \pm 1$ for $i = 1, \ldots, m$. Set*

$$F_{\alpha'}(x) = \Big(\sum_{i=1}^{m} \varepsilon_i x_i^2\Big)^{\alpha_1} \prod_{i=2}^{N-1} x_i^{\alpha_i}, \quad \alpha' = (\alpha_1, \ldots, \alpha_{N-1}), \quad |\alpha'| \leq d,$$

$$G_{\beta'}(x) = x_1\Big(\sum_{i=1}^{m} \varepsilon_i x_i^2\Big)^{\beta_1} \prod_{i=2}^{N-1} x_i^{\beta_i}, \quad \beta' = (\beta_1, \ldots, \beta_{N-1}), \quad |\beta'| \leq d-1.$$

Then the set $\mathcal{D} = \{F_{\alpha'}, G_{\beta'} : |\alpha'| \leq d, |\beta'| \leq d-1\}$ consists of $\binom{N+d}{N} - \binom{N+d-2}{N}$ independent polynomials on $\mathbb{R}^{N-1}$. Moreover, $\mathrm{span}(\mathcal{D})$ *is D-invariant.*

Proof. Without loss of generality we assume that $\Omega := \{(x_2, \ldots, x_m) \in \mathbb{R}^{m-1} : \sum_{i=2}^{m} \epsilon_i x_i^2 > 0\}$ is non-empty open set in $\mathbb{R}^{m-1}$, where $\sum_{i=2}^{m} \epsilon_i x_i^2$ is taken as 1 when $m = 1$. Otherwise, we work with $\sum_{i=1}^{m} -\epsilon_i x_i^2$ instead of $\sum_{i=1}^{m} \epsilon_i x_i^2$. We first see that

$$\begin{aligned} \sharp\mathcal{D} &= \sharp\{F_{\alpha'} : |\alpha'| \leq d\} + \sharp\{G_{\beta'} : |\beta'| \leq d-1\} \\ &= \binom{N-1+d}{N-1} + \binom{N-1+d-1}{N-1} = \binom{N+d}{N} - \binom{N+d-2}{N}. \end{aligned}$$

Next we show that $\mathcal{D}$ is linearly independent. We learn the method of Bos and de Marchi [8, p. 372-373]. Assume that

$$P_1\Big(x_2, \ldots, x_{N-1}, \sum_{i=1}^{m} \epsilon_i x_i^2\Big) + x_1 P_2\Big(x_2, \ldots, x_{N-1}, \sum_{i=1}^{m} \epsilon_i x_i^2\Big) = 0, \quad x_1, \ldots, x_{N-1} \in \mathbb{R}. \tag{6}$$

Here $P_1, P_2 \in \mathcal{P}(\mathbb{R}^{N-1})$. We need to show that $P_1 \equiv 0$ and $P_2 \equiv 0$. Since (6) is an algebraic identity, it also holds for all $x_1, \ldots, x_{N-1} \in \mathbb{C}$. In particular, taking $x_1 = \mathbf{i}t\sqrt{\sum_{i=2}^{m} \epsilon_i x_i^2}$, $t \in \mathbb{R}$, where $\mathbf{i}$ is the imaginary unit , we obtain the following identity for $(x_2, \ldots, x_{N-1}) \in \Omega \times \mathbb{R}^{N-m}$ and $t \in \mathbb{R}$,

$$P_1\Big(x_2, \ldots, x_{N-1}, (1 - \epsilon_1 t^2) \sum_{i=2}^{m} \epsilon_i x_i^2\Big) + \mathbf{i}t\sqrt{\sum_{i=2}^{m} \epsilon_i x_i^2} P_2\Big(x_2, \ldots, x_{N-1}, (1 - \epsilon_1 t^2) \sum_{i=2}^{m} \epsilon_i x_i^2\Big) \equiv 0. \tag{7}$$

Hence, both real part and imaginary part of the left hand side is identically zero,

$$P_k\Big(x_2, \ldots, x_{N-1}, (1 - \epsilon_1 t^2) \sum_{i=2}^{m} \epsilon_i x_i^2\Big) \equiv 0, \quad (x_2, \ldots, x_{N-1}) \in \Omega \times \mathbb{R}^{N-m}, \quad t \in \mathbb{R}, \quad k = 1, 2.$$

Since Ω is non-empty open set in $\mathbb{R}^{m-1}$, the above relation holds in $\mathbb{R}^{N-1}$. But $t \in \mathbb{R}$ is arbitrary, it follows that $P_1 \equiv 0$ and $P_2 \equiv 0$ in $\mathbb{R}^{N-1}$, and the claim is proved.

Now, to show that span($\mathcal{D}$) is D-invariant, it is sufficient to check that

$$\frac{\partial P}{\partial x_k} \in \operatorname{span}(\mathcal{D}), \quad \forall P \in \mathcal{D}, \quad 1 \le k \le N-1.$$

We first treat the case where $P = F_{\alpha'}$, $|\alpha'| \le d$. We have

$$\frac{\partial F_{\alpha'}}{\partial x_1}(x) = 2\alpha_1 \varepsilon_1 x_1 \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\alpha_1 - 1} \prod_{i=2}^{N-1} x_i^{\alpha_i}.$$

If $m+1 \le k \le N-1$, then

$$\frac{\partial F_{\alpha'}}{\partial x_k}(x) = \alpha_k \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\alpha_1} x_2^{\alpha_2} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{\alpha_k - 1} x_{k+1}^{\alpha_{k+1}} \cdots x_{N-1}^{\alpha_{N-1}}.$$

On the other hand, if $2 \le k \le m$, then

$$\frac{\partial F_{\alpha'}}{\partial x_k}(x) = \alpha_k \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\alpha_1} x_2^{\alpha_2} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{\alpha_k - 1} x_{k+1}^{\alpha_{k+1}} \cdots x_{N-1}^{\alpha_{N-1}} + 2\alpha_1 \varepsilon_k \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\alpha_1 - 1} x_2^{\alpha_2} \cdots x_{k-1}^{\alpha_{k-1}} x_k^{\alpha_k + 1} x_{k+1}^{\alpha_{k+1}} \cdots x_{N-1}^{\alpha_{N-1}}.$$

