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Equivalence Between Distribution Functions and Probability Measures on a LOTS

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Abstract. In this paper we give some conditions such that there is an equivalence between probability measures and distribution functions defined on a separable linearly ordered topological space like it happens in the classical case. What is more, we prove that there is a one-to-one relationship between a probability measure and the pseudo-inverse of its cumulative distribution function.

1. Introduction

The study of measures on topological spaces (see [6]) lies in the intersection of functional analysis, measure theory, general topology and probability theory and is a very wide research field, with multiple connection between fields.

In this paper we are concerned with linearly ordered topological spaces (LOTS). The study of measures on LOTS is also of interest (see [2, 5, 14, 20–22]).

On the other hand, the description of probability measures by using a cumulative distribution function (cdf) is standard in mathematics, due to the benefit and simplification that provides a cdf over a probability measure. Indeed, there are some studies where the equivalence between probability measures and distribution functions has been treated. For example, in [16] it is proved the equivalence between probability measures and fuzzy intervals in the real line. On the other hand, in a similar research line, probabilistic metric and normed spaces provide a certain relationship between topology and probability measures. See [3] for further reference about this topic.

However, as far as we know, there has been no attempt to extend the theory of a cdf to a more general framework, where a first natural step is to deal with a probability measure on an ordered space. This work collects some results on a theory of a cdf on a separable LOTS.

This theory was first described in [11], where we showed that the cdf plays a similar role to that played in the classical case and studied its pseudo-inverse, which allowed us to generate samples of the probability measure that we used to define the distribution function. The concept of distribution function was generalized in another way in [24].

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In [12], we extended a cdf defined on a separable LOTS, X, to its Dedekind-MacNeille completion, $DM(X)$. That completion is, indeed, a compactification. Moreover, we proved that each function satisfying the properties of a cdf on $DM(X)$ is the cdf of a probability measure defined on $DM(X)$. Indeed, if X is compact, a similar result can be obtained in this context. Finally, the compactification $DM(X)$ lets us generate samples of a distribution in X.

By following this research line, the next step is exploring some conditions on X such that, given a function F with the properties of a cdf, we can ensure that there exists a unique probability measure on X such that its cdf is F. Furthermore, we will show that there is a one-to-one relationship between the pseudo-inverse of a cdf and its probability measure.

2. Preliminaries

2.1. Measure theory

We first recall some definitions and results from [13].

Let X be a set. If $\mathcal{R}$ is a non-empty collection of subsets of X, we say that $\mathcal{R}$ is a ring if it is closed under complement and finite union. A non-empty collection of subsets of X, $\mathcal{A}$, is a σ-algebra if it is closed under complement and countable union and $X \in \mathcal{A}$.

For a given topological space, (X, τ), its Borel σ-algebra is defined as the σ-algebra generated by the open sets of X.

A set mapping is said to be σ-additive if $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for each countable collection $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$.

Given a measurable space $(\Omega, \mathcal{A})$, a measure μ is a non-negative and σ-additive set mapping defined on $\mathcal{A}$ such that $\mu(\emptyset) = 0$. Moreover, it holds that the restriction of a measure to a sub-σ-algebra is a measure.

Some properties of a measure are its monotonicity (which means that $\mu(A) \leq \mu(B)$, for each $A \subseteq B$) and the following results: (1) if A_n is a monotonically non-decreasing sequence of sets (which means that $A_n \subseteq A_{n+1}$, for each $n \in \mathbb{N}$) then $\mu(A_n) \to \mu(\bigcup_{n\in\mathbb{N}} A_n)$; (2) if A_n is monotonically non-increasing (which means that $A_{n+1} \subseteq A_n$, for each $n \in \mathbb{N}$) and $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(\bigcap_{n\in\mathbb{N}} A_n)$.

Definition 2.1. ([7, Def. 1.5.1.]) Suppose that μ is a non-negative set function on domain $\mathcal{A} \subset 2^X$. A set A is called μ-measurable (or Lebesgue measurable with respect to μ) if, for any $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $\mu^*(A \vartriangle A_\varepsilon) < \varepsilon$, where μ^* is the outer measure defined by $\mu^*(A) = \inf\{\sum_{n=1}^{\infty} \mu(A_n) : A_n \in \mathcal{A}, A \subset \bigcup_{n=1}^{\infty} A_n\}$ and $\vartriangle$ denotes the symmetric difference, that is, $A \vartriangle B = (A \setminus B) \cup (B \setminus A)$. The class of μ-measurable sets is denoted by $\mathcal{A}_\mu$.

Proposition 2.2. ([7, Section 1.5]) *Every set $A \in \mathcal{A}_\mu$ is the support of a measurable space by restricting μ to the class of μ-measurable subsets of A, which is a σ-algebra in A.*

Definition 2.3. ([7, Section 3.6]) Let X and Y be two spaces with σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $f : X \to Y$ be a measurable function. Then, for any bounded (or bounded from below) measure μ on $\mathcal{A}_1$, the formula $\mu \circ f^{-1}$ given by $\mu(f^{-1}(B))$, for each $B \in \mathcal{A}_2$, defines a measure on $\mathcal{A}_2$ called the image of the measure μ under the mapping f.

2.2. Ordered spaces and the Dedekind-MacNeille completion

First, we recall the definition of a linear order and a linearly ordered topological space:

Definition 2.4. ([19, Chapter 1]) A partially ordered set $(P, \leq)$ (that is, a set P with the binary relation $\leq$ that is reflexive, antisymmetric and transitive) is totally ordered if the elements of every pair $x, y \in P$ are comparable, namely, either $x \leq y$ or $y \leq x$. In this case, the order is said to be total or linear.

For further reference about partially ordered sets see, for example, [9] and, for more about for ordered sets, see [8].

On a linearly ordered set $(X, \leq)$, for each $a, b \in X$ with $a < b$, we define the set $]a, b] = \{x \in X : a < x \leq b\}$ and, analogously, we define $]a, b[, [a, b]$ and $[a, b[$ for all $a, b \in X$. Moreover, $(\leq a)$ is given by $(\leq a) = \{x \in X : x \leq a\}$. $(< a), (\geq a)$ and $(> a)$ are defined similarly.

The definition of the order topology is the following one:

Definition 2.5. ([1, Part II, 39]) Let X be a set which is linearly ordered by $\leq$, we define the order topology τ on X by taking the subbasis $\{(< a) : a \in X\} \cup \{(> a) : a \in X\}$.

Definition 2.6. ([17, Section 1]) A linearly ordered topological space (abbreviated LOTS) is a triple $(X, \tau, \leq)$ where $(X, \leq)$ is a linearly ordered set and where τ is the topology of the order $\leq$.

Remark 2.7. Note that an open basis of X with respect to τ is given by $\{]a, b[: a < b, a, b \in X\} \cup \{(< a) : a \in X\} \cup \{(> a) : a \in X\}$.

Definition 2.8. ([23, Def. 2.16 and 2.17]) Let P be an ordered set and let $A \subseteq P$. Then:

1. l is called a lower bound of A if, and only if, we have $l \leq a$, for each $a \in A$.
2. u is called an upper bound of A if, and only if, we have $u \geq a$, for each $a \in A$.

We denote by A^l and A^u, respectively, the set of lower and upper bounds of A.

Definition 2.9. ([23, Def. 3.18]) Let P be an ordered set and let $A \subseteq P$. Then:

1. The point u is called the lowest upper bound or supremum or join of A iff $u \in A^u$ and for each $p \in A^u$ we have that $p \geq u$.
2. The point l is called the greatest lower bound or infimum or meet of A iff $l \in A^l$ and for each $p \in A^l$ we have that $p \leq l$.

