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Multiple Nonnegative Solutions for a Class IVPs for Second Order ODEs

Svetlin Georgiev Georgiev[a], Mohamed Majdoub[b], Karima Mebarki[c]

[a] *Department of Differential Equations, Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria*
[b] *Department of Mathematics, College of Science, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam, Saudi Arabia*
[c] *Laboratory of Applied Mathematics, Faculty of Exact Sciences, A. Mira University, 6000 Bejaia, Algeria*

Abstract. We study a class of initial value problems for second order ODEs. The interesting points of our results are that the nonlinearity depends on the solution and its derivative and may change sign. Moreover, it satisfies general polynomial growth conditions. A new topological approach is applied to prove the existence of at least two nonnegative classical solutions. The arguments are based upon a recent theoretical result.

1. Introduction

ODEs have many important applications across fields as biology, mechanics, chemistry, design of electrical systems, stability of aircraft and many others. Over recent decades, there have been progress in the determination of unique solution for IVPs for ODEs and BVPs for ODEs and a progress in the determination of existence of multiple solutions for BVPs for ODE and multiple periodic solutions for ODEs (see [2], [3], [4], [6], [7], [9], [10], [13], [15], [17], [18], [19], [20], [21], [22], [23] and references therein). However, researches regarding nonunique solutions for IVPs for ODEs and their potential applications have been neglected by comparisons. One of the reason for the focus of scientists on multiple solutions for BVPs for ODEs is that for many BVPs we have constructed the Green function and using it we have a suitable integral representation of the solutions. As other reason, we will mention that many of the considered BVPs for ODEs are on a finite interval.

In this paper we will investigate for existence of at least two nonnegative solutions the following class IVPs for second order ODEs

$$\begin{aligned} y'' &= f(t, y, y'), \quad t > t_0, \\ y(t_0) &= y_0, \quad y'(t_0) = y_1, \end{aligned} \tag{1}$$

where $t_0, y_0, y_1 \in \mathbb{R}$ and $f : [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a given function. More specific assumptions on $t_0, y_0, y_1 \in \mathbb{R}$ and f will be made later. A classical theorem due to Peano [24] guarantee the existence of local solution

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Email addresses: svetlingeorgiev1@gmail.com (Svetlin Georgiev Georgiev), mmajdoub@iau.edu.sa (Mohamed Majdoub), mebarqi_karima@hotmail.fr and karima.mebarki@univ-bejaia.dz (Karima Mebarki)

when f is continuous. The uniqueness fails to hold as shown by the following example

$$y'' = \sqrt{|y|}, \quad y(0) = y'(0) = 0. \tag{2}$$

Note that the Cauchy problem (2) possesses, besides the trivial solution $y \equiv 0$, another global solution $y(t) = \frac{t^4}{144}$ for all $t \in \mathbb{R}$. The fact that the function $y \mapsto \sqrt{|y|}$ is not locally Lipschitz at $y = 0$ is the main reason that causes nonuniqueness. Indeed, when f is locally Lipschitz, we have the uniqueness of local solutions. See the Appendix for more details. The question of uniqueness and nonuniqueness for ODE's was extensively investigated and there is a vast literature dealing with this subject. See, among many, [1, 16–18].

Our main goal here is to prove that (1) has at least two nonnegative global solutions under suitable assumptions on the nonlinearity f and the initial data y_0, y_1. Hence, we suppose the following:

(H1) $t_0, y_0, y_1 \in \mathbb{R}$, $0 \le y_0, |y_1|$, $r = \max\{y_0, |y_1|\} > 0$,

(H2) $f \in \mathcal{C}([t_0, \infty) \times \mathbb{R}^+ \times \mathbb{R})$ and

$$|f(t, w_1, w_2)| < \sum_{j=1}^{l} \left(a_j(t)|w_1|^{p_j} + b_j(t)|w_2|^{p_j} \right), \quad t \ge t_0,$$

for any $(t, w_1, w_2) \in [t_0, \infty) \times \mathbb{R}^+ \times \mathbb{R}$, where $l \in \mathbb{N}$, $a_j, b_j \in \mathcal{C}([t_0, \infty))$ are nonnegative functions such that $A_j = \sup_{t \in [t_0, \infty)} a_j(t)$, $B_j = \sup_{t \in [t_0, \infty)} b_j(t)$ exist and $0 \le A_j, B_j < \infty$, $j \in \{1, \ldots, l\}$, $(A_1, \ldots, A_l, B_1, \ldots, B_l) \neq (0, \ldots, 0, 0, \ldots, 0)$, p_j, $j \in \{1, \ldots, l\}$, are given nonnegative constants so that $(p_1, \ldots, p_l) \neq (0, \ldots, 0)$.

In addition of above conditions, we suppose

(H3) the positive constant m is large enough, ε, A, r_1, L_1 and R_1 are positive constants such that

$$r_1 < L_1 < R_1, \quad \varepsilon > 1, \quad R_1 > \left(\frac{2}{5m} + 1 \right) L_1, \tag{3}$$

$$A \left(R_1 + r + \sum_{j=1}^{l} (A_j + B_j) R_1^{p_j} \right) \le \frac{L_1}{5}, \tag{4}$$

$$0 < r < L_1, \quad 0 < r < \sum_{j=1}^{l} (A_j + B_j) L_1^{p_j}, \quad r \ge \sup_{\substack{t \in [t_0, \infty) \\ r \le w_1, |w_2| < L_1}} |f(t, w_1, w_2)|, \tag{5}$$

(H4) there exists a nonnegative function $g \in \mathcal{C}([t_0, \infty))$ such that

$$A \quad \ge \quad 2 \int_{t_0}^{t} (1 + t - t_1 + (t - t_1)^2)(1 + t_1 - t_0 + (t_1 - t_0)^2) g(t_1) dt_1, \quad t \ge t_0.$$

In the last section, we will give an example for constants $y_0, y_1, m, \varepsilon, A, A_j, B_j$, $j \in \{1, \ldots, l\}$, r_1, L_1, R_1, r and for functions f and g that satisfy $(H1)$-$(H4)$. For the proof of our main result we use (3)-(5) of (H3). In Remark 3.1, we give a motivation for (5) of (H3) and we remove the case when $f(t, w_1, w_2) = \sum_{j=1}^{l} (a_j(t) w_1^{p_j} + b_j(t) w_2^{p_j})$.

Our main result is as follows.

Theorem 1.1. *Suppose* $(H1)$-$(H4)$. *Then the IVP* (1) *has at least two nonnegative solutions.*

Here we propose a new integral representation of the solutions of (1). A new topological approach is applied to prove the existence of at least two nonnegative classical solutions. The arguments are based upon a recent theoretical result.

The paper is organized as follows. In Section 2 we give some preliminary results which will be used for the proof of our main result. In Section 3 we prove our main result. In Section 4 we illustrate our result with an example. A conclusion is given in Section 5.

2. Auxiliary Results

Let X be a real Banach space.

Definition 2.1. *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0,\infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad diam(Y_j) \le \delta, \quad j \in \{1,\ldots,m\}\right\},$$

where $diam(Y_j) = \sup\{\|x-y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1,\ldots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [5].

Definition 2.3. *A mapping $K : X \to X$ is said to be k-set contraction for some number $k \ge 0$ if it is continuous, bounded and $\alpha(K(Y)) \le k\alpha(Y)$, for any bounded set $Y \subset X$.*

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [12]).

Definition 2.4. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that $\|Kx - Ky\|_Y \ge h\|x-y\|_X$ for any $x, y \in X$.*

Definition 2.5. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$. The following result will be used to prove our main result. We refer the reader to [8] and [11] for more details.

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \le k < h-1$ and $S(\overline{U}_3) \subset (I-T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \ne \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \ne \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx \ne (I-T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(ii) *there exists $\epsilon \ge 0$ such that $Sx \ne (I-T)(\lambda x)$, for all $\lambda \ge 1+\epsilon$, $x \in \partial U_2$ and $\lambda x \in \Omega$,*

(iii) *$Sx \ne (I-T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

Lemma 2.7. *Suppose (H1) and (H2). If $y \in C^1([t_0,\infty))$ satisfies the integral equation*

$$y(t) = (t-t_0)y_1 + y_0 + \int_{t_0}^{t} (t-s) f(s, y(s), y'(s))ds, \quad t \ge t_0,$$

then it is a solution to the IVP (1).

Proof. We have

$$\begin{aligned} y(t_0) &= y_0, \\ y'(t) &= y_1 + \int_{t_0}^{t} f(s, y(s), y'(s))ds, \\ y'(t_0) &= y_1, \\ y''(t) &= f(t, y(t), y'(t)), \quad t \geq t_0. \end{aligned}$$

This completes the proof. □

Lemma 2.8. *Suppose* (H1), (H2) *and* (H4). *If for a nonnegative function* $g \in C([t_0, \infty))$ *and for a positive constant* L_1 *a function* $y \in C^1([t_0, \infty))$ *satisfies the integral equation*

$$\begin{aligned} 0 &= \tfrac{L_1}{5} + \int_{t_0}^{t} (t - t_1)^2 g(t_1)\Big(- y(t_1) + (t_1 - t_0)y_1 + y_0 \\ &\quad + \int_{t_0}^{t_1} (t_1 - s) f(s, y(s), y'(s))ds\Big)dt_1, \quad t \geq t_0, \end{aligned} \tag{6}$$

then it is a solution to the IVP (1).

Proof. We differentiate trice the equation (6) and we get

$$0 = g(t)\left(-y(t) + (t - t_0)y_1 + y_0 + \int_{t_0}^{t} (t - s) f(s, y(s), y'(s))ds\right), \quad t \geq t_0,$$

whereupon

$$y(t) = (t - t_0)y_1 + y_0 + \int_{t_0}^{t} (t - s) f(s, y(s), y'(s))ds, \quad t \geq t_0.$$

Now, we apply Lemma 2.7 and we get the desired result. This completes the proof. □

Let $X = C^2([t_0, \infty))$ be endowed with the norm

$$\|y\| = \max\{\|y\|_\infty, \quad \|y'\|_\infty, \quad \|y''\|_\infty\},$$

provided it exists, where $\|y\|_\infty = \sup_{t \geq t_0} |y(t)|$. For $y \in X$, define the operator

$$\begin{aligned} Fy(t) &= \int_{t_0}^{t} (t - t_1)^2 g(t_1)\Big(- y(t_1) + (t_1 - t_0)y_1 + y_0 \\ &\quad + \int_{t_0}^{t_1} (t_1 - s) f(s, y(s), y'(s))ds\Big)dt_1, \quad t \geq t_0. \end{aligned}$$

Lemma 2.9. *Suppose* (H1), (H2) *and* (H4). *If* $y \in X$ *and* $\|y\| \leq b$ *for some positive constant* b, *then*

$$\|Fy\| \leq A\left(b + r + \sum_{j=1}^{l} (A_j + B_j) b^{p_j}\right).$$

Proof. We have

$$
\begin{aligned}
|Fy(t)| &\le \int_{t_0}^{t}(t-t_1)^2 g(t_1)\Big(|y(t_1)|+(t_1-t_0)|y_1|+|y_0| \\
&\quad + \int_{t_0}^{t_1}(t_1-s)|f(s,y(s),y'(s))|ds\Big)dt_1 \\
&\le \int_{t_0}^{t}(t-t_1)^2 g(t_1)\Big(|y(t_1)|+(t_1-t_0)|y_1|+|y_0| \\
&\quad + \sum_{j=1}^{l}\int_{t_0}^{t_1}(t_1-s)\left(a_j(s)|y(s)|^{p_j}+b_j(s)|y'(s)|^{p_j}\right)ds\Big) \\
&\le \int_{t_0}^{t}(t-t_1)^2 g(t_1)\left(b+(t_1-t_0+1)r+\sum_{j=1}^{l}(A_j+B_j)b^{p_j}\int_{t_0}^{t_1}(t_1-s)ds\right)dt_1 \\
&= b\int_{t_0}^{t}(t-t_1)^2 g(t_1)dt_1 + r\int_{t_0}^{t}(t-t_1)^2 g(t_1)(t_1-t_0+1)dt_1 \\
&\quad +\frac{1}{2}\sum_{j=1}^{l}(A_j+B_j)b^{p_j}\int_{t_0}^{t}(t-t_1)^2(t_1-t_0)^2 g(t_1)dt_1 \\
&\le A\left(b+r+\sum_{j=1}^{l}(A_j+B_j)b^{p_j}\right), \quad t\ge t_0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{d}{dt}Fy(t)\right| &\le 2\int_{t_0}^{t}(t-t_1)g(t_1)\Big(|y(t_1)|+(t_1-t_0)|y_1|+|y_0| \\
&\quad + \int_{t_0}^{t_1}(t_1-s)|f(s,y(s),y'(s))|ds\Big)dt_1 \\
&\le 2\int_{t_0}^{t}(t-t_1)g(t_1)\Big(|y(t_1)|+(t_1-t_0)|y_1|+|y_0| \\
&\quad + \sum_{j=1}^{l}\int_{t_0}^{t_1}(t_1-s)\left(a_j(s)|y(s)|^{p_j}+b_j(s)|y'(s)|^{p_j}\right)ds\Big) \\
&\le 2\int_{t_0}^{t}(t-t_1)g(t_1)\left(b+(t_1-t_0+1)r+\sum_{j=1}^{l}(A_j+B_j)b^{p_j}\int_{t_0}^{t_1}(t_1-s)ds\right)dt_1 \\
&= 2b\int_{t_0}^{t}(t-t_1)g(t_1)dt_1 + 2r\int_{t_0}^{t}(t-t_1)g(t_1)(t_1-t_0+1)dt_1 \\
&\quad +\sum_{j=1}^{l}(A_j+B_j)b^{p_j}\int_{t_0}^{t}(t-t_1)(t_1-t_0)^2 g(t_1)dt_1
\end{aligned}
$$

$$\leq \quad A\left(b + r + \sum_{j=1}^{l}(A_j + B_j)b^{p_j}\right), \quad t \geq t_0,$$

and

$$\begin{aligned}
\left|\frac{d^2}{dt^2}Fy(t)\right| &\leq 2\int_{t_0}^{t} g(t_1)\Big(|y(t_1)| + (t_1 - t_0)|y_1| + |y_0| \\
&\quad + \int_{t_0}^{t_1}(t_1 - s)|f(s, y(s), y'(s))|ds\Big)dt_1 \\
&\leq 2\int_{t_0}^{t} g(t_1)\Big(|y(t_1)| + (t_1 - t_0)|y_1| + |y_0| \\
&\quad + \sum_{j=1}^{l}\int_{t_0}^{t_1}(t_1 - s)\left(a_j(s)|y(s)|^{p_j} + b_j(s)|y'(s)|^{p_j}\right)ds\Big) \\
&\leq 2\int_{t_0}^{t} g(t_1)\left(b + (t_1 - t_0 + 1)r + \sum_{j=1}^{l}(A_j + B_j)b^{p_j}\int_{t_0}^{t_1}(t_1 - s)ds\right)dt_1 \\
&= 2b\int_{t_0}^{t} g(t_1)dt_1 + 2r\int_{t_0}^{t} g(t_1)(t_1 - t_0 + 1)dt_1 \\
&\quad + \sum_{j=1}^{l}(A_j + B_j)b^{p_j}\int_{t_0}^{t}(t_1 - t_0)^2 g(t_1)dt_1 \\
&\leq A\left(b + r + \sum_{j=1}^{l}(A_j + B_j)b^{p_j}\right), \quad t \geq t_0.
\end{aligned}$$

Consequently

$$\|Fy\| \leq A\left(b + r + \sum_{j=1}^{l}(A_j + B_j)b^{p_j}\right).$$

This completes the proof. □

3. Proof of the Main Result

Let

$$\widetilde{P} = \{y \in X : y \geq 0 \quad \text{on} \quad [t_0, \infty)\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $y \in X$, define the operators

$$\begin{aligned}
Ty(t) &= (1 + m\varepsilon)y(t) - \varepsilon\frac{L_1}{10}, \\
Sy(t) &= -\varepsilon Fy(t) - m\varepsilon y(t) - \varepsilon\frac{L_1}{10},
\end{aligned}$$

$t \in [t_0, \infty)$. Note that any fixed point $u \in X$ of the operator $T + S$ is a solution to the IVP (1). Define

$$\begin{aligned} U_1 = \mathcal{P}_{r_1} &= \{y \in \mathcal{P} : \|y\| < r_1\}, \\ U_2 = \mathcal{P}_{L_1} &= \{y \in \mathcal{P} : \|y\| < L_1\}, \\ U_3 = \mathcal{P}_{R_1} &= \{y \in \mathcal{P} : \|y\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m}\left(R_1 + r + \sum_{j=1}^{l}(A_j + B_j)R_1^{p_j}\right) + \frac{L_1}{5m}, \\ \Omega &= \overline{\mathcal{P}}_{R_2} = \{y \in \mathcal{P} : \|y\| \le R_2\}. \end{aligned}$$

1. For $y_1, y_2 \in \Omega$, we have
$$\|Ty_1 - Ty_2\| = (1 + m\varepsilon)\|y_1 - y_2\|,$$
whereupon $T : \Omega \to X$ is an expansive operator with a constant $1 + m\varepsilon > 1$.
2. For $y \in \overline{\mathcal{P}}_{R_1}$, we get
$$\begin{aligned} \|Sy\| &\le \varepsilon\|Fy\| + m\varepsilon\|y\| + \varepsilon\frac{L_1}{10} \\ &\le \varepsilon\left(A\left(R_1 + r + \sum_{j=1}^{l}(A_j + B_j)R_1^{p_j}\right) + mR_1 + \frac{L_1}{10}\right). \end{aligned}$$
Therefore $S(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.
3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$.Set
$$v_2 = v_1 + \frac{1}{m}Fv_1 + \frac{L_1}{5m}.$$
Note that by the inequality (4) in $(H3)$ and by Lemma 2.9, it follows that $Fv_1 + \frac{L_1}{5} \ge 0$ on $[t_0, \infty)$. We have $v_2 \ge 0$ on $[t_0, \infty)$ and
$$\begin{aligned} \|v_2\| &\le \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m} \\ &\le R_1 + \frac{A}{m}\left(R_1 + r + \sum_{j=1}^{l}(A_j + B_j)R_1^{p_j}\right) + \frac{L_1}{5m} \\ &= R_2. \end{aligned}$$
Therefore $v_2 \in \Omega$ and
$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon F v_1 - \varepsilon\frac{L_1}{10} - \varepsilon\frac{L_1}{10}$$
or
$$\begin{aligned} (I - T)v_2 &= -\varepsilon m v_2 + \varepsilon\frac{L_1}{10} \\ &= Sv_1. \end{aligned}$$
Consequently $S(\overline{\mathcal{P}}_{R_1}) \subset (I - T)(\Omega)$.

4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda > 0$ and $y \in \partial\mathcal{P}_{r_1} \cap (\Omega + \lambda u_0)$ or $y \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

$$Sy = (I - T)(y - \lambda u_0).$$

Then

$$-\epsilon F y - m\epsilon y - \epsilon\frac{L_1}{10} = -m\epsilon(y - \lambda u_0) + \epsilon\frac{L_1}{10}$$

or

$$-m\epsilon(y - \lambda u_0) = -\epsilon F y - m\epsilon y - \epsilon\frac{L_1}{5},$$

or

$$\lambda m\epsilon u_0 = -\epsilon F y - \epsilon\frac{L_1}{5},$$

which is a contradiction because $\lambda m\epsilon u_0 > 0$ and by the second inequality of ($H3$) and Lemma 2.9, we have $Fy + \frac{L_1}{5} \geq 0$ and then

$$-\epsilon F y - \epsilon\frac{L_1}{5} \leq 0.$$

5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial\mathcal{P}_{L_1}$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$Sx_1 = (I - T)(\lambda_1 x_1). \tag{7}$$

In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial\mathcal{P}_{L_1}$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (7) holds. Since $x_1 \in \partial\mathcal{P}_{L_1}$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1\right)L_1 < \lambda_1 L_1 = \lambda_1\|x_1\| \leq R_1.$$

Moreover,

$$-\epsilon F x_1 - m\epsilon x_1 - \epsilon\frac{L_1}{10} = -\lambda_1 m\epsilon x_1 + \epsilon\frac{L_1}{10},$$

or

$$Fx_1 + \frac{L_1}{5} = (\lambda_1 - 1)m x_1.$$

From here,

$$2\frac{L_1}{5} \geq \left\|Fx_1 + \frac{L_1}{5}\right\| = (\lambda_1 - 1)m\|x_1\| = (\lambda_1 - 1)mL_1$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 2.6 hold. Hence, the IVP (1) has at least two solutions u_1 and u_2 so that

$$r_1 \leq \|u_1\| < L_1 < \|u_2\| \leq R_1.$$

Remark 3.1. 1. *If $u_1 = r$ is a solution of considered IVP, then $r_1 < r < L_1$. And if $u_2 = r$ is a solution of considered IVP, then $L_1 < r < R_1$.*

2. *For any solution u of the IVP* (1) *we have that*

$$\|u\| \geq y_0, |y_1| \quad or \quad \|u\| \geq r.$$

Since $r_1 < \|u_1\| < L_1$ and $L_1 < \|u_2\| < R_1$, we have to have $0 \leq r < L_1$ and $\|u_2\| > r$. Since we have non trivial solutions and if for some solution u of the IVP (1) *we have $\|u\| = r$, then we have to have $r > 0$.*

3. *If $\|u_1\| = |u_1''(t_1)| = r$ for some $t_1 \in [t_0, \infty)$, then $u_1(t)$, $|u_1'(t)| < L_1$, $t \in [t_0, \infty)$, and*

$$\begin{aligned} r &= |u_1''(t_1)| = |f(t_1, u_1(t_1), u_1'(t_1))| \\ &\leq \sum_{j=1}^{l} \left(a_j(t_1)|u_1(t_1)|^{p_j} + b_j(t_1)|u_1'(t_1)|^{p_j}\right) < \sum_{j=1}^{l} (A_j + B_j) L_1^{p_j} \end{aligned}$$

and

$$\begin{aligned} r &= |u_1''(t_1)| = \sup_{t \in [t_0, \infty)} |u_1''(t)| \\ &= \sup_{t \in [t_0, \infty)} |f(t, u_1(t), u_1'(t))|, \end{aligned}$$

which is true because we have supposed (see the last condition of $(H3)$*)*

$$r \geq \sup_{\substack{t \in [t_0, \infty) \\ r \leq w_1, |w_2| < L_1}} |f(t, w_1, w_2)|.$$

4. *If*

$$f(t, w_1, w_2) = a_1(t) w_1^{p_1}, \quad t \in [t_0, \infty), \quad w_1 \in \mathbb{R}^+, w_2 \in \mathbb{R} \tag{8}$$

where $a_1 \in C([t_0, \infty))$ and there exists $0 \leq A_1 = \sup_{t \in [t_0, \infty)} |a_1(t)| < \infty$, by $(H3)$, we get

$$A_1 L_1^{p_1} > r \geq \sup_{\substack{t \in [t_0, \infty) \\ r \leq w_1 < L_1}} \left| a_1(t) w_1^{p_1} \right| = A_1 L_1^{p_1},$$

which is a contradiction. Therefore, when f has the form (8)*, we can not apply our main result. In particular, when $p_1 = 0$ or $p_1 = 1$, we have*

$$A_1 > A_1,$$

which is a contradiction. If

$$f(t, w_1, w_2) = b_1(t) w_2^{p_1}, \quad t \in [t_0, \infty), \quad w_1 \in \mathbb{R}^+, w_2 \in \mathbb{R}, \tag{9}$$

where $b_1 \in C([t_0, \infty))$ and there exists $0 \leq B_1 = \sup_{t \in [t_0, \infty)} |b_1(t)| < \infty$, by $(H3)$, we get

$$B_1 L_1^{p_1} > r \geq \sup_{\substack{t \in [t_0, \infty) \\ r \leq |w_2| < L_1}} \left| b_1(t) w_2^{p_1} \right| = B_1 L_1^{p_1},$$

which is a contradiction. Therefore, when f has the form (9), we can not apply our main result. In particular, when $p_1 = 0$ or $p_1 = 1$, we have

$$B_1 > B_1,$$

which is a contradiction. If

$$f(t, w_1, w_2) = a_1(t) + a_2(t)w_1 + b_1(t)w_2, \quad t \in [t_0, \infty), \quad w_1 \in \mathbb{R}^+, w_2 \in \mathbb{R}, \tag{10}$$

where $a_j, b_1 \in C([t_0, \infty))$, $j \in \{1, 2\}$, there exist $0 \le A_j = \sup_{t\in[t_0,\infty)} |a_j(t)| < \infty$, $j \in \{1, 2\}$, $0 \le B_1 = \sup_{t\in[t_0,\infty)} |b_1(t)| < \infty$, then we consider the IVPs

$$\begin{aligned} y'' &= f_j(t, y, y'), \quad t > t_0, \\ y(t_0) &= \frac{y_0}{3}, \quad y'(t_0) = \frac{y_1}{3}, \quad j \in \{1, 2, 3\}, \end{aligned}$$

where $f_1(t, w_1, w_2) = a_1(t)$, $f_2(t, w_1, w_2) = a_2(t)w_1$, $f_3(t, w_1, w_2) = b_1(t)w_2$, and for every one of them we can not apply our main result. Hence, for (10) we can not apply our main result. If

$$f(t, w_1, w_2) = \sum_{j=1}^{n} \left(a_j(t)w_1^{p_j} + b_j(t)w_2^{p_j} \right), \quad w_1 \in \mathbb{R}^+, w_2 \in \mathbb{R} \tag{11}$$

where a_j, b_j, $j \in \{1, \ldots, l\}$, satisfy (H2), then by the last condition of (H3), we get

$$\sum_{j=1}^{l} (A_j + B_j)L_1^{p_j} > r \ge \sup_{\substack{t \in [t_0, \infty) \\ r \le w_1, |w_2| < L_1}} \left| \sum_{j=1}^{n} \left(a_j(t)w_1^{p_j} + b_j(t)w_2^{p_j} \right) \right| = \sum_{j=1}^{l} (A_j + B_j)L_1^{p_j},$$

which is impossible. Therefore we can not apply our main result for (11).

4. An Example

Let

$$\begin{aligned} l &= 2, \quad t_0 = 0, \quad p_1 = \frac{3}{5}, \quad p_2 = 0, \qquad R_1 = \frac{3}{10^{10}}, \quad r = \frac{4}{3 \cdot 10^{10}}, \quad A_1 = \frac{2}{10^{10}}, \\ A_2 &= \left(\frac{4}{3 \cdot 10^{10}} \right)^{\frac{3}{5}}, \quad L_1 = \frac{2}{10^{10}}, \quad r_1 = \frac{1}{10^{12}}, \quad m = 10^{50}, \quad \epsilon = 50, \\ A &= \frac{1}{10^{10}}, \quad R = 100, \quad B_1 = B_2 = 0. \end{aligned}$$

Then

$$R_1 = \frac{3}{10^{10}} < \frac{5}{10^{10}} = \epsilon \frac{L_1}{20}, \quad r_1 < L_1 < R_1, \quad r_1 = \frac{1}{10^{12}} < \frac{L_1}{5} = \frac{2}{5 \cdot 10^{10}}.$$

Also,

$$\begin{aligned} A\left(R_1 + r + \sum_{j=1}^{l} A_j R_1^{p_j} \right) &= \frac{1}{10^{10}} \left(\frac{3}{10^{10}} + \frac{4}{3 \cdot 10^{10}} + \frac{2}{10^{10}} \left(\frac{3}{10^{10}} \right)^{\frac{3}{5}} + \left(\frac{4}{3 \cdot 10^{10}} \right)^{\frac{3}{5}} \right) \\ &< \frac{1}{10^{11}} \\ &< \frac{L_1}{5}. \end{aligned}$$

Also,

$$R_1 > \left(\frac{2}{5m} + 1\right) L_1.$$

Now, we will construct the function g in $(H4)$. Let

$$h(x) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2) h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2) l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$\begin{aligned} (1 + s + s^2)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) &\leq C_1, \\ (1 + s + s^2)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) &\leq C_1, \end{aligned}$$

$t \in [0, \infty)$, $s \in \mathbb{R}$. Note that by [25](pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^{20}}, \quad s \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ so that

$$C_2 \geq \int_{t_0}^{t} (1 + t - t_1 + (t - t_1)^2)(1 + t_1 + t_1^2) Q(t_1) dt_1, \quad t \in [t_0, \infty).$$

Now, we take

$$g(t) = \frac{1}{10^{20} C_2} Q(t), \quad t \in [t_0, \infty).$$

Then

$$\begin{aligned} A &= \frac{1}{10^{10}} \\ &\geq \int_{t_0}^{t} (1 + t - t_1 + (t - t_1)^2)(1 + t_1 + t_1^2) g(t_1) dt_1, \quad t \in [t_0, \infty). \end{aligned}$$

Now, consider the IVP

$$\begin{aligned} y'' &= h(t)\left(y - \tfrac{4}{3\cdot 10^{10}}\right)^{\frac{3}{5}}, \quad t \in (0,\infty), \\ y(0) &= \tfrac{4}{3\cdot 10^{10}}, \quad y'(0) = 0, \end{aligned} \tag{12}$$

where

$$h(t) = \begin{cases} \frac{1}{10^{10}}(9t^2 - 9t + 2), & t \in [0,1], \\ \frac{2}{10^{10}}, & t > 1. \end{cases}$$

Next,

$$0 < r < L_1, \quad r = \frac{4}{3\cdot 10^{10}} < \frac{2}{10^{10}} \cdot \left(\frac{2}{10^{10}}\right)^{\frac{3}{5}} + \left(\frac{4}{3\cdot 10^{10}}\right)^{\frac{3}{5}} = A_1 L_1^{p_1} + A_2$$

and

$$r = \frac{4}{3\cdot 10^{10}} \geq \sup_{\substack{t \in [0,\infty) \\ \frac{4}{3\cdot 10^{10}} \leq w_1 < \frac{2}{10^{10}}}} \left| h(t)\left(w_1 - \frac{4}{3\cdot 10^{10}}\right)\right| = \frac{2}{10^{10}} \cdot \frac{2}{3\cdot 10^{10}} = \frac{4}{3\cdot 10^{10}}.$$

We have that $(H1)$-$(H4)$ hold. The IVP (12) has two nonnegative solutions $u_1(t) = \frac{4}{3\cdot 10^{10}}$, $t \in [0,\infty)$, and

$$u_2(t) = \begin{cases} \frac{1}{10^{\frac{55}{2}}}(t(1-t))^5 + \frac{4}{3\cdot 10^{10}}, & t \in [0,1], \\ \frac{4}{3\cdot 10^{10}}, & t \in (1,\infty). \end{cases}$$

5. Conclusions

In this paper we investigate a class of initial value problems for second order ODEs. The nonlinear term depends on the solution and its derivative and may change its sign, and it satisfies general polynomial growth conditions. We prove existence of at least two nonnegative solutions of the considered class of second order ODEs. The proof of the main result in the paper is based upon a recent theoretical result. The main result in this paper can be used for some classes second order PDEs.

Appendix A. Existence and uniqueness for ODE

Consider the ordinary differential equation

$$y' = F(t, y) \tag{A.1}$$

where $F : \mathbf{E} \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, with $\mathbf{E}$ an open set. We recall the following existence and uniqueness results for (A.1).

Theorem A.1. *[16, Theorem 3.1, p. 18] If $F(t, y)$ is continuous in $\mathbf{E}$ and locally lipschitzian with respect to y in $\mathbf{E}$, then for any $(t_0, y_0) \in \mathbf{E}$, there exists a unique local solution $y(t)$ of (A.1) satisfying $y(t_0) = y_0$.*

Note that if F is a C^1 function then all assumptions required in Theorem A.1 are satisfied. We also recall the following extension result.

Theorem A.2. *[17, Theorem 3.1, p. 12] Let $F(t, y)$ be continuous on an open set $\mathbf{E}$ and let $y(t)$ be a solution of (A.1) on some interval. Then $y(t)$ can be extended (as a solution) over a maximal interval of existence (t_*, t^*). Also, if (t_*, t^*) is a maximal interval of existence, then $y(t)$ tends to the boundary $\partial\mathbf{E}$ of $\mathbf{E}$ as $t \to t_*$ and $t \to t^*$.*

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