Above computations evidently give $\frac{\partial F_{\alpha'}}{\partial x_k} \in \operatorname{span}(\mathcal{D})$ for all $k = 1, \ldots, N-1$. The same arguments apply to the case $P = G_{\beta'}$, $|\beta'| \le d-1$. Here, we only verify that $\frac{\partial G_{\beta'}}{\partial x_1} \in \operatorname{span}(\mathcal{D})$. Since $\epsilon_1 x_1^2 = \sum_{i=1}^{m} \epsilon_i x_i^2 - \sum_{i=2}^{m} \epsilon_i x_i^2$, we have

$$\begin{aligned} \frac{\partial G_{\beta'}}{\partial x_1}(x) &= \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\beta_1} \prod_{k=2}^{N-1} x_k^{\beta_k} + 2\beta_1 \varepsilon_1 x_1^2 \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\beta_1 - 1} \prod_{k=2}^{N-1} x_k^{\beta_k} \\ &= \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\beta_1} \prod_{k=2}^{N-1} x_k^{\beta_k} + 2\beta_1 \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\beta_1} \prod_{k=2}^{N-1} x_k^{\beta_k} \\ &- 2\beta_1 \sum_{j=2}^{m} \varepsilon_j \Big(\sum_{i=1}^{m} \epsilon_i x_i^2\Big)^{\beta_1 - 1} x_2^{\beta_2} \cdots x_{j-1}^{\beta_{j-1}} x_j^{\beta_j + 2} x_{j+1}^{\beta_{j+1}} \cdots x_{N-1}^{\beta_{N-1}}. \end{aligned}$$

The last polynomial belongs to $\operatorname{span}\{F_{\alpha'} : |\alpha'| \le d\}$ since $|\beta'| \le d-1$. Hence, $\frac{\partial G_{\beta'}}{\partial x_1} \in \operatorname{span}(\mathcal{D})$, which completes the proof.
□

Lemma 3.3. *Let $1 \le r \le s \le N$. Let $q(\mathbf{x}) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2 - 1$ be an irreducible quadratic polynomial on $\mathbb{R}^N$ and $\mathbf{a} \in V$ with $V = V(q)$. Here $x_{r+1}^2 + \cdots + x_s^2$ is taken as 0 when $r = s$. Then there exists a linear automorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\Phi(V) = V \quad \text{and} \quad \Phi(\mathbf{a}) = (1, 0, \ldots, 0).$$

Proof. Let us take affine isomorphisms ϕ_1, ϕ_2, ϕ_3 such that

$$\phi_1 : \mathbb{R}^r \to \mathbb{R}^r, \quad \phi_1(a_1, \ldots, a_r) = \Big(\sqrt{\sum_{i=1}^{r} a_i^2}, 0, \ldots, 0\Big) =: (b, 0, \ldots, 0);$$

$$\phi_2 : \mathbb{R}^{s-r} \to \mathbb{R}^{s-r}, \quad \phi_2(a_{r+1}, \ldots, a_s) = \Big(\sqrt{\sum_{i=r+1}^{s} a_i^2}, 0, \ldots, 0\Big) =: (c, 0, \ldots, 0), \quad r < s;$$

$$\phi_3 : \mathbb{R}^{N-s} \to \mathbb{R}^{N-s}, \quad \phi_3(x_{s+1}, \ldots, x_N) = (x_{s+1} - a_{s+1}, \ldots, x_N - a_N), \quad s < N.$$

Define an affine automorphism Φ_1 of $\mathbb{R}^N$ by

$$\Phi_1 = \begin{cases} (\phi_1, \phi_2, \phi_3) & \text{if} \quad 1 \le r < s < N \\ (\phi_1, \phi_2) & \text{if} \quad 1 \le r < s = N \\ (\phi_1, \phi_3) & \text{if} \quad r = s < N \\ \phi_1 & \text{if} \quad r = N \end{cases}$$

Assume that $1 \le r < s < N$. We have

$$\Phi_1(x_1, \ldots, x_N) = \Big(\phi_1(x_1, \ldots, x_r), \phi_2(x_{r+1}, \ldots, x_s), \phi_3(x_{s+1}, \ldots, x_N)\Big).$$

It is easily seen that

$$\Phi_1(V) = V \quad \text{and} \quad \Phi_1(\mathbf{a}) = (b, 0, \ldots, 0, c, 0 \ldots, 0) =: \mathbf{a}^*.$$

We consider the linear mapping $\Phi_2 : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\Phi_2(x_1, \ldots, x_N) = (bx_1 - cx_{r+1}, x_2, \ldots, x_r, cx_1 - bx_{r+1}, \ldots, x_N).$$

Since $b^2 - c^2 = 1$, Φ_2 is an automorphism. Evidently, $\Phi_2(\mathbf{a}^*) = (1, 0, \ldots, 0)$. Moreover, if we set $\mathbf{y} = \Phi_2(\mathbf{x})$ then

$$y_1^2 + \cdots + y_r^2 - y_{r+1}^2 - \cdots - y_s^2 = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2.$$

Therefore, $\Phi_2(V) = V$, and hence it suffices to choose $\Phi = \Phi_2 \circ \Phi_1$. The case where $1 \le r < s = N$ can be done in the same way as above. In two remaining cases, we take $\Phi = \Phi_1$. The details of the proof are left to the reader. □

Lemma 3.4. *Every point on the irreducible quadratic hypersurface $V = V(q)$ is D-invariant of order d for $d \ge 1$, where $q(\mathbf{x}) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2 - 1$, $1 \le r \le s \le N$.*

Proof. By Lemmas 2.6 and 3.3, it suffices to prove the assertion for $\mathbf{a} = (1, 0, \ldots, 0)$. For convenience, we write $q(\mathbf{x}) = \sum_{i=1}^{s} \epsilon_i x_i^2 - 1$ with $\epsilon_1 = 1, \epsilon_i = \pm 1$ for $i \ge 2$. A local parametrization of V at $\mathbf{a}$ is $\mathcal{L} = (0, U, R)$ where U is a neighborhood of 0 in $\mathbb{R}^{N-1}$ and $R : U \to \mathbb{R}^N$ given by

$$R(y_2, \ldots, y_N) = \Big(\sqrt{1 - \sum_{i=2}^{s} \epsilon_i y_i^2}, y_2, \ldots, y_N\Big), \quad y = (y_2, \ldots, y_N) \in U. \tag{8}$$

Since $x_s^2 = \epsilon_s - \epsilon_s \sum_{i=1}^{s-1} \epsilon_i x_i^2$, we can write each polynomial $p \in \mathcal{P}^d(V)$ into the form

$$p(\mathbf{x}) = p_1(x_1 - 1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_N) + x_s p_2(x_1 - 1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_N), \tag{9}$$

where $p_1 \in \mathcal{P}^d(\mathbb{R}^{N-1})$, $p_2 \in \mathcal{P}^{d-1}(\mathbb{R}^{N-1})$. Hence

$$\begin{aligned} (p \circ R)(y_2, \ldots, y_N) &= p_1\Big(\sqrt{1 - \sum_{i=2}^{s} \epsilon_i y_i^2} - 1, y_2, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N\Big) \\ &+ y_s p_2\Big(\sqrt{1 - \sum_{i=2}^{s} \epsilon_i y_i^2} - 1, y_2, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N\Big). \end{aligned} \tag{10}$$

We have

$$\sqrt{1-\sum_{i=2}^{s}\epsilon_i y_i^2}-1=-\frac{1}{2}\sum_{i=2}^{s}\epsilon_i y_i^2+\text{(terms of higher 0-order)}. \tag{11}$$

It follows from (10) that $\mathcal{P}^d_{\mathcal{L}\downarrow}$ contains the following polynomials

$$P_{\overline{\alpha}}(y):=\Big(\sum_{i=2}^{s}\epsilon_i y_i^2\Big)^{\alpha_1}\prod_{i=2,i\neq s}^{N}y_i^{\alpha_i},\quad \overline{\alpha}=(\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_N),\quad |\overline{\alpha}|\le d$$

and

$$Q_{\overline{\beta}}(y):=y_s\Big(\sum_{i=2}^{s}\epsilon_i y_i^2\Big)^{\beta_1}\prod_{i=2,i\neq s}^{N}y_i^{\beta_i},\quad \overline{\beta}=(\beta_1,\dots,\beta_{s-1},\beta_{s+1},\dots,\beta_N),\quad |\overline{\beta}|\le d-1.$$

Remark that the s-index does not appear in $\overline{\alpha}$ and $\overline{\beta}$. Let us set

$$\mathcal{D}_d=\{P_{\overline{\alpha}},Q_{\overline{\beta}}:|\overline{\alpha}|\le d,|\overline{\beta}|\le d-1\}.$$

By Lemma 3.2, $\mathcal{D}_d$ consists of $\binom{N+d}{N}-\binom{N+d-2}{N}$ independent polynomials on $\mathbb{R}^{N-1}$ and span$(\mathcal{D}_d)$ is D-invariant. On the other hand

$$\binom{N+d}{N}-\binom{N+d-2}{N}=m_d(V)=\dim\mathcal{P}^d_{\mathcal{L}\downarrow}.$$

It follows that $\mathcal{D}_d$ forms a basis for $\mathcal{P}^d_{\mathcal{L}\downarrow}$ and $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant. The proof is complete. □

Lemma 3.5. *Let $q(\mathbf{x})=x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_s^2$, $1\le r<s\le N$, $s\ge 3$, be an irreducible quadratic polynomial on $\mathbb{R}^N$ and $\mathbf{a}\in V^0$ with $V=V(q)$. Then there exists a linear automorphism $\Phi:\mathbb{R}^N\to\mathbb{R}^N$ such that*

$$\Phi(V)=V\quad\textit{and}\quad\Phi(\mathbf{a})=(1,0,\dots,0,1,0,\dots,0),$$

where the first and s-th entries are equal to 1.

Proof. Since $\mathbf{a}=(a_1,\dots,a_N)\in V^0$, $\sqrt{\sum_{i=1}^r a_i^2}=\sqrt{\sum_{i=r+1}^s a_i^2}$ denoted shortly by b is different from 0. Let us take linear isomorphisms ϕ_1,ϕ_2,ϕ_3 such that

$$\varphi_1:\mathbb{R}^r\to\mathbb{R}^r,\quad \varphi_1(a_1,\dots,a_r)=(\sqrt{\sum_{i=1}^r a_i^2},0,\dots,0)=:(b,0,\dots,0);$$

$$\varphi_2:\mathbb{R}^{s-r}\to\mathbb{R}^{s-r},\quad \varphi_2(a_{r+1},\dots,a_s)=(0,\dots,0,\sqrt{\sum_{i=r+1}^s a_i^2})=(0,\dots,0,b);$$

$$\varphi_3:\mathbb{R}^{N-s}\to\mathbb{R}^{N-s},\quad \varphi_3(x_{s+1},\dots,x_N)=(x_{s+1}-a_{s+1},\dots,x_N-a_N),\quad s<N.$$

We define a linear automorphism of $\mathbb{R}^N$ by

$$\Phi_1=\begin{cases}(\varphi_1,\varphi_2,\varphi_3) & \text{if}\quad 1\le r<s<N,\\ (\varphi_1,\varphi_2) & \text{if}\quad 1\le r<s=N.\end{cases}$$

Then $\Phi_1(V)=V$ and $\Phi_1(\mathbf{a})=(b,0,\dots,0,b,0,\dots,0)$, where the first and s-th entries are equal to b. Let Φ_2 be the dilation defined by $\Phi_2(\mathbf{x})=\frac{\mathbf{x}}{b}$. Then $\Phi_2(V)=V$ and

$$\Phi_2(b,0,\dots,0,b,0,\dots,0)=(1,0,\dots,0,1,0,\dots,0).$$

The composed mapping $\Phi:=\Phi_2\circ\Phi_1$ satisfies the desired properties, and the proof is complete. □

Lemma 3.6. *Every regular point on the irreducible quadratic hypersurface $V = V(q)$ is D-invariant of order d for $d \geq 1$, where $q(\mathbf{x}) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2$, $1 \leq r < s \leq N$, $s \geq 3$.*