Some results that we need from [1], with respect to the order topology, are collected in the next proposition:

Proposition 2.10. ([1, Part II, 39]) *Let X be a linearly ordered space, then:*

1. *The order topology on X is compact if, and only if, the order is complete, that is, if, and only if, every nonempty subset of X has a greatest lower bound and a least upper bound.*
2. *X is T_5.*

Definition 2.11. ([10, Section 4.1]) We say that a set $A \subseteq X$ is decreasing (respectively increasing) if given $a \in A$ and $x < a$ (respectively $x > a$), then $x \in A$.

The next two results will be useful in the context of decreasing and increasing sets.

Lemma 2.12. ([12, Lemma 3.8]) *Let X be a separable LOTS and $A \subseteq X$. If A is decreasing (respectively increasing) and it does not have a maximum (respectively a minimum), then there exists an increasing sequence (respectively decreasing sequence) $a_n \in A$, such that $\bigcup_{n \in \mathbb{N}} (\leq a_n) = A$ (respectively $\bigcup_{n \in \mathbb{N}} (\geq a_n) = A$).*

Lemma 2.13. ([12, Lemma 4.1]) *Let X be a separable LOTS and $F : X \to \mathbb{R}$ be a non-decreasing function and let (a_n) be an increasing (respectively decreasing) sequence on a decreasing (respectively increasing) set $A \subseteq X$ such that $A = \bigcup (\leq a_n)$ (respectively $A = \bigcup (\geq a_n)$), then $F(a_n) \to \sup F(A)$ (respectively $F(a_n) \to \inf F(A)$).*

2.3. The cdf on a separable LOTS

In what follows we will assume that X is a separable LOTS and collect all results and definitions that we need from [11]. First, we recall that, given $x \in X$, it is left-isolated (respectively right-isolated) if $(< x) = \emptyset$ (respectively $(> x) = \emptyset$) or there exists $z \in X$ such that $]z, x[= \emptyset$ (respectively there exists $z \in X$ such that $]x, z[= \emptyset$). Moreover, if $x \in X$ is both right- and left-isolated, then it is said to be isolated.

Moreover, given $x \in X$ and ν a topology defined on X, we say that a sequence (x_n) is monotonically right ν-convergent (respectively monotonically left ν-convergent) to x if $x_n \xrightarrow{\nu} x$ and $x < x_{n+1} < x_n$ (respectively $x_n < x_{n+1} < x$), for each $n \in \mathbb{N}$.

Points in X can be characterized in terms of monotonically sequences as the next result shows.

Proposition 2.14. ([11, Prop. 3.11]) *Let $x \in X$. Then x is not left-isolated (respectively right-isolated) if, and only if, there exists a monotonically left τ-convergent (respectively monotonically right τ-convergent) to x sequence.*

Given a probability measure μ on X, we define the mapping $F : X \to [0, 1]$ given by $F(x) = \mu(\leq x)$ as the cdf of μ. Its properties are collected in the next proposition.

Proposition 2.15. ([11, Prop. 4.6]) *Let F be a cdf. Then:*

1. *F is monotonically non-decreasing.*
2. *F is right τ-continuous.*
3. *If there does not exist* $\min X$*, then* $\inf F(X) = 0$*.*
4. $\sup F(X) = 1$*.*

It holds that the fact that $\mu(\{x\}) = 0$, for each $x \in X$ implies that F is τ-continuous at x. Moreover, if x is isolated, then F is continuous at x.

From a probability measure μ on X, we also define the mapping $F_- : X \to [0, 1]$, by $F_-(x) = \mu(< x)$, for each $x \in X$. F_- is related to the F according to the equality $F_-(x) = \sup F(< x)$. It also holds that $F(x) = F_-(x) + \mu(\{x\})$. The properties of F_- are those we collect in the next result.

Proposition 2.16. ([11, Prop. 4.18]) *Let μ be a probability measure on X and F its cdf, then:*

1. *F_- is monotonically non-decreasing.*
2. *F_- is left τ-continuous.*
3. $\inf F_-(X) = 0$*.*
4. *If there does not exist the maximum of X, then* $\sup F_-(X) = 1$*.*

Proposition 2.17. ([11, Prop. 4.17]) *Let μ be a probability measure on X and F its cdf. Let $x \in X$ and (x_n) be a monotonically left τ-convergent sequence to x then $F(x_n) \to F_-(x)$.*

Concerning the Dedekind-MacNeille completion we cite some definitions and results from [23] next:

The Dedekind-MacNeille completion of X consists of cuts. Cuts are all subsets $A \subseteq X$ for which $(A^u)^l = A$. Indeed, recall that, given an ordered set P, its Dedekind-MacNeille completion is defined by $DM(P) = \{A \subseteq P : A = (A^u)^l\}$ ordered by inclusion, that is, given $A, B \in DM(X)$, it holds that $A \leq B$ if, and only if, $A \subseteq B$. It is also referred to as the MacNeille completion or the completion by cuts.

From now on, given a LOTS X, we will denote the order topology on $DM(X)$ by τ'.

Proposition 2.18. ([12, Cor. 3.4]) *If X is a separable LOTS, then $DM(X)$ is also a separable LOTS.*

Let P be an ordered set. Then $DM(P)$ is a complete lattice. Moreover, the map $\phi_{DM} : P \to DM(P)$, which is defined by $\phi_{DM}(p) = (\leq p)$ is an embedding that preserves all suprema and infima that exist in P. Recall that, given P, Q two ordered sets, then the mapping $f : P \to Q$ is called an (order) embedding if, and only if, f is injective and for all $p_1, p_2 \in P$, we have $p_1 \leq p_2$ if, and only if, $f(p_1) \leq f(p_2)$.

Note that $DM(X)$ is a compactification of X and that the order embedding is also a topological embedding. In fact, $DM(X)$ is the smallest order-compactification of X ([4], [15]).

For further reference about cuts and the Dedekind-McNeille completion see, respectively, [18] and [19].

According to [12], the pseudo-inverse of a cdf can be naturally defined on $DM(X)$ as follows:

Definition 2.19. ([12, Def. 5.1]) Let F be a cdf. We define the pseudo-inverse of F as $G : [0,1] \to DM(X)$ given by $G(r) = A$, for each $r \in [0,1]$, where $A = B^l$ and $B = \{x \in X : F(x) \geq r\}$.

G is monotonically non-decreasing and left τ-continuous.

Let F be the cdf of the probability measure μ on X, then F can be extended to a cdf on $DM(X)$, $\widetilde{F}$, that is defined from the probability measure $\widetilde{\mu}$ which is defined by $\widetilde{\mu}(A) = \mu(\phi^{-1}(A))$, for each $A \in \sigma(DM(X))$. What is more, it holds that $\widetilde{F} \circ \phi = F$ and $\widetilde{F}_- \circ \phi = F_-$ (see [12, Lemma 4.6] and [12, Cor. 4.11]). According to the properties that we proved in [11] about the pseudo-inverse of a cdf and, by taking into account that G is the pseudo-inverse of $\widetilde{F}$ if we extend F to $DM(X)$, we can relate G to F and $\widetilde{F}$ as the next proposition shows.

Proposition 2.20. *Let F be a cdf and let $x \in X$ and $r \in [0,1]$. Then:*

1. *([12, Prop. 7.4.2]) $G(F(x)) \leq \phi(x)$.*
2. *([12, Prop. 7.4.3]) $\widetilde{F}(G(r)) \geq r$.*
3. *([12, Prop. 7.4.4]) $G(r) \leq \phi(x)$ if, and only if, $r \leq F(x)$.*
4. *([12, Prop. 7.4.5]) $F(x) < r$ if, and only if, $G(r) > \phi(x)$.*
5. *([12, Prop. 5.6]) $G(r) = \inf\{A \in DM(X) : \widetilde{F}(A) \geq r\}$.*

In [11] it is shown the uniqueness of a measure with respect to its cdf.

Proposition 2.21. ([11, Prop. 7.3]) *Let F_μ and F_δ be the cdfs of the measures μ and δ satisfying $F_\mu = F_\delta$, then $\mu = \delta$ on the Borel σ-algebra of (X, τ).*

Moreover, a cdf F can be defined from F_- as the next result states:

Proposition 2.22. ([11, Prop. 4.13]) *Let F be a cdf, then $F(x) = \inf F_-(> x)$, for each $x \in X$ with $x \neq \max X$.*

Additionally, we can prove that the value of $F(x)$ can be obtained from the pseudo-inverse:

Proposition 2.23. *Let X be a separable LOTS. If F is the cdf of a probability measure on X, then $F(x) = \sup G^{-1}(\leq \phi(x))$, for each $x \in X$.*

Proof. Let $x \in X$. By Proposition 2.20.3, it holds that $G(r) \leq \phi(x)$ if, and only if, $F(x) \geq r$, for each $r \in [0,1]$. Hence, $\sup G^{-1}(\leq \phi(x)) = \sup\{r \in [0,1] : G(r) \leq \phi(x)\} = \sup\{r \in [0,1] : F(x) \geq r\} = F(x)$. □

Now, we prove the uniqueness of the measure with respect to F_- and the pseudo-inverse.

Corollary 2.24. *Let F_μ and F_δ be respectively the cdfs of the measures μ and δ. If $F_{\mu-} = F_{\delta-}$, then $\mu = \delta$ on the Borel σ-algebra of (X, τ).*

Proof. By Proposition 2.22, $F(x) = \inf F_-(x)$, for each $x \in X$ and, consequently, $F_\mu = F_\delta$. Hence, by Proposition 2.21, $\mu = \delta$ on the Borel σ-algebra of (X, τ). □

Corollary 2.25. *Let F_μ and F_δ be respectively the cdfs of the measures μ and δ. If $G_\mu = G_\delta$, then $\mu = \delta$ on the Borel σ-algebra of (X, τ).*

Proof. By Proposition 2.23, $F(x) = \sup G^{-1}(\leq \phi(x))$, for each $x \in X$ and, consequently, $F_\mu = F_\delta$. Hence, by Proposition 2.21, $\mu = \delta$ on the Borel σ-algebra of (X, τ). □

3. Defining a probability measure from a cdf

In [12] it was proven that each cdf on a compact separable LOTS, X, is defined from a unique probability measure on X with respect to its Borel σ-algebra, as the next theorem states:

Theorem 3.1. ([12, Th. 7.7]) *Let X be a compact separable LOTS and $F : X \to [0, 1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$. Then there exists a unique probability μ on $\sigma(X)$ such that $F = F_\mu$.*

Moreover, the next results were proven in order to justify that each cdf on X can be extended to a cdf defined on $DM(X)$ from a probability measure on this last space.

Lemma 3.2. ([12, Lemma 3.1]) *Let $A \in DM(X)$ be a left-isolated or right-isolated cut such that $A \neq \min DM(X)$ (respectively $A \neq \max DM(X)$), then there exists $x \in X$ such that $A = (\leq x)$.*

The next result lets us claim that each cdf can be extended to a cdf on the Dedekind-MacNeille completion and that there exists a probability measure on $DM(X)$ such that its cdf is the last one.

Theorem 3.3. ([12, Th. 7.8]) *Let X be a separable LOTS and $F : X \to [0, 1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$. Then the function $\widetilde{F} : DM(X) \to [0, 1]$ given by $\widetilde{F}(A) = \inf F(A^u)$ ($\widetilde{F}(A) = 1$ if $A^u = \emptyset$) is the cdf of a probability measure on $DM(X)$. Moreover, $\widetilde{F}$ is an extension of F to $DM(X)$.*

Lemma 3.4. *Let X be a separable LOTS and let (x_n) be a τ-convergent sequence to x. Suppose that there exists $z \in X$ such that $x_n \leq z$ for each $n \in \mathbb{N}$, then $x \leq z$.*

Proof. Suppose that $x > z$. The convergence of (x_n) gives us that there exists $n_0 \in \mathbb{N}$ such that $x_n > z$, for each $n \geq n_0$, a contradiction with the fact that $x_n \leq z$, for each $n \in \mathbb{N}$. □

In this section we explore some conditions such that given a function, F, with the properties of a cdf on a separable LOTS, then there exists a probability measure, μ, on X such that $F_\mu = F$. Indeed, the converse relationship between a measure and its cdf is well-known. According to [11] and [12], the cdf of a probability measure on a separable LOTS, X, is right τ-continuous, monotonically non-decreasing and $\sup F(X) = 1$ and $\inf F(X) = 0$ if there does not exist $\min X$ (see Proposition 2.15). In addition to this properties, a cdf verifies some other conditions stated in the next proposition.

In what follows, when we write a statement like $\sup F_-(A) = \inf F_-(A^u)$, for each $A \in DM(X)$, we mean for each $A \in DM(X)$ such that the expression makes sense. In this case A must be nonempty (so that $\sup F_-(A)$ makes sense) and A^u must be nonempty (so that $\inf F_-(A^u)$ makes sense). Note that A can be empty if X does not have a minimum and $A = \min DM(X)$ and A^u can be empty if X does not have a maximum and $A = \max DM(X)$.

Proposition 3.5. *Let X be a separable LOTS and $F : X \to [0, 1]$ a cdf defined from a probability measure μ, then:*

1. ([12, Prop. 4.3]) $\sup F(A) = \inf F(A^u)$, *for each $A \in DM(X)$.*
2. $\sup F_-(A) = \inf F_-(A^u)$, *for each $A \in DM(X)$.*
3. $\sup F(A) = \sup F_-(A)$, *for each $A \in DM(X) \setminus \phi(X)$.*
4. $\inf F(A^u) = \inf F_-(A^u)$, *for each $A \in DM(X) \setminus \phi(X)$.*
5. $\sup G^{-1}(< A) = \sup F(A)$, *for each $A \in DM(X) \setminus \phi(X)$.*
6. $\inf F(A^u) = \inf G^{-1}(> A)$, *for each $A \in DM(X) \setminus \phi(X)$.*
7. $\sup G^{-1}(< A) = \inf G^{-1}(> A)$, *for each $A \in DM(X) \setminus \phi(X)$.*

Proof. The first item was proven in [12] so we just have to show the others:

(2) Let $A \in DM(X)$. In case that $A \in \phi(X)$, the equality is clear. Now, let $A \in DM(X) \setminus \phi(X)$, then $A \cap A^u = \emptyset$. Since A is decreasing and it does not have a maximum, by Lemma 2.12, there exists an increasing sequence (a_n) in A such that $A = \bigcup(\leq a_n)$. Analogously, since A^u is increasing and it does not

have a minimum, we can consider a decreasing sequence in A^u, (b_n), such that $A^u = \bigcup(\geq b_n)$. Moreover, Lemma 2.13 lets us claim that $F_-(a_n) \to \sup F_-(A)$ and $F_-(b_n) \to \inf F_-(A^u)$. Now, note that $[a_n, b_n[$ is a monotonically non-increasing sequence, which implies that $[a_n, b_n[\to \bigcap_{n\in\mathbb{N}}[a_n, b_n[$. Indeed, $\bigcap_{n\in\mathbb{N}}[a_n, b_n[= \emptyset$, which gives us that $\mu([a_n, b_n[) \to \mu(\emptyset) = 0$, that is, $F_-(b_n) - F_-(a_n) \to 0$. Both convergences $F_-(a_n) \to \sup F_-(A)$ and $F_-(b_n) \to \inf F_-(A^u)$ let us conclude that $\inf F_-(A^u) = \sup F_-(A)$.

(3) Let $A \in DM(X) \setminus \phi(X)$ with $A \neq \min DM(X)$. By Lemma 3.2, A is not isolated.

$\geq$) This inequality is clear if we take into account that $F_-(x) \leq F(x)$, for each $x \in X$.

$\leq$) Since A is not left-isolated and $A \neq \min DM(X)$, we can consider a monotonically non-decreasing sequence (A_n) in $DM(X)$, such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A_{n+1} \setminus A_n$, for each $n \in \mathbb{N}$. Note that $a_n \in A$ since $A_n \subset A$, for each $n \in \mathbb{N}$. Given $a \in A$, then there exists $n \in \mathbb{N}$ such that $a_n > a$ and, hence, it follows that $F(a) \leq F_-(a_n) \leq \sup F_-(A)$, which lets us conclude that $\sup F(A) \leq \sup F_-(A)$.

(4)] Let $A \in DM(X) \setminus \phi(X)$ with $A \neq \max DM(X)$. By Lemma 3.2, A is not isolated.

$\geq$) This inequality is clear if we take into account that $F_-(x) \leq F(x)$, for each $x \in X$.

$\leq$) Since A is not right-isolated and $A \neq \max DM(X)$, we can consider a monotonically non-increasing sequence A_n such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A^u_{n+1} \setminus A^u_n$, for each $n \in \mathbb{N}$. Note that $a_n \in A^u$ since $A \subset A_n$, for each $n \in \mathbb{N}$. Given $a \in A^u$, then there exists $n \in \mathbb{N}$ such that $a_n < a$ and, hence, it follows that $\inf F(A^u) \leq F(a_n) \leq F_-(a)$, which lets us conclude that $\inf F(A^u) \leq \inf F_-(A^u)$.

(5)] Let $A \in DM(X) \setminus \phi(X)$.

$\geq$) Let $a \in A$, by Proposition 2.23, $F(a) = \sup G^{-1}(\leq \phi(a)) \leq \sup G^{-1}(< A)$, so we have that $\sup F(A) \leq \sup G^{-1}(< A)$.

$\leq$) Let $r \in G^{-1}(< A)$, then $G(r) < A$, so we can consider $a \in A \setminus G(r)$, and hence, $G(r) \leq \phi(a)$. By Proposition 2.23, $F(a) = \sup\{r' \in [0, 1] : G(r') \leq \phi(a)\}$. Note that $r \leq F(a)$ and, consequently, $r \leq \sup F(A)$, which lets us conclude that $\sup G^{-1}(< A) \leq \sup F(A)$.

(6) Let $A \in DM(X) \setminus \phi(X)$.

$\leq$) Suppose that $\inf F(A^u) > \inf G^{-1}(> A)$, then there exists $r \in [0, 1]$ such that $r < \inf F(A^u)$ and $G(r) > A$. Since $r < \inf F(A^u)$, $r < F(a)$, for each $a \in A^u$. By Proposition 2.20.3, $G(r) \leq \phi(a)$, for each $a \in A^u$, which means that $G(r) \leq A$, a contradiction with the fact that $G(r) > A$.

$\geq$) Suppose that $\inf F(A^u) < \inf G^{-1}(> A)$. Now, let $r = \inf G^{-1}(> A)$ and consider a sequence (r_n) such that $\inf F(A^u) < r_n < r$ and $r_n \to r$. By the left-continuity of G, $G(r_n) \to G(r)$. Moreover, the fact that $r_n < r$ implies that $G(r_n) \leq A$. Consequently, $G(r) \leq A$ by Lemma 3.4. What is more, the fact that $\inf F(A^u) < r$ means that there exists $a \in A^u$ such that $F(a) < r$ and, by Proposition 2.20.4, $G(r) > \phi(a) > A$. $G(r) > A$ is a contradiction with the fact that $G(r) \leq A$. Thus, $\inf F(A^u) \geq \inf G^{-1}(> A)$.

(7) It immediately follows from items 1, 5 and 6. $\square$

Moreover, a cdf always satisfies:

Proposition 3.6. *Let X be a separable LOTS and $F : X \to [0, 1]$ be a cdf. It follows that $G(0) = \min DM(X)$. Moreover, if X does not have a maximum then, $G^{-1}(\max DM(X)) \subseteq \{1\}$, and, if X does not have a minimum, then $G^{-1}(\min DM(X)) = \{0\}$.*

Proof. First, we prove that $G(0) = \min DM(X)$.

By Proposition 2.20.5, we have that $G(0) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq 0\}$. Since $\widetilde{F}$ is a cdf, it holds that $\widetilde{F}(C) \geq 0$ for each $C \in DM(X)$. Moreover, the fact that $DM(X)$ is compact means that $\inf\{C \in DM(X) : \widetilde{F}(C) \geq 0\}$ is, indeed, a minimum, which lets us conclude that $G(0) = \min DM(X)$.

Now, suppose that X does not have a minimum. This implies, by Proposition 2.15, that $\inf F(X) = 0$. We prove that $G^{-1}(\min DM(X)) \subseteq \{0\}$.

Suppose that there exists $r \in]0, 1]$ such that $G(r) = \min DM(X)$. By Proposition 2.20.5, $G(r) = \inf\{C \in DM(X) : \widetilde{F}(C) \geq r\}$. Thus, given $x \in X$ it holds that $\phi(x) > \min DM(X)$ due to the fact that there does not exist the minimum of X. Hence, $F(x) = \widetilde{F}(\phi(x)) \geq \widetilde{F}(G(r)) \geq r$ (see Proposition 2.20.2). Consequently, $F(x) \geq r$, for each $x \in X$, a contradiction with the fact that $\inf F(X) = 0$.

Finally, consider the case in which X does not have a maximum. We prove that $G^{-1}(\max DM(X)) \subseteq \{1\}$. Suppose that there exists $r \in [0, 1[$ such that $G(r) = \max DM(X)$. Note that, given $x \in X$, it holds that $\phi(x) < \max DM(X) = G(r)$ due to the fact that there does not exist the maximum of X. Now, by Proposition 2.20.4, $\phi(x) < G(r)$ implies that $F(x) < r$, a contradiction with the fact that $\sup F(X) = 1$. We conclude that $G^{-1}(\max DM(X)) \subseteq \{1\}$. □

Example 3.7. Let $X = \mathbb{Q}_0^+$, that is, the set of non-negative rationals, and $F : X \to [0, 1]$ the function given by $F(x) = 1 - e^{-x}$, for each $x \in X$. Consider $\leq$ as the usual order in X and suppose that there exists a probability measure, μ, on X such that $F_\mu = F$. Hence, $1 = \mu(X) = \mu(\bigcup_{x \in X}\{x\}) \leq \sum_{x \in X} \mu(\{x\}) = 0$, a contradiction, which means that there does not exist any probability measure such that its cdf is F. Note that in this case $DM(X) \setminus \phi(X)$ is not countable.

The last example suggests considering the countability of $DM(X) \setminus \phi(X)$ in order to be able to get a probability measure on X such that its cdf is a function satisfying the properties in Proposition 2.15.

The main result of this section is the following one:

Theorem 3.8. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and $F : X \to [0, 1]$ a monotonically non-decreasing and right τ-continuous function satisfying $\sup F(X) = 1$ and $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X)$. Moreover, $\inf F(X) = 0$ if there does not exist the minimum of X. Then there exists a unique probability measure on X, μ, such that $F = F_\mu$.*

Proof. By Theorem 3.3, the function $\widetilde{F} : DM(X) \to [0, 1]$ given by $\widetilde{F}(A) = \inf F(A^u)$, for each $A \in DM(X)$ is an extension of F, which means that $\widetilde{F}(\leq x) = F(x)$, for each $x \in X$ and $\widetilde{F}$ is the cdf of a probability measure, $\widetilde{\mu}$, on $DM(X)$. Now, define the measure μ by $\mu(A) = \widetilde{\mu}(\phi(A))$, for each $A \subseteq X$. We show that $\phi(X)$ is measurable with respect to the Borel σ-algebra of $DM(X)$. Indeed, note that given $A \in DM(X)$, $\{A\}$ is closed with respect to the order topology of $DM(X)$, which means that $\{A\} \in \sigma(DM(X))$. Hence, the fact that $DM(X) \setminus \phi(X)$ is countable lets us claim that $DM(X) \setminus \phi(X)$ is the countable union of elements in $\sigma(DM(X))$, which implies that $DM(X) \setminus \phi(X) \in \sigma(DM(X))$. Hence, its complement belongs to the Borel σ-algebra of $DM(X)$, that is $\phi(X) \in \sigma(DM(X))$, which lets us conclude that $\phi(X)$ is measurable. Hence, Proposition 2.2 lets us claim that $\widetilde{\mu}$ is a measure on $\sigma(\phi(X))$. Now, considering the map $\phi^{-1} : \phi(X) \to X$, Definition 2.3 gives us that μ is a measure with respect to $\sigma(X)$.

Now, we prove a claim which is crucial to show that μ is a probability measure on X.

Claim 3.9. *Let $A \in DM(X) \setminus \phi(X)$ be such that $A \neq \min DM(X)$, then $\widetilde{F}_-(A) = \sup F(A)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$, then Lemma 3.2 lets us claim that A is not left-isolated. Now, by Proposition 2.14, there exists a sequence (A_n) in $DM(X)$ such that $A_n \xrightarrow{\tau'} A$ and $A_n < A_{n+1} < A$, for each $n \in \mathbb{N}$. Now, let $a_n \in A_n \setminus A_{n-1}$, for each $n \geq 2$.

$\leq$) Let $B \in DM(X)$ be such that $B < A$, then, by the definition of a_n, there exists $n \in \mathbb{N}$ such that $B < \phi(a_n) < A$. What is more, $\sup F(A) \geq F(a_n) = \widetilde{F}(\phi(a_n)) \geq \widetilde{F}(B)$ where we have used the monotonicity of $\widetilde{F}$ as a cdf. Hence, $\sup F(A) \geq \sup_{B<A} \widetilde{F}(B) = \widetilde{F}_-(A)$, for each $A \in DM(X) \setminus \phi(X)$.

$\geq$) Let $a \in A$ then $\phi(a) < A$. Moreover, $F(a) = \widetilde{F}(\phi(a)) \leq \sup_{B<A} \widetilde{F}(B) = \widetilde{F}_-(A)$. Therefore, $\widetilde{F}_-(A) \geq F(a)$, for each $a \in A$, which means that $\widetilde{F}_-(A) \geq \sup F(A)$. □

Finally we prove that $\mu(X) = 1$. Note that we can write $DM(X) = \phi(X) \cup (DM(X) \setminus \phi(X))$, which implies that $\widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X) \cup (DM(X) \setminus \phi(X))$. Now, the σ-additivity of $\widetilde{\mu}$ gives us that $\widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X) \cup (DM(X) \setminus \phi(X))) = \widetilde{\mu}(\phi(X)) + \widetilde{\mu}(DM(X) \setminus X)$. Since $\widetilde{\mu}$ is a probability measure on $DM(X)$, we have that $\widetilde{\mu}(DM(X)) = 1$. The fact that $DM(X) \setminus \phi(X)$ is countable implies that $\widetilde{\mu}(DM(X) \setminus \phi(X)) = 0$. Indeed, to prove that, we first show the next claim:

Claim 3.10. *$\widetilde{\mu}(\{A\}) = 0$, for each $A \in DM(X) \setminus \phi(X)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. Then $\widetilde{\mu}(\{A\}) = \widetilde{F}(A) - \widetilde{F}_-(A)$. Now, we distinguish two cases depending on whether $A = \min DM(X)$ or not:

Suppose that $A = \min DM(X)$, in which case $A^u = X$. Note that there does not exist the minimum of X. Indeed, if there exists $\min X$, then $\phi(\min X) = A$, which contradicts the fact that $A \in DM(X) \setminus \phi(X)$. Hence, the definition of $\widetilde{F}$ and the initial assumption that $\inf F(X) = 0$ let us claim that $\widetilde{F}(A) = \inf F(A^u) = \inf F(X) = 0$. On the other hand, since $\widetilde{F}$ is a cdf on $DM(X)$, it holds, by Proposition 2.16, that $\inf \widetilde{F}_-(DM(X)) = 0$ and, consequently, $\widetilde{F}_-(A) = 0$. Therefore, $\widetilde{\mu}(\{A\}) = 0$.

If $A \neq \min DM(X)$, by taking into account the definition of $\widetilde{F}$ and Claim 3.9, it follows that $\widetilde{\mu}(\{A\}) = \inf F(A^u) - \sup F(A)$. Finally, $\inf F(A^u) - \sup F(A) = 0$ by the initial assumption in the theorem, which lets us conclude that $\widetilde{\mu}(\{A\}) = 0$. □

Hence, by the previous claim and the σ-additivity of $\widetilde{\mu}$ as a measure, we can write $\widetilde{\mu}(DM(X) \setminus \phi(X)) = \widetilde{\mu}(\bigcup_{A\in DM(X)\setminus\phi(X)}\{A\}) = \sum_{A\in DM(X)\setminus\phi(X)} \widetilde{\mu}(\{A\}) = 0$. Consequently, $1 = \widetilde{\mu}(DM(X)) = \widetilde{\mu}(\phi(X)) + \widetilde{\mu}(DM(X) \setminus \phi(X)) = \widetilde{\mu}(\phi(X)) = \mu(X)$.

The uniqueness of the measure immediately follows from Proposition 2.21. □

To end with this section we introduce some results whose main goal is to define a probability measure from a function F_- satisfying the properties that we collect in Proposition 2.16 and from a function G satisfying the properties of the pseudo-inverse of a cdf.

Corollary 3.11. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and let $F_- : X \to [0,1]$ be a monotonically non-decreasing, left τ-continuous function such that $\inf F_-(X) = 0$ and $\sup F_-(A) = \inf F_-(A^u)$, for each $A \in DM(X)$. Moreover, $\sup F_-(X) = 1$ if there does not exist the maximum of X. Then there exists a unique probability measure on X, μ, such that $F_{\mu-} = F_-$.*

Proof. Let us define $F : X \to [0,1]$ by $F(x) = \inf F_-(> x)$ if $(> x) \neq \emptyset$ and $F(x) = 1$ if $x = \max X$.

First of all, we prove the next claims which are crucial in the rest of the proof:

Claim 3.12. *$F_-(x) \leq F(x)$, for each $x \in X$.*

Proof. It immediately follows from the definition of F and the monotonicity of F_-. □

Claim 3.13. *Let $a, b \in X$ be such that $a < b$, then $F(a) \leq F_-(b)$.*

Proof. Let $a, b \in X$ be such that $a < b$ then $\inf F_-(> a) \leq F_-(b)$, that is, $F(a) \leq F_-(b)$. □

Claim 3.14. *Let (x_n) be a monotonically right τ-convergent sequence to x. Then $F_-(x_n) \to F(x)$.*

Proof. Let (x_n) be a monotonically right τ-convergent sequence to x. By the previous claim, it holds that $F(x) \leq F_-(x_n)$ since $x < x_n$. Note that $F_-(x_n)$ is a monotonically non-increasing sequence with a lower bound, $F(x)$. Hence, $F_-(x_n) \to r'$ for some $r' \geq F(x)$. Note that $F_-(x_n) \geq r'$, for each $n \in \mathbb{N}$. Now, suppose that $F(x) < r'$, then, by the definition of F, there exists $y > x$ such that $F_-(y) < r'$. Since $x_n \to x$, there exists $m \in \mathbb{N}$ such that $x < x_m < y$ and, hence, $F_-(x_m) \leq F_-(y) < r'$, which contradicts the fact that $F_-(x_n) \geq r'$ for each $n \in \mathbb{N}$. Consequently, $r' = F(x)$. □

Secondly, we show that F is a cdf. Indeed,

1. The fact that F_- is monotonically non-decreasing gives us that F satisfies that property too.

2. F is right τ-continuous. Let (x_n) be a monotonically right τ-convergent sequence to x, then by Claim 3.13, we have that $F(x) \leq F_-(x_{n+1})$ and $F(x_{n+1}) \leq F_-(x_n)$. Moreover, Claim 3.12 gives us that $F_-(x_{n+1}) \leq F(x_{n+1})$. Hence, if we join all previous inequalities, it follows that $F(x) \leq F_-(x_{n+1}) \leq F(x_{n+1}) \leq F_-(x_n)$. Finally, by taking limits and using the fact that $F_-(x_n) \to F(x)$ (see Claim 3.14), we conclude that $F(x_n) \to F(x)$, that is, F is right τ-continuous.

3. $\sup F(X) = 1$. We distinguish two cases depending on whether there exists the maximum of X or not:

(a) Suppose that there does not exist $\max X$, then by Claim 3.12, it holds that $F_-(x) \leq F(x)$, for each $x \in X$, which gives us that $\sup F_-(X) \leq \sup F(X)$. By taking into account that $\sup F_-(X) = 1$, we conclude that $\sup F(X) = 1$.

(b) If there exists $\max X$, then by the definition of F we have that $F(\max X) = 1$ and, consequently, $\sup F(X) = 1$.

4. $\inf F(X) = 0$ if there does not exist the minimum of X. Since X is increasing and it does not have a minimum, by Lemma 2.12 we can consider a decreasing sequence (a_n) in X such that $X = \bigcup(\geq a_n)$. Moreover, the fact that F is monotonically non-decreasing lets us claim, by Lemma 2.13, that $F(a_n) \to \inf F(X)$. What is more, the monotonicity of F_- implies that $F_-(a_n) \to \inf F_-(X) = 0$. By Claim 3.12, we have that $\inf F_-(X) \leq \inf F(X)$ and, by Claim 3.13 it holds that $F(a_{n+1}) \leq F_-(a_n)$. Therefore, the next inequality follows $0 \leq \inf F(X) \leq F(a_{n+1}) \leq F_-(a_n)$. By taking limits, we conclude that $\inf F(X) = 0$.

Note that it is obvious that $\sup F(A) = \inf F(A^u)$, for each $A \in \phi(X)$. Now, we prove a claim that will be crucial to get the equality $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X) \setminus \phi(X)$.

Claim 3.15. *Let $A \in DM(X) \setminus \phi(X)$, then $\sup F(A) = \sup F_-(A)$ and $\inf F(A^u) = \inf F_-(A^u)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$ then A is not isolated by Lemma 3.2. First we prove that $\sup F(A) = \sup F_-(A)$, for each $n \in \mathbb{N}$

$\geq$) It is clear if we take into account Claim 3.12 which gives us that $F_-(x) \leq F(x)$, for each $x \in X$.

$\leq$) Since A is not left-isolated, we can consider a monotonically non-decreasing sequence (A_n) such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A_{n+1} \setminus A_n$, for each $n \in \mathbb{N}$. Note that $a_n \in A$ since $A_n \subseteq A$, for each $n \in \mathbb{N}$. Given $a \in A$, then there exists $n \in \mathbb{N}$ such that $a_n > a$ and, hence by Claim 3.13 it follows that $F(a) \leq F_-(a_n) \leq \sup F_-(A)$, which lets us conclude that $\sup F(A) \leq \sup F_-(A)$.

Now, we prove the equality $\inf F(A^u) = \inf F_-(A^u)$.

$\geq$) It immediately follows from Claim 3.12 due to the fact that $F_-(x) \leq F(x)$, for each $x \in X$.

$\leq$) Since A is not right-isolated, we can consider a monotonically non-increasing sequence (A_n) such that $A_n \xrightarrow{\tau'} A$. Now, let $a_n \in A^u_{n+1} \setminus A^u_n$, for each $n \in \mathbb{N}$. Note that $a_n \in A^u$ since $A \subset A_n$, for each $n \in \mathbb{N}$. Given $a \in A^u$, then there exists $n \in \mathbb{N}$ such that $a_n < a$ and, hence by Claim 3.13 it follows that $\inf F(A^u) \leq F(a_n) \leq F_-(a)$, which lets us conclude that $\inf F(A^u) \leq \inf F_-(A^u)$. □

The previous claim gives us that $\sup F(A) = \sup F_-(A)$ and $\inf F(A^u) = \inf F_-(A^u)$, for each $A \in DM(X) \setminus \phi(X)$, which means that the condition $\sup F_-(A) = \inf F_-(A^u)$, for each $A \in DM(X) \setminus \phi(X)$ implies that $\inf F(A^u) = \sup F(A)$, for each $A \in DM(X) \setminus \phi(X)$. Hence, Theorem 3.8 lets us conclude that F is the cdf of a probability measure, μ, defined on X.

Finally we show that $F_{\mu-} = F_-$. For that purpose, given $x \in X$, we distinguish two cases depending on whether x is left-isolated or not:

1. Suppose that x is not left-isolated, then, by Proposition 2.14, there exists a monotonically left τ-convergent sequence to x. Let (x_n) be that sequence. On the one hand, since $F = F_\mu$ is a cdf, we have that $F_\mu(x_n) \to F_{\mu-}(x)$ (see Proposition 2.17). Moreover, Claim 3.12 gives us that $F_-(x_n) \leq F(x_n)$ and, by Claim 3.13, $F(x_n) \leq F_-(x_{n+1})$. Hence, if we join the previous inequalities, we have that $F_\mu(x_n) = F(x_n) \leq F_-(x_{n+1}) \leq F(x_{n+1}) = F_\mu(x_{n+1})$. Now, by taking limits in the previous expression, since $F_\mu(x_n) \to F_{\mu-}(x)$, we have that $F_-(x_{n+1}) \to F_{\mu-}(x)$.

On the other hand, the left τ-continuity of F_- means that $F_-(x_n) \to F_-(x)$.

The facts that $F_-(x_n) \to F_{\mu-}(x)$ and $F_-(x_n) \to F_-(x)$ let us conclude that $F_-(x) = F_{\mu-}(x)$.

2. Suppose that x is left-isolated, then it can happen:

(a) There exists $z \in X$ such that $]z, x[= \emptyset$. Note that the fact that F_μ is the cdf defined from μ gives us that $F_{\mu-}(x) = \mu(< x) = \mu(\leq z) = F_\mu(z)$. Now, Theorem 3.8 lets us claim that $F_\mu(z) = F(z)$. By the definition of F, it holds that $F(z) = \inf F_-(> z) = \inf F_-(\geq x) = F_-(x)$, which finishes the proof.

(b) If $(< x) = \emptyset$, then $x = \min X$ and, consequently, $F_{\mu-}(x) = 0 = F_-(x)$ by hypothesis.

Hence $F_{\mu-} = F_-$. The uniqueness of the measure immediately follows from Corollary 2.24. □

Corollary 3.16. *Let X be a separable LOTS such that $DM(X) \setminus \phi(X)$ is countable and let $G : [0,1] \to DM(X)$ be a monotonically non-decreasing and left τ-continuous function such that $\sup G^{-1}(< A) = \inf G^{-1}(> A)$, for each $A \in DM(X) \setminus \phi(X)$, $G(0) = \min DM(X)$, $G^{-1}(\max DM(X)) \subseteq \{1\}$ if there does not exist the maximum of X and $G^{-1}(\min DM(X)) = \{0\}$ if there does not exist the minimum of X. Then there exists a unique probability measure on X, μ, such that G is the pseudo-inverse of F_μ.*

Proof. First of all, we use the fact that $DM(X)$ is separable as a consequence of the separability of X (see Proposition 2.18).

Let us define $F : X \to [0,1]$ by $F(x) = \sup\{r \in [0,1] : G(r) \leq \phi(x)\} = \sup G^{-1}(\leq \phi(x))$.

Note that $0 \in G^{-1}(\leq \phi(x))$ for each $x \in X$, since $G(0) = \min DM(X)$, and hence F is well defined.

First of all, we prove a claim which will be crucial to show the right continuity of F.

Claim 3.17. *Let $x \in X$ and $r \in [0,1]$, then $F(x) < r$ if, and only if, $G(r) > \phi(x)$.*

Proof. First, note that if $r = 0$ the statement is trivial, so we can suppose that $r > 0$.

$\Rightarrow$) Suppose that $F(x) < r$, then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} < r$, which means that $r \notin \{r' \in [0,1] : G(r') \leq \phi(x)\}$, which implies that $G(r) > \phi(x)$.

$\Leftarrow$) Suppose now that $G(r) > \phi(x)$. We distinguish two cases:

1. Suppose that $F(x) > r$, then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} > r$, which means that there exists $r' \in [0,1]$ with $r' > r$ such that $G(r') \leq \phi(x)$. Hence, the monotonicity of G gives us that $G(r) \leq G(r') \leq \phi(x)$. Thus, $G(r) \leq \phi(x)$, a contradiction with the initial assumption.

2. Suppose now that $F(x) = r$ and let (r_n) be a left convergent sequence to r with $r_n \in [0, r[$ for each $n \in \mathbb{N}$, then $\sup\{r' \in [0,1] : G(r') \leq \phi(x)\} > r_n$, for each $n \in \mathbb{N}$. Hence, given $n \in \mathbb{N}$, there exists $r' \in [0,1]$ with $r' > r_n$ and such that $G(r') \leq \phi(x)$. Hence, the monotonicity of G gives us that $G(r_n) \leq G(r') \leq \phi(x)$. Thus, $G(r_n) \leq \phi(x)$, for each $n \in \mathbb{N}$. Since G is left τ-continuous by hypothesis, by taking limits and using Lemma 3.4, we conclude that $G(r) \leq \phi(x)$, which is a contradiction with the initial assumption. □

Secondly, we show that F is a cdf. For this purpose we start proving its properties as cdf:

1. F is monotonically non-decreasing. Indeed, it immediately follows from the monotonicity of G and the definition of F.

2. F is right τ-continuous. Let (x_n) be a monotonically right τ-convergent sequence to x. Note that $F(x) \leq F(x_n)$ for each $n \in \mathbb{N}$ and $F(x_{n+1}) \leq F(x_n)$, that is, $F(x_n)$ is a monotonically non-increasing sequence with a lower bound, which means that $F(x_n) \to r'$ for some $r' \geq F(x)$. Suppose that $r' > F(x)$, then there exists $r \in [0,1]$ such that $F(x) < r < r'$. The previous claim gives us that $\phi(x) < G(r)$ and $G(r) \leq G(r')$ since G is monotonically non-decreasing. Since (x_n) is a monotonically right τ-convergent sequence to x, there exists $n \in \mathbb{N}$ such that $\phi(x_n) < G(r)$. By the previous claim, this fact implies that $F(x_n) < r$, which contradicts the fact that $F(x_n) \geq r$, for each $n \in \mathbb{N}$. Hence, $F(x) = r'$.

3. $\sup F(X) = 1$. Note that if there exists the maximum of X, then $F(\max X) = 1$ by definition of F. Suppose that there does not exist the maximum of X and that $\sup F(X) \neq 1$. Then we can consider $r \in [0,1[$ such that $r > \sup F(X)$. Now, we claim that $G(r) = \max DM(X)$. Indeed, suppose that $G(r) \neq \max DM(X)$, then we can choose $x \in X$ such that $\phi(x) > G(r)$ and, hence, by Claim 3.17, we have that $F(x) \geq r$ which contradicts the fact that $r > \sup F(X)$. Consequently, $G(r) = \max DM(X)$, which is a contradiction with the initial assumption $G^{-1}(\max DM(X)) \subseteq \{1\}$.

4. $\inf F(X) = 0$ if there does not exist the minimum of X. Suppose that $\inf F(X) \neq 0$, then we can consider $r \in]0,1]$ such that $r < \inf F(X)$. Now, we claim that $G(r) = \min DM(X)$. Indeed, suppose that $G(r) \neq \min DM(X)$, then we can choose $x \in X$ such that $\phi(x) < G(r)$ and, hence, by Claim 3.17, we have that $F(x) < r$, a contradiction with the fact that $r < \inf F(X)$. Consequently, $G(r) = \min DM(X)$, which is a contradiction with the initial assumption $G^{-1}(\min DM(X)) = \{0\}$.

Now, we prove a claim that will be crucial to get the equality $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X) \setminus \phi(X)$.

Claim 3.18. *Let $A \in DM(X) \setminus \phi(X)$, then $\sup F(A) = \sup G^{-1}(< A)$ and $\inf F(A^u) = \inf G^{-1}(> A)$.*

Proof. Let $A \in DM(X) \setminus \phi(X)$. First we prove that $\sup F(A) = \sup G^{-1}(< A)$.

$\leq$) Let $a \in A$, then $F(a) = \sup G^{-1}(\leq \phi(a)) \leq \sup G^{-1}(< A)$, so we have that $\sup F(A) \leq \sup G^{-1}(< A)$.

$\geq$) Let $r \in G^{-1}(< A)$, then $G(r) < A$, so we can consider $a \in A \setminus G(r)$, that is, $G(r) \leq \phi(a)$. Now, according to the definition of F from G, $F(a) = \sup\{r' \in [0,1] : G(r') \leq \phi(a)\}$. Note that $r \leq F(a)$. What is more, $r \leq F(a) \leq \sup F(A)$, which implies that $\sup G^{-1}(< A) \leq \sup F(A)$.

Now, we prove the equality $\inf F(A^u) = \inf G^{-1}(> A)$. Given $A \in DM(X) \setminus \phi(X)$:

$\leq$) Suppose that $\inf F(A^u) > \inf G^{-1}(> A)$, then there exists $r \in [0,1]$ such that $r < \inf F(A^u)$ and $G(r) > A$. Since $r < \inf F(A^u)$, $r < F(a)$, for each $a \in A^u$. By Claim 3.17, $G(r) \leq \phi(a)$, for each $a \in A^u$, which means that $G(r) \leq A$, a contradiction with the fact that $G(r) > A$.

$\geq$) Suppose that $\inf F(A^u) < \inf G^{-1}(> A)$, then there exists $b \in A^u$ such that $F(b) < \inf G^{-1}(> A)$. Since $\inf G^{-1}(> A) = \sup G^{-1}(< A)$ by hypothesis, we have that $F(b) < \sup G^{-1}(< A)$. Hence, there exists $r \in G^{-1}(< A)$ such that $F(b) < r$. Equivalently, there exists $r \in [0,1]$ with $G(r) < A$ such that $F(b) < r$. Now, Claim 3.17 gives us that $G(r) > \phi(b)$. The fact that $(< A)$ is decreasing together with the facts that $G(r) < A$ and $G(r) > \phi(b)$ let us conclude that $b \in A$, a contradiction. □

By the previous claim, the condition $\sup G^{-1}(< A) = \inf G^{-1}(> A)$, for each $A \in DM(X) \setminus \phi(X)$ implies that $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X) \setminus \phi(X)$, so Theorem 3.8 lets us conclude that F is the cdf of a probability measure, μ, defined on X.

Now, we prove another claim that will help us in showing the equality $G_\mu = G$. For that purpose, and by taking into account that F is a cdf, we will use its extension to $DM(X)$, $\widetilde{F}$.

Claim 3.19. *$\widetilde{F}(G(r)) \geq r$, for each $r \in [0,1]$.*

Proof. Let $r \in [0,1]$ and suppose that $\widetilde{F}(G(r)) < r$, then $\inf_{x \in G(r)^u} F(x) < r$. Hence, there exists $x \in G(r)^u$ such that $F(x) < r$. Now, by Claim 3.17 it follows that $G(r) > \phi(x)$, which means that $x \notin G(r)^u$, a contradiction. Consequently, $\widetilde{F}(G(r)) \geq r$. □

Finally, we show that $G_\mu = G$.

$\geq$) Let $r \in [0,1]$ and $A \in DM(X)$ such that $\widetilde{F}(A) \geq r$. Now, let $x \in A^u$, then $\widetilde{F}(\phi(x)) \geq \widetilde{F}(A) \geq r$. The fact that $\widetilde{F}$ is an extension of F gives us that $\widetilde{F}(\phi(x)) = F(x)$. Since $F(x) \geq r$, by Claim 3.17, we have that $G(r) \leq \phi(x)$. By the arbitrariness of x, we conclude that $G(r) \leq A$ and, consequently, $\inf\{A \in DM(X) : \widetilde{F}(A) \geq r\} \geq G(r)$, that is, $G_\mu(r) \geq G(r)$.

$\leq$) By Claim 3.19, $r \leq \widetilde{F}(G(r))$, for each $r \in [0,1]$. Now, by taking into account that G_μ is the pseudo-inverse of $\widetilde{F}$ as a cdf, its monotonicity gives us that $G_\mu(r) \leq G_\mu(\widetilde{F}(G(r)))$. Finally, by taking into account Proposition 2.20.1, it follows that $G_\mu(\widetilde{F}(G(r))) \leq G(r)$ so we can conclude that $G_\mu(r) \leq G(r)$.

The uniqueness of the measure immediately follows from Corollary 2.25. □

Once we have proven that a measure can be determined from F_- and G when given some conditions on them and $DM(X)$, we get two immediate results:

Corollary 3.20. *Let X be a compact separable LOTS and $F_- : X \to [0,1]$ a monotonically non-decreasing, left τ-continuous function such that $\inf F_-(X) = 0$. Then there exists a unique probability measure μ on the Borel σ-algebra of X such that $F_{\mu-} = F_-$.*

Proof. Note that the fact that X is compact means that $DM(X) = \phi(X)$ which implies that $DM(X) \setminus \phi(X) = \emptyset$. Since given $A \in DM(X)$, there exists $a \in X$ such that $A = \phi(a)$, it is clear that $\sup F_-(A) = \inf F_-(A^u)$. Hence, by taking into account the hypothesis on F_- and Corollary 3.11, we conclude that there exists a probability measure μ on the Borel σ-algebra of X such that $F_{\mu-} = F_-$. Moreover, Corollary 2.24 ensures that μ is unique. □

Corollary 3.21. *Let X be a compact separable LOTS and $G : X \to [0,1]$ a monotonically non-decreasing and left τ-continuous function satisfying $G(0) = \min X$ and $G(1) = \max X$. Then there exists a unique probability measure μ on the Borel σ-algebra of X such that G is the pseudo-inverse of F_μ.*

Proof. Since X is compact, $DM(X) = \phi(X)$ which implies that $DM(X) \setminus \phi(X) = \emptyset$. Now, by taking into account the hypothesis on G and Corollary 3.16, we conclude that there exists a probability measure μ on the Borel σ-algebra of X such that G is the pseudo-inverse of F_μ. Moreover, Corollary 2.25 ensures that μ is unique. □

4. Examples

Next, we show some examples in which it is possible to define a probability measure on X from a function satisfying the properties of a cdf by taking into account the theory that has been developed in the previous sections.

Example 4.1. Let $X = (\{0\} \cup \mathbb{N}) \times [0,1]$ and $\leq$ be the lexicographical order in X. Consider the function $F : X \to [0,1]$ given by $F(x,y) = 1 - \frac{1}{2}e^{-(x+y)} - \frac{1}{2}e^{-x}$, for each $(x,y) \in X$.

Note that in this case $DM(X) \setminus \phi(X) = \{X\}$. Roughly speaking, $DM(X)$ coincides with the one-point compactification of X since the only cut that we add when we consider $DM(X)$ is X.

We have already seen that $DM(X) \setminus \phi(X)$ is countable. Moreover, by the definition of F it holds that F is a monotonically non-decreasing and right τ-continuous function (indeed, F is continuous) satisfying $\sup F(X) = 1$ and $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X)$. Finally, Theorem 3.8 lets us conclude that there exists a unique probability measure μ on X such that F is its cdf.

Example 4.2. Let $X = (\{0\} \cup \mathbb{N}) \times [0,1]$ and $\leq$ be the lexicographical order in X. Consider the function $F_- : X \to [0,1]$ given by $F_-(x,y) = 1 - \frac{1}{2}e^{-(x+y)} - \frac{1}{2}e^{-x}$, for each $(x,y) \in X \setminus \{(x,0) : x \in \mathbb{N}\}$ and $F_-(x,0) = 1 - \frac{e+1}{2}e^{-x}$, for each $x \in \mathbb{N}$.

We have already seen that $DM(X) \setminus \phi(X)$ is countable. Note that F_- is continuous in $X \setminus \{(x,0) : x \in \mathbb{N}\}$ so it is left τ-continuous. Moreover, given $(x,0)$ for some $x \in \mathbb{N}$, it holds that F_- is left τ-continuous at $(x,0)$ since this point is left-isolated.

On the other hand, by the definition of F_- it holds that F_- is monotonically non-decreasing and it satisfies $\sup F_-(X) = 1$ and $\sup F_-(A) = \inf F_-(A^u)$, for each $A \in DM(X)$. Finally, Corollary 3.11 lets us conclude that there exists a unique probability measure, μ, on X such that $F_{\mu-} = F_-$.

Example 4.3. Let $X = (\{0\} \cup \mathbb{N}) \times [0,1]$ and $\leq$ be the lexicographical order in X. Consider the function $G : DM(X) \to [0,1]$ given by $G(r) = (\leq \min\{(x,y) \in X : x + y \geq \ln(1-r)^{-1}\})$, for each $r \in [0,1[$ and $G(1) = X$. Note that G satisfies the conditions of Corollary 3.16, which means that there exists a probability measure μ on X such that G is the pseudo-inverse of F_μ.

Indeed, by taking into account Proposition 2.23, we can define F_μ by $F_\mu(x,y) = 1 - e^{-(x+y)}$, for each $(x,y) \in X$.

The next example shows a function that is not a cdf.

Example 4.4. Let $X = (\{0\} \cup \mathbb{N}) \times]0,1[$ and $\leq$ be the lexicographical order in X. Consider the function $F : X \to [0,1]$ given by $F(x,y) = 1 - \frac{1}{2^x}$, for each $(x,y) \geq (1,0)$ and $F(x,y) = 0$ otherwise.

Note that $DM(X) \setminus \phi(X)$ is countable, F is right τ-continuous, monotonically non-decreasing, $\inf F(X) = 0$ and $\sup F(X) = 1$.

However, if we consider the cut $A = (< (1,1))$, the condition $\sup F(A) = \inf F(A^u)$ does not hold. In this case $A^u = (> (2,0))$. Note that $\sup F(A) = \frac{1}{2}$ and $\inf F(A^u) = \frac{3}{4}$. Hence, $\sup F(A) \neq \inf F(A^u)$ and, by Proposition 3.5.1, F is not the cdf of a probability measure on X.

In order to end with this section, we introduce a simple real example where our theory is essential to get the probability distribution:

Example 4.5. Consider three cdfs that are the lifetime of three different light bulbs. The distributions are exponential with means 800, 1000 and 1200 hours. Consider a system with three light bulbs one of each type. