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On Some Sharp Embedding Theorems in Area Nevanlinna Spaces and Related Problems

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Abstract. We provide some new sharp embedding theorems for analytic area Nevanlinna spaces in the unit disk extending some previously known assertions in various directions. Various related results (embedding theorems for area Nevanlinna type spaces) in some other general domains and polydomains will be discussed and provided by us.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let also T_Ω be a tubular domain over symmetric cone in $\mathbb{C}^n$. Let further $\mu(\tilde{\mu})$ be positive Borel measure on T_Ω (on Ω). Let also $\triangle^\alpha$, $\alpha \geq 0$, be determinant function on T_Ω, and $\delta^\alpha(w) = dist^\alpha(w, \partial\Omega)$, $\alpha > -1$. Let dv (dV) be the normalized Lebesgues measure on T_Ω (on Ω). $B(z,r)$ ($\widetilde{B}(z,r)$) $z \in \Omega$ ($z \in T_\Omega$) be Bergman or Kobayashi ball in tubular domains over symmetric cones or pseudoconvex bounded domains in $\mathbb{C}^n$, (see [1], [2]). Let N^p_α be the classical area Nevanlinna space of T_Ω (on Ω). Then we can state the following based on recent techniques developed in recent papers [1], [2].

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

Namely it is easy to show that if for some $\tilde{\alpha}$, $\tilde{\alpha} > 0$,

$$\mu(B(z,r)) \leq c_0(\triangle^{\tilde{\alpha}}(Imz)),\ r > 0,\ z \in T_\Omega.$$

Then

$$\int_{T_\Omega} (\log^+ |f(z)|)^p \, d\mu(z) \leq c\|f\|^p_{N^\varrho_\alpha} = c\int_{T_\Omega} (\log^+ |f(w)|)^p (\triangle^\alpha(Imw))dv(z),$$

for $\alpha > -1$ and for $1 \leq p < \infty$, and also

$$\int_\Omega (\log^+ |f(z)|)^p d\tilde{\mu}(z) \leq \tilde{c}\|f\|^p_{N^p_\alpha} = \tilde{c}\int_\Omega (\log^+ |f(w)|)^p \, \delta^\alpha(w)dV(w)$$

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for $1 \le p < \infty$, $\alpha > -1$. If the following condition holds $\tilde{\mu}(\tilde{B}(z,r)) \le c_2(\delta^{\tilde{\alpha}}(z))$, $z \in \Omega$, $r > 0$, for some $\widetilde{\alpha}$ and for some positive constants c, c_0, $\widetilde{c}$, c_2. Same type results can be provided in the polydisk and in bounded symmetric domains based on the same known technique (see, for example, [17] for unit ball case). These type of conditions on measure will be called Carleson type conditions.

Let w be a function from a set S of all positive growing functions, $w \in L^1(0,1)$ such that there are two numbers $m_w > 0$, $M_w > 0$ and a number $q_w \in (0,1)$ such that

$$m_w \le \frac{w(\lambda\tau)}{w(\tau)} < M_w,\ \tau \in (0,1),\ \lambda \in [q_w, 1],$$

(this condition will be used in proofs of $2^\circ \Rightarrow 1^\circ$ in theorems 2.1 and 2.2). Let $w \in S$, then there are measurable functions $\varepsilon(x)$, $q(x)$ so that

$$w(x) = exp\left\{q(x) + \int_x^1 \frac{\varepsilon(u)}{u}du\right\},\ x \in (0,1).$$

This characterization gives various examples of funkcions from S class. See properties of these classes in [15].

We need the following simple estimate for proofs of theorems 2.1-2.3, (see [6]):
Let $s > 1$, $w \in S$ then

$$\int_0^1 \frac{w(1-r)}{(1-\rho r)^s}dr \le \frac{Cw(1-\rho)}{(1-\rho)^{s-1}},\ \rho \in (0,1),$$

(this condition will be used in proofs of $1^\circ \Rightarrow 2^\circ$ in theorems 2.1 and 2.2).

Let $T(r,f)$ be, as usual, Nevanlinna characteristic of analytic function f

$$T(r,f) = \int_T \log^+ |f(r\xi)| d\xi,\ r \in (0,1).$$

Let define $\triangle_{k,s}$ as a standard dyadic cube in the unit disk (see [8])

$$\triangle_{k,s} = \left\{z \in \mathcal{D} : 1 - \frac{1}{2^k} \le |z| < 1 - \frac{1}{2^{k+1}}, \frac{2\pi s}{2^{k+1}} \le \arg z < \frac{2\pi(s+1)}{2^{k+1}}\right\},$$

$s = -2^{k+1}, \dots, 2^{k+1}-1$, $k = 0, 1, 2, \dots$.