Proof. Analysis similar to that in the proof of Lemma 3.3 shows that we need only to prove the assertion for $\mathbf{a} = (1, 0, \ldots, 0, 1, 0 \ldots, 0)$ in which the first and s-th entries are equal to 1. For simplicity, we write $q(\mathbf{x}) = \sum_{i=1}^{s} \epsilon_i x_i^2$ with $\epsilon_1 = 1, \epsilon_s = -1$ and $\epsilon_i = \pm 1$. Let $a = (0, \ldots 0, 1, 0, \ldots, 0) \in \mathbb{R}^{N-1}$ where only s-th entry is equal to 1. We denote by $\mathcal{L} = (a, U, R)$ the local parametrization of V at $\mathbf{a}$, U is a neighborhood of a in $\mathbb{R}^{N-1}$ and $R : U \to \mathbb{R}^N$ defined by

$$R(y_2, \ldots, y_N) = \Big(\sqrt{-\sum_{i=2}^{s} \epsilon_i y_i^2}, y_2, \ldots, y_N\Big), \quad y = (y_2, \ldots, y_N) \in U.$$

For each $\alpha \in \mathbb{N}^N$, $|\alpha| \leq d$, let us consider the polynomial

$$q_\alpha(\mathbf{x}) = (x_1 - x_s)^{\alpha_1} x_2^{\alpha_2} \cdots x_{s-1}^{\alpha_{s-1}} (x_s - 1)^{\alpha_s} x_{s+1}^{\alpha_{s+1}} \cdots x_N^{\alpha_N}.$$

Evidently,

$$(q_\alpha \circ R)(y) = \Big(\sqrt{-\sum_{i=2}^{s} \epsilon_i y_i^2} - y_s\Big)^{\alpha_1} (y_s - 1)^{\alpha_s} \prod_{i=2, i\neq s}^{N} y_i^{\alpha_i}.$$

We have

$$\begin{aligned}
\sqrt{-\sum_{i=2}^{s} \epsilon_i y_i^2} - y_s &= \sqrt{1 + 2(y_s - 1) + (y_s - 1)^2 - \sum_{i=2}^{s-1} \epsilon_i y_i^2} - y_s \\
&= -y_s + 1 + \frac{1}{2}\Big(2(y_s - 1) + (y_s - 1)^2 - \sum_{i=2}^{s-1} \epsilon_i y_i^2\Big) \\
&- \frac{1}{8}\Big(2(y_s - 1) + (y_s - 1)^2 - \sum_{i=2}^{s-1} \epsilon_i y_i^2\Big)^2 + (\text{terms of higher } a\text{-order}) \\
&= -\frac{1}{2} \sum_{i=2}^{s-1} \epsilon_i y_i^2 + (\text{terms of higher } a\text{-order}).
\end{aligned}$$

It follows that

$$(q_\alpha \circ R)_{a,\downarrow}(y) = (-1/2)^{\alpha_1} \Big(\sum_{i=2}^{s-1} \epsilon_i y_i^2\Big)^{\alpha_1} (y_s - 1)^{\alpha_s} \prod_{i=2, i\neq s}^{N} y_i^{\alpha_i}, \quad |\alpha| \leq d.$$

By definition, we obtain

$$\begin{aligned}
\mathcal{P}^d_{\mathcal{L}\downarrow} &\supset \operatorname{span}\Big\{\Big(\sum_{i=2}^{s-1} \epsilon_i y_i^2\Big)^{\alpha_1} (y_s - 1)^{\alpha_s} \prod_{i=2, i\neq s}^{N} y_i^{\alpha_i}, \quad |\alpha| \leq d\Big\} \\
&= \operatorname{span}\Big\{\Big(\sum_{i=2}^{s-1} \epsilon_i y_i^2\Big)^{\alpha_1} \prod_{i=2}^{N} y_i^{\alpha_i}, \quad |\alpha| \leq d\Big\} \\
&=: \mathcal{Q}_d.
\end{aligned}$$

Let us consider the special polynomials in $\mathcal{Q}_d$

$$\widehat{P_{\widehat{\alpha}}}(y) := \Big(\sum_{i=2}^{s-1} \epsilon_i y_i^2\Big)^{\alpha_1} \prod_{i=2, i\neq s-1}^{N} y_i^{\alpha_i}, \quad \widehat{\alpha} = (\alpha_1, \ldots, \alpha_{s-2}, \alpha_s, \ldots, \alpha_N), \quad |\widehat{\alpha}| \leq d.$$

and

$$\widehat{Q}_{\widehat{\beta}}(y) := y_{s-1}\Big(\sum_{i=2}^{s-1}\epsilon_i y_i^2\Big)^{\beta_1}\prod_{i=2,i\neq s-1}^{N} y_i^{\beta_i},\quad \widehat{\beta}=(\beta_1,\ldots,\beta_{s-2},\beta_s,\ldots,\beta_N),\quad |\widehat{\beta}|\le d-1.$$

By Lemma 3.2, the set

$$\widehat{\mathcal{D}}_d = \{\widehat{P}_{\widehat{\alpha}}, \widehat{Q}_{\widehat{\beta}} : |\widehat{\alpha}|\le d, |\widehat{\beta}|\le d-1\}$$

are linearly independent and $\sharp\widehat{\mathcal{D}}_d = m_d(V)$. Consequently,

$$m_d(V) = \dim\operatorname{span}\Big(\widehat{\mathcal{D}}_d\Big) \le \dim \mathcal{Q}_d \le \dim \mathcal{P}^d_{\mathcal{L}\downarrow} = m_d(V).$$

It follows that

$$\mathcal{P}^d_{\mathcal{L}\downarrow} = \mathcal{Q}_d = \operatorname{span}\Big(\widehat{\mathcal{D}}_d\Big).$$

Lemma 3.2 also implies that $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant. The proof is complete. □

Lemma 3.7. *Let $\mathbf{x}=(x,x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}$, $q(\mathbf{x}) = x_N - \sum_{i=1}^s \epsilon_i x_i^2$ with $1\le s\le N-1$ and $\epsilon_i=\pm1$ for $i=1,\ldots,s$. Then every point on $V(q)$ is D-invariant of order d for $d\ge 1$.*

Proof. Let $\mathbf{a}=(a,a_N)$ be a point on $V(q)$. We denote by $\mathcal{L}=(a,\mathbb{R}^{N-1},R)$ the trivial parametrization of $V(q)$ at $\mathbf{a}$,

$$R(x_1,\ldots,x_{N-1}) = (x_1,\ldots,x_{N-1},\sum_{i=1}^s \epsilon_i x_i^2),\quad (x_1,\ldots,x_{N-1})\in\mathbb{R}^{N-1}.$$

Define

$$p_\alpha(\mathbf{x}) = \prod_{i=1}^{N-1}(x_i-a_i)^{\alpha_i}\Big(x_N - \sum_{i=1}^s \epsilon_i(2x_ia_i - a_i^2)\Big)^{\alpha_N},\quad |\alpha|\le d.$$

Then

$$(p_\alpha\circ R)_{a,\downarrow}(x) = \prod_{i=1}^{N-1}(x_i-a_i)^{\alpha_i}\Big(\sum_{i=1}^s \epsilon_i(x_i-a_i)^2\Big)^{\alpha_N},\quad |\alpha|\le d$$

and hence

$$\begin{aligned}\mathcal{P}^d_{\mathcal{L}\downarrow} &\supset \operatorname{span}\Big\{\prod_{i=1}^{N-1}(x_i-a_i)^{\alpha_i}\Big(\sum_{i=1}^s \epsilon_i(x_i-a_i)^2\Big)^{\alpha_N} : |\alpha|\le d\Big\}\\ &= \operatorname{span}\Big\{\prod_{i=1}^{N-1} x_i^{\alpha_i}(\sum_{i=1}^s\epsilon_i x_i^2)^{\alpha_N} : |\alpha|\le d\Big\}.\end{aligned}$$

Analysis similar to that in the proof of Lemma 3.6 shows that

$$\mathcal{P}^d_{\mathcal{L}\downarrow} = \operatorname{span}\Big\{\widetilde{P}_{\widetilde{\alpha}}, \widetilde{Q}_{\widetilde{\beta}} : |\widetilde{\alpha}|\le d, |\widetilde{\beta}|\le d-1\Big\} = \operatorname{span}\Big\{\prod_{i=1}^{N-1} x_i^{\alpha_i}(\sum_{i=1}^s\epsilon_i x_i^2)^{\alpha_N} : |\alpha|\le d\Big\}. \tag{12}$$

where

$$\widetilde{P}_{\widetilde{\alpha}}(x) = \prod_{i=2}^{N-1} x_i^{\alpha_i}(\sum_{i=1}^s \epsilon_i x_i^2)^{\alpha_N},\quad \widetilde{\alpha}=(\alpha_2,\ldots,\alpha_N), |\widetilde{\alpha}|\le d \tag{13}$$

and

$$\widetilde{Q}_{\widetilde{\beta}}(x) = x_1 \prod_{i=2}^{N-1} x_i^{\beta_i} (\sum_{i=1}^{s} \epsilon_i x_i^2)^{\beta_N}, \quad \widetilde{\beta} = (\beta_2, \dots, \beta_N), |\widetilde{\beta}| \le d-1. \tag{14}$$

Hence, by Lemma 3.2, $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant, and the proof is complete. □

Proof. [Proof of Theorem 3.1] By performing a change of variables, we may assume that q is one of the three forms defined in Lemmas 3.4, 3.6 and 3.7. The assertion follows directly from these three lemmas. □

Proposition 3.8. *Let $N \ge 3$ and $q(\mathbf{x})$ be an irreducible quadratic polynomial on $\mathbb{R}^N$ with $V^0 \neq \emptyset$, $V = V(q)$. Let $\mathcal{L}$ be a local parametrization of V at a regular point $\mathbf{a} \in V^0$. Then, for $d \ge 1$,*

$$\mathcal{P}^d(\mathbb{R}^{N-1}) \subseteq \mathcal{P}^d_{\mathcal{L}\downarrow} \quad \text{and} \quad \mathcal{P}^{d+1}(\mathbb{R}^{N-1}) \nsubseteq \mathcal{P}^d_{\mathcal{L}\downarrow}.$$

Proof. Looking at Proposition 2.4 and relation (5), and performing a change of variables, we need only prove the claim for q and $\mathbf{a}$ in Lemmas 3.4, 3.6 and 3.7. Since the bases of the least spaces on the three lemmas are of the same form, we need only to prove the claim for q and $\mathbf{a}$ in the last lemma. Let

$$V = \Big\{\mathbf{x} : x_N = \sum_{i=1}^{s} \epsilon_i x_i^2 : x = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}\Big\}, \quad \epsilon_i = \pm 1,$$

$\mathbf{a} \in V$ and $\mathcal{L}$ the trivial parametrization. By (12) it is obvious that $x_1^{\alpha_1} \cdots x_{N-1}^{\alpha_{N-1}}$ belongs to $\mathcal{P}^d_{\mathcal{L}\downarrow}$ for $\alpha_1 + \cdots + \alpha_{N-1} \le d$. Therefore,

$$\mathcal{P}^d(\mathbb{R}^{N-1}) \subseteq \mathcal{P}^d_{\mathcal{L}\downarrow}.$$

It is well-known that the number of independent homogenenous polynomials of degree $d+1$ in $\mathbb{R}^{N-1}$ is equal to $\binom{d+N-1}{N-2}$. On the other hand, from (13) we see that, in the set $\{\widetilde{P}_{\widetilde{\alpha}} : |\widetilde{\alpha}| \le d\}$, there are exactly h_1 homogenenous polynomials of degree $d+1$, where

$$h_1 = \binom{N+d-4}{N-3} + \binom{N+d-6}{N-3} + \cdots + \binom{N+d-2[\frac{d+1}{2}]-2}{N-3}.$$

Similarly, there are exactly h_2 homogenenous polynomials of degree $d+1$ in the set $\{\widetilde{Q}_{\widetilde{\beta}} : |\widetilde{\beta}| \le d-1\}$, where

$$h_2 = \binom{N+d-5}{N-3} + \binom{N+d-7}{N-3} + \cdots + \binom{N+d-2[\frac{d}{2}]-3}{N-3}.$$

It follows that the basis of $\mathcal{P}^d_{\mathcal{L}\downarrow}$ contains $(h_1 + h_2)$ homogenenous polynomials of degree $d+1$. A direct computation gives

$$\begin{aligned} h_1 + h_2 &= \binom{N+d-4}{N-3} + \binom{N+d-5}{N-3} + \cdots + \binom{N-3}{N-3} \\ &= \binom{N+d-3}{N-2} < \binom{N+d-1}{N-2}. \end{aligned}$$

Hence, we can find a homogenenous polynomial of degree $d+1$ that does not belong to $\mathcal{P}^d_{\mathcal{L}\downarrow}$, and the proof is complete. □

The concept of local approximation order was introduced by de Boor and Ron. Let $\mathbb{A}_0$ be the space of all analytic functions at the origin in $\mathbb{R}^{N-1}$. Let $\mathcal{H}$ be a finite-dimensional linear subspace of $\mathbb{A}_0$. The local approximation order of $\mathcal{H}$ is the largest integer m for which, for every $f \in C^\infty(\mathbb{R}^{N-1})$, there exists $h \in \mathcal{H}$ such that

$$(f-h)(x) = O(\|x\|^m) \quad \text{as} \quad x \to 0.$$

Let $\mathcal{L} = (0, U, R)$ be a local parametrization of V at a regular point $\mathbf{a} \in V^0$. Evidently, $\mathcal{P}^d_{\mathcal{L}} = \mathcal{P}^d(\mathbb{R}^N) \circ R$ is a finite-dimensional linear subspace of $\mathbb{A}_0$. Proposition 3.8 and [4, Corollary 2.14] give the following result.

Corollary 3.9. *Let $N \geq 3$ and $q(\mathbf{x})$ be an irreducible quadratic polynomial on $\mathbb{R}^N$ with $V^0 \neq \emptyset$, $V = V(q)$. Let $\mathcal{L} = (0, U, R)$ be a local parametrization of V at a regular point $\mathbf{a} \in V^0$. Then the local approximation order of $\mathcal{P}^d_{\mathcal{L}}$ is $d+1$.*

Example 3.10. *Let V be the unit sphere in $\mathbb{R}^3$, $V = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, and $\mathbf{a} = (1, 0, 0)$. Let $\mathcal{L}_1 = (0, U_1, R_1)$ be a local parametrization of V at $\mathbf{a}$ where $R_1 : U_1 \to V$ given by*

$$R_1(y,z) = \left(\sqrt{1-y^2-z^2}, y, z\right), \quad (y,z) \in U_1 = \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 < 1\}.$$

From computations in Lemma 3.4 we have

$$\mathcal{P}^1_{\mathcal{L}_1\downarrow} = \operatorname{span}\left\{1, y, z, y^2 + z^2\right\}.$$

If $\mathcal{L}_2 = (0, U_2, R_2)$ is a local parametrization of V at $\mathbf{a}$ defined by

$$R_2(y,z) = \left(\sqrt{1-(y+z)^2-z^2}, y+z, z\right), \quad (y,z) \in U_2 = \{(y,z) \in \mathbb{R}^2 : (y+z)^2 + z^2 < 1\},$$

then

$$\mathcal{P}^1_{\mathcal{L}_2\downarrow} = \operatorname{span}\left\{1, y, z, y^2 + 2yz + 2z^2\right\}$$

which differs from $\mathcal{P}^1_{\mathcal{L}_1\downarrow}$. A similar example can be found in [9, Example 1].

4. Local Taylor interpolation at D-invariant points

We first recall the construction of the local Taylor interpolation introduced by Bos and Calvi [6]. Let $V = \{\mathbf{x} \in \mathbb{R}^N : q(\mathbf{x}) = 0\}$ where q is an irreducible polynomial on $\mathbb{R}^N$ such that $V^0 \neq \emptyset$. Let $\mathcal{L} = (a, U, R)$ be a local parametrization of V at $\mathbf{a} \in V^0$. For $Q(x) = \sum_\alpha c_\alpha (x-a)^\alpha$ belonging to the least space, we associate a local differential operator $Q_{\mathcal{L}}(D)$ defined on the space of sufficiently smooth functions on a neighborhood of $\mathbf{a}$ by

$$Q_{\mathcal{L}}(D)(f) := \sum_\alpha c_\alpha D^\alpha (f \circ R)(a).$$

Here, in this section, the multi-indices are all in $\mathbb{N}^{N-1}$. Remark that if $a = 0$, then we can write

$$Q_{\mathcal{L}}(D)(f) = Q(D)(f \circ R)(0).$$

Let $\operatorname{Dif}(\mathcal{L}, d)$ be the linear space spanned by local differential operators of degree at most d,

$$\operatorname{Dif}(\mathcal{L}, d) = \operatorname{span}\left\{Q_{\mathcal{L}}(D) : Q \in \mathcal{P}^d_{\mathcal{L}\downarrow}\right\}.$$

Lemma 3.2 in [6] points out that $\operatorname{Dif}(\mathcal{L}, d)\Big|_{\mathcal{P}^d(V)}$ is identical with the dual space of $\mathcal{P}^d(V)$. In particular, $\dim \operatorname{Dif}(\mathcal{L}, d) = m_d(V)$. This result leads to the definition of the local Taylor interpolation polynomial (see [6, Theorem 3.3]).

Theorem 4.1. *Let q be a non-constant irreducible real polynomial such that $V^0(q) \neq \emptyset$. Let $\mathbf{a} \in V^0(q)$ and $\mathcal{L}$ be a local parametrization of $V(q)$ at $\mathbf{a}$. Then for every sufficiently smooth function f in a neighborhood of $\mathbf{a}$, there exists a unique polynomial p in $\mathcal{P}^d(V)$ such that*

$$\mu(p) = \mu(f), \quad \forall \mu \in \mathrm{Dif}(\mathcal{L}, d).$$

The polynomial p is called the $\mathcal{L}$-Taylor interpolation polynomial of f at $\mathbf{a}$ to the order d and denoted by $\mathbf{T}^d_{\mathcal{L}}(f)$.

The following theorem is an extension of [7, Theorem 3.2]. A similar result was obtained in [13] for the complex case. Note that our proof is different from [13]. Here $\mathcal{A}_{\mathbf{a}}$ is the space of analytic functions in neighborhoods of $\mathbf{a}$ in V^0.

Theorem 4.2. *A point $\mathbf{a} \in V^0$ is D-invariant of order d if and only if, for any (or some) local parametrization $\mathcal{L}$ of V at $\mathbf{a}$, $\ker \mathbf{T}^d_{\mathcal{L}} = \{f \in \mathcal{A}_{\mathbf{a}} : \mathbf{T}^d_{\mathcal{L}}(f) = 0\}$ is an ideal in $\mathcal{A}_{\mathbf{a}}$.*

Proof. Without loss of generality we assume that $a = 0$. Suppose that $\mathbf{a}$ is D-invariant. Let $f \in \ker \mathbf{T}^d_{\mathcal{L}}$. Hence $P(D)(f \circ R)(0) = 0$ for all $P \in \mathcal{P}^d_{\mathcal{L}\downarrow}$. Since $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant, $(D^\alpha P)(D)(f \circ R)(0) = 0$ for all $P \in \mathcal{P}^d_{\mathcal{L}\downarrow}$ and $\alpha \in \mathbb{N}^{N-1}$. For $g \in \mathcal{A}_{\mathbf{a}}$, using the Leibniz-Hörmander formula (see for instance [12, p. 177] or [11, p. 243]), we have

$$\begin{aligned} P(D)\big((fg) \circ R\big)(0) &= P(D)\big((f \circ R)(g \circ R)\big)(0) \\ &= \sum_{|\alpha| \le \deg P} (D^\alpha P)(D)(f \circ R)(0) \frac{1}{\alpha!} D^\alpha (g \circ R)(0) \\ &= 0. \end{aligned}$$

Conversely, assume that $\ker \mathbf{T}^d_{\mathcal{L}}$ is an ideal in $\mathcal{A}_{\mathbf{a}}$ and that there exist $P \in \mathcal{P}^d_{\mathcal{L}\downarrow}$ and $0 < |\alpha| < \deg P$ such that $D^\alpha P \notin \mathcal{P}^d_{\mathcal{L}\downarrow}$. We look for a contradiction. We take a basis $\mathcal{B}_d = \{p_1, \ldots, p_m\}$ for $\mathcal{P}^d(V)$ with $m = m_d(V)$. Let $\{\mu_1, \ldots, \mu_m\} \subset \mathrm{Dif}(\mathcal{L}, d)$ be the dual basis for $\mathcal{B}_d$, i.e., $\mu_i(p_j) = \delta_{ij}$ for $1 \le i, j \le m$. We can find $Q_i \in \mathcal{P}^d_{\mathcal{L}\downarrow}$ such that $\mu_i(f) = Q_i(D)(f \circ R)(0)$ for $i = 1, \ldots, m$. It is easily seen that

$$\mathbf{T}^d_{\mathcal{L}}(f) = \sum_{i=1}^m \mu_i(f) p_i, \quad f \in \mathcal{A}_{\mathbf{a}}. \tag{15}$$

We will denote by $\widehat{\mu}$ the differential operator defined by $\widehat{\mu}(f) = (D^\alpha P)(D)(f \circ R)(0)$, $f \in \mathcal{A}_{\mathbf{a}}$. Assume that

$$\widehat{\mu}\big(\mathbf{T}^d_{\mathcal{L}}(f)\big) = \widehat{\mu}(f), \quad \forall f \in \mathcal{A}_{\mathbf{a}}.$$

From (15) we obtain

$$\widehat{\mu}(f) = \sum_{i=1}^m \widehat{\mu}(p_i) \mu_i(f) = \sum_{i=1}^m c_i \mu_i(f), \quad \forall f \in \mathcal{A}_{\mathbf{a}},$$

where $c_i = \widehat{\mu}(p_i)$. In other words,

$$\Big(D^\alpha P - \sum_{i=1}^m c_i Q_i\Big)(D)(f \circ R)(0) = 0, \quad \forall f \in \mathcal{A}_{\mathbf{a}}. \tag{16}$$

Suppose that $D^\alpha P - \sum_{i=1}^m c_i Q_i$ contains a nonzero term $Q(x) = b_\gamma x^\gamma$. By the immersion theorem (see for instance [11, Theorem 4.3.1]) there exists $f_0 \in \mathcal{A}_{\mathbf{a}}$ such that $f_0 \circ R = Q$ in a neighborhood of $0 \in \mathbb{R}^{N-1}$. An easy computation shows that (16) does not hold for f_0, a contradiction. It follows that $D^\alpha P = \sum_{i=1}^m c_i Q_i$, which contradicts the assumption $D^\alpha P \notin \mathcal{P}^d_{\mathcal{L}\downarrow}$. Therefore, we can find $f \in \mathcal{A}_{\mathbf{a}}$ such that

$$(D^\alpha P)(D)\big(f \circ R\big)(0) \neq (D^\alpha P)(D)\big(\mathbf{T}^d_{\mathcal{L}}(f) \circ R\big)(0).$$

Hence, if we set $h = f - \mathbf{T}^d_{\mathcal{L}}(f)$, then $h \in \ker \mathbf{T}^d_{\mathcal{L}}$ and $(D^\alpha P)(D)(h \circ R)(0) \neq 0$. Using the immersion theorem again, we can find $g \in \mathcal{A}_{\mathbf{a}}$ such that $g \circ R = x^\alpha$. We easily see that

$$\frac{1}{\beta!} D^\beta (g \circ R)(0) = \delta_{\alpha\beta}, \quad \forall \beta \in \mathbb{N}^{N-1},$$

where $\delta_{\alpha\beta}$ stands for the Kronecker symbol. It follows that

$$\begin{aligned} P(D)\Big((gh) \circ R\Big)(0) &= \sum_{|\beta| \le \deg P} (D^\beta P)(D)(h \circ R)(0) \frac{1}{\beta!} D^\beta (g \circ R)(0) \\ &= (D^\alpha P)(D)(h \circ R)(0) \neq 0. \end{aligned}$$

Hence $gh \notin \ker \mathbf{T}^d_{\mathcal{L}}$, contrary to the hypothesis. □

Remark 4.3. *Theorem 4.2 remains true when we replace the space $\mathcal{A}_{\mathbf{a}}$ by the space $C^\ell(\{a\})$ of all functions of class C^ℓ in neighborhoods of* **a**, *where*

$$\ell := \max\{\deg p : p \in \mathcal{P}^d_{\mathcal{L}\downarrow}\}.$$

In [6, p. 42], the authors point out that ℓ does not depend on $\mathcal{L}$.

Proposition 4.4. *If* **a** *is D-invariant of order d and $\mathcal{L}$ is a local parametrization of V at* **a**, *then for suitably defined functions f and g we have*
a) $\mathbf{T}^d_{\mathcal{L}}\Big(f \mathbf{T}^d_{\mathcal{L}}(g)\Big) = \mathbf{T}^d_{\mathcal{L}}(fg)$;
b) $\mathbf{T}^d_{\mathcal{L}}\Big(\mathbf{T}^d_{\mathcal{L}}(f) \mathbf{T}^d_{\mathcal{L}}(g)\Big) = \mathbf{T}^d_{\mathcal{L}}(fg)$.

Proof. It is sufficient to prove the first assertion. The second assertion is an immediate consequence of the first one. Without loss of generality we assume that $a = 0$. For every $P \in \mathcal{P}^d_{\mathcal{L}\downarrow}$, using the Leibniz-Hörmander formula again, we obtain

$$\begin{aligned} P(D)\Big(\big(f\mathbf{T}^d_{\mathcal{L}}(g)\big) \circ R\Big)(0) &= P(D)\Big((f \circ R)(\mathbf{T}^d_{\mathcal{L}}(g) \circ R)\Big)(0) \\ &= \sum_{|\alpha| \le \deg P} (D^\alpha P)(D)\Big(\mathbf{T}^d_{\mathcal{L}}(g) \circ R\Big)(0) \frac{1}{\alpha!} D^\alpha (f \circ R)(0). \end{aligned}$$

Now, since $\mathcal{P}^d_{\mathcal{L}\downarrow}$ is D-invariant, $D^\alpha P \in \mathcal{P}^d_{\mathcal{L}\downarrow}$. Hence, the interpolation conditions of the local Taylor operator give

$$(D^\alpha P)(D)\Big(\mathbf{T}^d_{\mathcal{L}}(g) \circ R\Big)(0) = (D^\alpha P)(D)\Big(g \circ R\Big)(0).$$

Consequently,

$$\begin{aligned} P(D)\Big(\big(f\mathbf{T}^d_{\mathcal{L}}(g)\big) \circ R\Big)(0) &= \sum_{|\alpha| \le \deg P} (D^\alpha P)(D)\Big(g \circ R\Big)(0) \frac{1}{\alpha!} D^\alpha (f \circ R)(0) \\ &= P(D)\Big((fg) \circ R\Big)(0). \end{aligned}$$

The last equation implies the desired relation. □

Remark 4.5. *Let V be an irreducible quadratic hypersurface in $\mathbb{R}^N$ with $V^0 \neq \emptyset$, $N \geq 3$. Let $\mathcal{L} = (0, U, R)$ be a local parametrization of V at a regular point* $\mathbf{a} \in V^0$ *and $f \in \mathcal{A}_{\mathbf{a}}$. By Proposition 3.8 and Theorem 4.1, we have*

$$D^\alpha\Big(\mathbf{T}^d_{\mathcal{L}}(f) \circ R\Big)(0) = D^\alpha (f \circ R)(0), \quad |\alpha| \le d.$$

Using the usual Taylor expansion of $\mathbf{T}^d_{\mathcal{L}}(f) \circ R$ and $f \circ R$ about 0, we can write

$$f \circ R(x) - \mathbf{T}^d_{\mathcal{L}}(f) \circ R(x) = O(\|x\|^{d+1}) \quad \text{as} \quad x \to 0. \tag{17}$$

There exist positive constants M_1, M_2 and a neighborhood U_1 of $0 \in \mathbb{R}^{N-1}$ with $U_1 \subset U$ such that

$$M_1\|x\| \le \|R(x) - R(0)\| \le M_2\|x\|, \quad x \in U_1.$$

The estimation in (17) implies

$$f(\mathbf{x}) - \mathbf{T}^d_{\mathcal{L}}(f)(\mathbf{x}) = O(\|\mathbf{x}-\mathbf{a}\|^{d+1}), \quad \mathbf{x} \in V, \quad \mathbf{x} \to \mathbf{a}.$$

Remark 4.6. *Let V be an irreducible quadratic curve in $\mathbb{R}^2$. Since any point $\mathbf{a} \in V$ is d-Taylorian for $d \ge 1$, the space $\mathcal{P}^d_{\mathcal{L}\downarrow}$ contains all monomials of degree less than $m_d(V)$. While it is well known that $m_d(V) = 2d+1$ as the degree d parts of Hilbert functions of V. Then all the derivatives of $\left(f - \mathbf{T}^d_{\mathcal{L}}\right) \circ R(t)$ of order $0, 1, \ldots, 2d$ at 0 vanish, where $(0, U, R)$ is a local parametrization of V at $\mathbf{a}$. This implies that $\left(f - \mathbf{T}^d_{\mathcal{L}}\right) \circ R(t) = O(|t|^{2d+1})$ by the usual Taylor expansion. Using the same arguments in Remark 4.5, we obtain*

$$f(\mathbf{x}) - \mathbf{T}^d_{\mathcal{L}}(f)(\mathbf{x}) = O(\|\mathbf{x}-\mathbf{a}\|^{2d+1}), \quad \mathbf{x} \in V, \quad \mathbf{x} \to \mathbf{a}.$$

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