Let further $|\triangle_{k,s}|$ be Lebegues measure of $\triangle_{k,s}$.

In the unit disk $\mathcal{D}$, $T = \partial\mathcal{D} = \{|z| = 1\}$ the following sharp results were provided recently in [6].

Theorem A. *Let μ be finite nonnegative Borel measure defined on subsets of $\mathcal{D}$. Let $1 \le p < \infty$. Then the following are equivalent:*

1° $\int_{\mathcal{D}} (\ln^+ |f(\xi)|)^p d\mu(\xi) \le c_3 \int_0^1 w(1-r) T^p(r,f) dr < \infty$,

2° $\mu(\triangle_l(\theta)) \le c_4 w(l) l^{p+1}$, $\theta \in [-\pi, \pi]$, $l \in (0,1)$,

$\triangle_l(\theta) = \{z \in \mathcal{D} :\ 1 - l < |z| < 1,\ |argz - \theta| \le \frac{l}{2}\}$.

Theorem B. *Let μ be a finite nonnegative Borel measure defined on subset of $\mathcal{D}$. Let $0 < p < 1$, $r_k = 1 - \frac{1}{2^k}$, $k = 0, 1, 2, \dots$, Then the following are equivalent.*

1° $\int_{\mathcal{D}} (\ln^+ |f(\xi)|)^p d\mu(\xi) \le c_5 \int_0^1 w(1-r) T^p(r,f) dr < +\infty$.

2° $\sum_{s=-2^k}^{2^k-1} \mu(\triangle_{k,s})^{\frac{1}{1-p}} \le c_6 (1-r_k)^{\frac{1+p}{1-p}} \left(w(1-r_k)^{\frac{1}{1-p}}\right)$.

Various basic properties of Nevanlinna type spaces can be seen in [4]. Results of this paper can be extended partially to more general area Nevanlinna type spaces studied in [9].

We use heavily some nice technique developed in [6] to extend the sharp results.

We refer to [3] for similar type results concerning sharp embeddings in area Nevanlinna type spaces in the unit disk. See also [15] for various new embeddings in area Nevanlinna spaces in the unit disk.

2. Main results

The goal of this note to extend those sharp results in theorems A and B using similar ideas to other values of parameters. Namely we obtained the following sharp embedding theorems for area Nevanlinna type spaces in the unit disk $\mathcal{D}$.

Theorem 2.1. *Let $q \le p$, $p > 1$. Let μ be positive Borel measure on $\mathcal{D}$. Then the following are equivalent:*

1° $\int_{\mathcal{D}}(\log^+|f(z)|)^p d\mu(z) \le c_8 \int_0^1 w(1-r)\left(\int_{-\pi}^{\pi}\log^+|f(r\xi)|d\xi\right)^q dr.$

2° $\mu(\triangle_l) \le c_7 w(l)^{\frac{p}{q}} l^{\frac{p}{q}+p}.$

Note obviously for $q = p$ we obtain immediately Theorem A.

The following result is another sharp extension of Theorem A.

Theorem 2.2. *Let $q \le 1$ and let μ be positive Borel measure on $\mathcal{D}$. Then the following conditions are equivalent:*

1° $\int_{\mathcal{D}}\log^+|f(z)|d\mu(z) \le c_9 \int_0^1 w(1-r)\left(\int_{-\pi}^{\pi}\log^+|f(r\xi)|d\xi\right)^q dr.$

2° $\mu(\triangle_l) \le c_{10} w(l)^{\frac{1}{q}} l^{\frac{1}{q}+1}.$

The following sharp result is a direct extension of Theorem B.

We have the following result.

Theorem 2.3. *Let μ be positive Borel measure on $\mathcal{D}$. Let $q \le p$, $p < 1$. Then the following conditions are equivalent:*

$$\int_{\mathcal{D}}(\log^+|f(z)|)^p\, d\mu(z) \le C_2 \int_0^1 \left(\int_T \log^+|f(r\xi)|d\xi\right)^q w(1-r)dr. \tag{1}$$

$$\left(\sum_{s=-2^k}^{2^{k-1}} \mu(\triangle_{k,s})^{\frac{1}{1-p}}\right)^{1-p} \le C_1 w(1-r_k)^{\frac{p}{q}}(1-r_k)^{\frac{(q+1)p}{q}} \tag{2}$$

Remark 2.4. *Note (2) type condition, namely 2° from Theorem B, with $p = q$ is also sufficient for embedding of the type*

$$\int_{\mathcal{D}}(\log^+|f(z)|)^{\tilde{p}} d\mu(z) \le C \int_0^1 \left(\int_T \log^+|f(r\xi)|d\xi\right)^q w(1-r)dr$$

where $q \le 1$, $\widetilde{p} > 1$, $q \le \tilde{p}$. This can be seen from our proof.

Remark 2.5. *Similar sharp theorems with very similar proof can be obtained if we replace the right side in Theorem 2.1 and Theorem 2.2 by spaces with quasi norms*

$$\sum_{k\ge 0}\left(\int_{1-2^{-k}}^{1-2^{-(k+1)}} w(1-r)\left(\int_T \log^+|f(r\xi)|d\xi\right)^q dr\right)^s,$$

$$\int_0^1\left(\int_{|z|\le r}\log^+|f(w)|(1-|w|)^\alpha dm_2(w)\right)^q w(1-r)dr,$$

(readers can easily recover such theorems based on our proofs below) for $0 < s \le 1$, $\alpha > -1$, $0 < q < \infty$, where dm_2 is a Lebesgues measure in $\mathcal{D}$, where w is a weight from a S function class, (see [6] for these weights) with some additional restrictions on parameters.

Sufficiency in Theorems 2.1 and 2.2:
Let B be unit ball in $\mathbb{C}^n$. Let μ also be a positive Borel measure in B, S be unit sphere (see [6], [17]) and let also $M_p^p(f,r) = \int_S |f(r\xi)|^p d\xi,\ r \in (0,1),\ p \in (0,\infty)$. First the sufficiency condition on μ measure follows directly from the following result (from the first side unrelated result).

$$\int_B \left|\tilde{f}(z)\right|^p d\mu(z) \le C_3 \int_0^1 \left(M_q^\tau(\tilde{f},r)\right)(1-r)^{-\tau s-1}dr$$

if and only if

$$\mu(\mathcal{D}(a,r) \le C_4 \left(1-|a|^2\right)^{n\frac{p}{q}-sp},\ a \in B,$$

(see [7]), where $\mathcal{D}(a,r)$ is a Bergman ball in B and $s<0, q<p$ or $q=p, \tau \le p, s<0$.

The proof of this fact (sufficiency part) is uses only the fact that $|\tilde{f}(z)|^p$ is subharmonic for $p \ge 0$, so the same proof can be passed for $\tilde{f} = (\log^+|f(z)|)^p$ if $p \ge 1$.

This can be seen after careful analysis of proofs in [7], [8]. Note also for $p \ge 1$ we have that

$$\int_B (\log^+|f(z)|)^p d\mu(z) \le C_5 \|\log^+|f(z)|\|^p_{F_s^{q,\tau}}$$

for $s<0, q<p$ or $s<0, q=p, \tau \le p$ if

$$\mu(\mathcal{D}(a,r)) \le C_6(1-|a|^2)^{\frac{np}{q}-sp},\ a \in B,$$

where

$$\|f\|^q_{F_s^{q,\tau}} = \int_S \left(\int_0^1 |f(r\xi)|^\tau (1-r)^{-\tau s-1}dr\right)^{\frac{q}{\tau}} d\xi,\ s<0,\ q,\tau \in (0,\infty)$$

(see [7], [8]).

These assertions are obviously more general than those in the unit disk in Theorems A and B (sufficiency of condition). Proofs of Theorem 2.1 and 2.2 for $F_\alpha^{p,q}$ spaces (sufficiency case) can be also provided.

We provide direct proof following ideas from [6], [8].

Using dyadic partition $\triangle_{k,s}$ of unit disk and standard arguments we have

$$M = \int_{\mathcal{D}} (\ln^+|f(\xi)|)^p d\mu(\xi) \le C_7 \sum_{k=0}^{+\infty} w(1-r_k)^{\frac{p}{q}}(1-r_k)^{\frac{p}{q}} \sum_{s=-2^k}^{2^{k-1}} \ln^+|f(\xi_{k,s})|(1-r_k)^p,$$

where $\xi_{k,s}$ is a center of $\triangle_{k,s}$. Then following [6], [8] and standard arguments we have

$$M \le \int_0^1 w(1-r)^{\frac{p}{q}}(1-r)^{\frac{p}{q}-1}(T(r,f))^p dr = J.$$

Note now $J^{\frac{q}{p}} \le \int_0^1 w(1-r)\,(T(r,f))^q\,dr,\ q \le p$.

The last estimate follows from properties of w and the fact that $T(r,f)$ is growing (see [6], [8]). For the proof of sufficiency of Theorem 2.2 we do the same and arrive at

$$M_1 \le C_8 \int_0^1 \left(w^{\frac{1}{q}}(1-r)\right)(1-r)^{\frac{1}{q}-1}T(r,f)dr = J_1,$$

and hence $J_1^q \le \int_0^1 w(1-r)(T(r,f))^q dr, q \le 1$. Since $T(r,f)$ is growing function we used that

$$\left(\int_0^1 w(1-r)G(r)dr\right)^s \le C_9 \int_0^1 w^s(1-r)(1-r)^{s-1}G^s(r)dr,\ s \le 1,$$

for every growing G function on $(0, 1)$, (see [6], [9]).

Note our proof has some similarities with known standard proofs (see [6], [9] and references there). The scope of this arguments is valid (so we get some extension of theorems) also in the unit ball and unit polydisk, where complete analogues of $\triangle_{k,s}$ dyadic cubes also exists, see [8] for the unit polydisk case and [17] for the unit ball case.

Necessity in Theorems 2.1 and 2.2:

Following the proof of theorem in [6] and using standard arguments we have to put $g(z) = u(z) + iv(z)$. Let also $u^+(z) = \max(0, u(z))$, $u^-(z) = \max(0, -u(z))$. We have

$$\int_{\mathcal{D}} (u^+(z))^p d\mu(z) \leq C_{10} \int_0^1 w(1-r)\,(T^q(u,r))\,dr,$$

(see [6]).

The same is valid for $u^-(z)$ as a result we have the same for $\int_{\mathcal{D}} |u(\xi)|^p d\mu(\xi)$ (see [6]). Following arguments of [6] the same is valid for $\int_{\mathcal{D}} |v(\xi)|^p d\mu(\xi)$ and as a result we have

$$\left(\int_{\mathcal{D}} |g(\xi)|^p d\mu(\xi)\right)^{\frac{1}{p}} \leq K = C_{11} \left(\int_0^1 w(1-r) \left(\int_{-\pi}^{\pi} |g(r\xi)| d\xi\right)^q dr\right)^{\frac{1}{q}}$$

for any q, and $q \leq p < \infty$. Putting here standard test function and following standard arguments (see [17]) we have immediately what we need in both theorems (Theorem 2.1 and 2.2).

Since if $\exp(\pm g(z)) = F(z)$, $M(g) < \infty$, then

$$\int_0^1 w(1-r) \left(\int_T \log^+ |F(r\xi)| d\xi\right)^p dr < \infty$$

where

$$M(g) = \int_0^1 w(1-r) \left(\int_T |g(r\xi)| d\xi\right)^p dr < \infty,$$

(see [6]).

Note first $K \leq cw(1-a)(1-a)^{1-p(\beta-1)}$. Indeed, we also have

$$\int_{\mathcal{D}} |g(z)|^p d\mu(z) \geq \frac{\mu(\Delta_l)}{l^{\beta p}},\ l = 1-a,$$

for $g(z) = \dfrac{(1-|a|^2)^\beta}{(1-az)^{2\beta}}$, $\beta > \beta_0$, $a \in (0, 1)$ (see [17]) and the estimate from above is also trivial. We refer to [17] for very similar standard arguments and some elementary properties of w, $w \in S$ also must be used, see [6]. □

Proof of Theorem 2.3:

Sufficiency:

We follow the proof Theorem B (this is $q = p$ case in our theorem). We note very similar arguments can be seen in [7], [8].

We have

$$M = \int_{\mathcal{D}} \ln^+ |f(\xi)|^p d\mu(\xi) \leq c \sum_{k=0}^{+\infty} \sum_{s=-2^k}^{2^{k+1}} |\ln^+ f(\xi_{k,s})|^p \mu(\triangle_{k,s}),\ p < 1,$$

where $\xi_{k,s}$ is a max point of $f(\xi)$ in $\triangle_{k,s}$. Then by Hardy-Littlewood theorem

$$(\ln^+ |f(\xi_{k,s})|)^p \leq \frac{c_1}{|\triangle_{k,s}|} \int_{r_k}^{r_k+1} \int_{\alpha_{k,s}}^{\alpha_{k,s+1}} \left|\ln^+ f(\rho\xi)\right|^p d\xi d\rho,\ p < 1$$

$\alpha_{k,s} = \frac{\pi s}{2^k}$, $r_k = 1 - \frac{1}{2^k}$, $s = -2^k, \dots, 2^k$, $k \geq 0$.

Applying twice Hölder's inequality, see also for similar arguments [7], [8]. We arrive at the following estimate

$$M \leq c_2 \sum_{k=0}^{\infty} \frac{1}{|\triangle_{k,s}|} \int_{r_k}^{r_{k+1}} \left(\int_{-\pi}^{\pi} \ln^+ |f(\rho\xi)| d\xi \right)^p \left(\sum_{s=-2^k}^{2^k-1} \mu(\triangle_{k,s})^{\frac{1}{1-p}} (\alpha_{k+1,s} - \alpha_{k,s}) \right)^{1-p} d\rho.$$

Note now using condition on measure we arrive at the following estimate easily

$$M \leq c_3 \int_0^1 (w(1-r))^t \cdot (T(r,f))^p dr,$$

for some $t = t(p,q)$.

Note that

$$\left(\sum_{s=-2^k}^{2^k-1} (\mu(\triangle_{k,s}))^{\frac{1}{1-p}} \right)^{(1-p)} \frac{(1-r_k)^{1-p}}{|\triangle_{k,s}|} \leq c_4 \frac{w(1-r_k)^{\frac{p}{q}} (1-r_k)^{\nu+1-p}}{|\triangle_{k,s}|} \leq$$

$$\leq w(1-r_k)^{\frac{p}{q}} (1-r_k)^{\frac{p}{q}-1}, \ \nu = \frac{(q+1)p}{q}.$$

And we obviously have what we need. Since $\frac{q}{p} \leq 1$. We follow arguments of the end of the previous theorem to arrive at estimate we need the last estimate.

Necessity:

Let us show the reverse in Theorem 2.3. This proof is also somewhat standard and use standard arguments with Rademacher function. Indeed, we have the following now chain of estimates (see also [6]). Let first

$$f(z,t) = \sum_{s=0}^{2^k-1} \frac{c_{k,s}\varphi_s(t)}{(1-\bar{z}_{k,s}z)^n}, \ z \in \mathcal{D}, \ t \in (0,1),$$

where $c_{k,s}$ is arbitrary sequence of complex numbers, and $z_{k,s}$ is the center of $\triangle_{k,s}$, $n > n_0$, $n \in \mathbb{N}$, $\varphi_s(t)$ is a Rademacher function of s order.

Put $F(z,t) = \exp(f(z,t))$, $t \in (0,1)$, and if our embedding holds, then we have

$$\int_{\mathcal{D}} \left| f(z,t) \right|^p d\mu(z) \leq c_5 \int_0^1 w(1-r) \left(\int_{-\pi}^{\pi} |f(re^{i\theta}, t)| d\theta \right)^q dr, \ p \geq 1, \ q \leq p.$$

Repeating almost arguments from [6] we arrive at the following estimate

$$\sum_{s=0}^{2^k-1} |c_{k,s}|^p (\mu(\triangle_{k,s})) \leq c_6 \left(w(1-r_k)^{\frac{p}{q}} \right) \left((1-r_k)^{\frac{q+1}{q}p} \right) \left(\sum_{s=0}^{2^k-1} |c_{k,s}| \right)^p$$

$|c_{k,s}| = |b_s|^{\frac{1}{p}}$, (we have to integrate by unit interval, change the order of integration and use some well-known properties of Rademacher function). Then by Hahn-Banach we have that

$$\sup_{\|b\|_{l^{1/p}} \leq 1} \sum_{s=0}^{2^k-1} |b_s| \mu(\triangle_{k,s}) = \left(\sum_{s=0}^{2^k-1} \mu(\triangle_{k,s})^{\frac{1}{1-p}} \right)^{p-1} \leq c_7 \left(w(1-r_k)^{\frac{p}{q}} \right) (1-r_k)^{p+1}.$$

But this is exactly what we need for our proof. Our theorem is proved now. □

Remark 2.6. *It is an open problem to show such type sharp results in more complicated domains that is to show that reverse to (1) and (2) are valid or not. Note that sufficient conditions on measure in the unit ball can be obtained also for various Herz-Nevanlinna type spaces from known sharp embeddings in the ball for analytic Herz spaces (see [12] and see discussion above also).*

Remark 2.7. *Some results are valid even in polyballs and more difficult domains (see next section).*

Note concerning more general spaces

$$\sum_{k\geq 0}\left(\int_{1-2^{-k}}^{1-2^{-(k+1)}}(w(1-r))\left(\int_T \log^+|f(r\xi)|d\xi\right)^q dr\right)^s,\ q>0,\ s>o.$$

We can use at the last step an obvious embedding

$$\left(\int_0^1 (w(1-r))\left(\int_T \log^+|f(r\xi)|d\xi\right)^q dr\right)^s \leq$$

$$\leq C\sum_{k\geq 0}\left(\int_{1-2^{-k}}^{1-2^{-(k+1)}}(w(1-r))\left(\int_T \log^+|f(r\xi)|d\xi\right)^q dr\right)^s,\ s\leq 1,\ q>0,$$

to get somewhat more general sharp versions of our sharp Theorems 2.1, 2.2 and 2.3. The same type embedding is valid for

$$\|f\|_{v,s}=\int_0^1\left(\int_{|z|\leq R}\log^+|f(r\xi)|(1-r)^v dm_2(r\xi)\right)^s dR,\ s\leq 1,\ v>-1,$$

spaces. We have now that obviously

$$\|f\|_{v,1}\leq C\int_0^1\left(\int_{|z|\leq R}\log^+|f(r\xi)|(1-r)^v dm_2(r\xi)\right)^s (1-R)^{s-1}dR,\ s\leq 1,\ v>-1.$$

This and an obvious relation between $\|f\|_{v,1}$ and area Nevanlinna N^p_α spaces also provide some sharp extensions of our sharp results from Theorems 2.1, 2.2. We leave this task to interested readers.

3. Some remarks on embeddings for mixed norm analytic area Nevanlinna type spaces in product domains in $\mathbb{C}^n$

In this section similar results will be given in some polydomains. We start with simpler domain the polydisk, then turn to some objects in more general domains. We consider very general Nevanlinna type spaces in tubular domains over symmetric cones and in bounded strongly pseudoconvex domains with sooth boundary.

Let

$$\mathcal{D}^n=\{z=(z_1,\dots,z_n):|z_j|<1,1\ \leq j\leq n\}$$

be the unit polydisc of n-dimensional complex space $\mathbb{C}^n$, T^n be the Shilov boundary of $\mathcal{D}^n$, $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $j=1,\dots,n$, $\vec{\omega}(t)=(\omega_1(t),\dots,\ \omega_n(t))$, $t\in(0,1)$, where $\omega_j(t)$ is positive integrable functions on $(0,1)$. We denote by $N^{\vec{p}}(\vec{\omega})$ the set of all holomorphic functions in $\mathcal{D}^n$ for which

$$\|f\|_{A^{\vec{p}}(\vec{\omega})}=\Big(\int\limits_{\mathcal{D}}\Big(\dots(\int\limits_{\mathcal{D}}(\log^+\big|f(z_1,\dots,z_n)\big|)^{p_1}\omega_1(1-|z_1|)dm_2(z_1))^{\frac{p_2}{p_1}}\times$$

$$\times\dots\Big)^{\frac{p_n}{p_{n-1}}}\omega_n(1-|z_n|)dm_2(z_n)\Big)^{\frac{1}{p_n}}<+\infty,$$

where m_2 is planar normalized Lebesgue measure on $\mathcal{D} := \mathcal{D}^1$. Assume further $\vec{\mu} = (\mu_1, \dots, \mu_n)$, where μ_j is the Borel nonnegative finite measure on $\mathcal{D}$, $L^{\vec{p}}(\vec{\mu})$ is relevant space with mixed norm that is, the space of all measurable functions on $\mathcal{D}^n$ for which

$$\|f\|_{L^{\vec{p}}(\vec{\mu})} = \Big(\int\limits_{\mathcal{D}} \Big(...(\int\limits_{\mathcal{D}} (\log^+ \big|f(\zeta_1, \dots, \zeta_n)\big|)^{p_1} d\mu_1)^{\frac{p_2}{p_1}} ...\Big)^{\frac{p_n}{p_{n-1}}} d\mu_n\Big)^{\frac{1}{p_n}} < +\infty.$$

We consider in this paper new analytic spaces on products of tubular domains $T_\Omega \times \cdots \times T_\Omega = T_\Omega^m$. We denote by $H(T_\Omega \times \cdots \times T_\Omega) = H\left(T_\Omega^m\right)$, $m \in \mathbb{N}$, the space of analytic functions by each variable on T_Ω^m.

Let

$$L_{\vec{\alpha}}^{\vec{p}}\left(T_\Omega^m\right) = \{f \in L_{loc}^1\left(T_\Omega^m\right) :$$

$$\left(\int\limits_{T_\Omega} \cdots \left(\int\limits_{T_\Omega} (\log^+ \big|f(z_1, \dots, z_m)\big|)^{p_1} \triangle^{\alpha_1}(z_1) dv(z_1) \right)^{\frac{p_2}{p_1}} \cdots \triangle^{\alpha_m}(z_m)\, dv(z_m) \right)^{\frac{1}{p_m}} < \infty\},$$

where $\triangle$ is a determinant of T_Ω (see [2] for $m = 1$ case), dv is Lebesgues measure on T_Ω, $0 < p_i < \infty$, $\alpha_i > -1$, $i = 1, \dots, m$. Let $N_{\vec{\alpha}}^{\vec{p}}\left(T_\Omega^m\right) = L_{\vec{\alpha}}^{\vec{p}}\left(T_\Omega^m\right) \cap H\left(T_\Omega^m\right)$, $m \in \mathbb{N}$, be a new area Nevanlinna space in T_Ω^m with mixed norm.

These are Banach spaces if $\min\limits_j(p_j) \geq 1$ and complete metric spaces for other values of p.

Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$, μ_j be Borel nonnegative measures on $T_\Omega \subset \mathbb{C}^k$, $k \in \mathbb{N}$, $\alpha_j > -1$, $j = 1, \dots, n$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n) \in \mathbb{R}_+^n$ with $0 < p_j \leq q_j$, $p_j > 1$, $j = 1, \dots, n$. We can following proofs of [16] find sufficient conditions on measures (Carleson type conditions) for the following embeddings:

$$\|f\|_{L^{\vec{q}}(\vec{\mu})} \leq C\,(\vec{\mu})\, \|f\|_{N_{\vec{\alpha}}^{\vec{p}}}.$$

In the case of measures ν defined on T_Ω^n the following observation holds:

Let $p_j \leq q < +\infty$, $p_j > 1$, $\alpha_j > -1$, $j = 1, \dots, n$, ν be the Borel nonnegative measure on T_Ω^n. We can following proofs of [16] find sufficient conditions on measures (Carleson type conditions) for the following embeddings:

$$\Big(\int\limits_{T_\Omega^n} (\log^+ \big|f(z)\big|)^q d\nu(z) \Big)^{\frac{1}{q}} \leq C(\nu) \big\|f\big\|_{N_{\vec{\alpha}}^{\vec{p}}}.$$

Also, we consider in this paper new analytic spaces on products of pseudoconvex domains $\Omega \times \cdots \times \Omega = \Omega^m$. We denote by $H(\Omega \times \cdots \times \Omega) = H(\Omega^m)$, $m \in \mathbb{N}$, the space of analytic functions by each variable on Ω^m.

Let

$$L_{\vec{\alpha}}^{\vec{p}}(\Omega^m) = \{f \in L_{loc}^1(\Omega^m) :$$

$$\left(\int\limits_{\Omega} \cdots \left(\int\limits_{\Omega} (\log^+ \big|f(z_1, \dots, z_m)\big|)^{p_1} \delta^{\alpha_1}(z_1) dV(z_1) \right)^{\frac{p_2}{p_1}} \cdots \delta^{\alpha_m}(z_m)\, dV(z_m) \right)^{\frac{1}{p_m}} < \infty\},$$

$0 < p_i < \infty$, $\alpha_i > -1$, $i = 1, \dots, m$. Let $N_{\vec{\alpha}}^{\vec{p}}(\Omega^m) = L_{\vec{\alpha}}^{\vec{p}}(\Omega^m) \cap H(\Omega^m)$, $m \in \mathbb{N}$, (see [1] for $m = 1$ case), be a new area Nevanlinna space in Ω^m with mixed norm.

These are Banach spaces if $\min\limits_j(p_j) \geq 1$ and complete metric spaces for other values of p.

Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$, μ_j be Borel nonnegative measures on $\Omega \subset \mathbb{C}^m$, $m \in \mathbb{N}$, $\alpha_j > -1$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n) \in \mathbb{R}_+^n$ with $p_j > 1$, $0 < p_j \leq q_j$, $j = 1, \dots, n$. We can following proofs of [11] find sufficient conditions on measures (Carleson type conditions) for the following embeddings:

$$\|f\|_{L_{\vec{\mu}}^{\vec{q}}} \leq C_0\,(\vec{\mu})\, \|f\|_{N_{\vec{\alpha}}^{\vec{p}}}.$$

In the case of measures μ defined on $\Omega^n = \Omega \times \cdots \times \Omega$ we have following observation:

Let $p_j \leq q < +\infty$, $p_j > 1$, $\alpha_j > -1$, $j = 1, \ldots, n$, μ be the Borel nonneqative measure on Ω^n. Let $\vec{z} = (z_1, ..., z_n)$. We can following proofs of [11] find sufficient conditions on measures (Carleson type conditions) for the following embeddings:

$$\Big(\int_{\Omega^n} (\log^+ \left|f(\vec{z})\right|)^q d\mu(\vec{z})\Big)^{\frac{1}{q}} \leq C_2(\vec{\mu})\left\|f\right\|_{N^{\vec{p}}_{\vec{\alpha}}}.$$

Proofs of these results are using heavily the fact that $(\log^+|f|)^p$ for $p > 1$ is subharmonic in these domains (or n -subharmonic in products of these domains).

Based on sharp embeddings in Herz spaces we can discuss some similar questions in analytic Nevanlinna-Herz type spaces in tubular domains over symmetric cones and pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. We mean some conditions on positive Borel measure μ for which the following estimate is valid

$$\int_{T_\Omega} (\log^+|f(z)|)^s d\mu(z) \leq \widetilde{C} \int_{T_\Omega} \left(\int_{B(\widetilde{z},r)} (\log^+|f(z)|)^q \triangle^\alpha(Im\, z) dv(z)\right)^{\frac{p}{q}} dv(\widetilde{z}),$$

with some restrictions on s, q, p, α and similar embeddings in pseudoconvex domains. See, for example, sharp embeddings in Herz spaces in tubular domains over symmetric cones in [14] and we refer for similar sharp embeddings in Herz spaces in pseudoconvex domains to [13].

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let $d(z) = dist(z, \partial\Omega)$. Then there is a neighborhood U of $\overline{\Omega}$ and $\rho \in C^\infty$ such that $\Omega = \{z \in U : \rho(z) > 0\}$, $|\nabla\rho(z)| \geq c > 0$ for $z \in \partial\Omega$, $0 < \rho(z) < 1$ for $z \in \Omega$ and $-\rho$ is strictly plurisubharmonic in a neighborhood U_0 of $\partial\Omega$. Note that $d(z) \asymp \rho(z)$, $z \in \Omega$. Then there is an $r_0 > 0$ such that the domains $\Omega_r = \{z \in \Omega : \rho(z) > r\}$ are also smoothly bounded strictly pseudoconvex domains for all $0 \geq r \geq r_0$. Let $d\sigma$ be the Lebesgue measure on $\partial\Omega_r$, (see [11]).

Since $\int_{\partial\Omega_r} (\log^+|f(z)|)^p d\sigma(z)$ is growing for $p > 1$, we can easily find embeddings relating these quasinorms for different p, q

$$\int_0^\rho \left(\int_{\partial\Omega_r} (\log^+|f(z)|)^p d\sigma(z)\right)^{\frac{q}{p}} r^\alpha dr$$

for $\alpha > -1$, $1 < p < \infty$, $0 < q < \infty$. Such quasinorms were studied actively in [5].

Let G be simply connected region on $\mathbb{C}$, ∂G it is boundary. $H(G)$ is the space of all analytic functions on G, $\ln^+ x = \max_{x>0}(\ln x, 0)$ and $d(w, \partial G)$ is a distance from w to ∂G.

We define analytic area Nevanlinna classes in these general domains

$$N^p_\alpha(G) = \{f \in H(G) : \int_G d^\alpha(w, \partial G)\,(\ln^+|f(w)|)^p\, dm_2(w) < \infty\}, 0 < p < \infty, \alpha > -1.$$

Let $G_{\vec{\beta}}$ be bounded simply connected region of complex plane. $G_{\vec{\beta}}$ is (B_n) region if $\partial G_{\vec{\beta}}$ is equal to $\bigcup_{s=1}^m \Gamma_s, m \in \mathbb{N}$, where Γ_s is a smooth arc for which the "angle points" (if they exist) are equal to $\left(\dfrac{\pi}{\beta_j}\right)$, $j = 1, \ldots, m$.

In [15] $N^p_\alpha(G_{\vec{\beta}})$ classes were studied by authors.

Our results maybe also extended to these general area Nevanlinna function spaces in the unit disk. We refer to [15] for some results in these classes of functions.

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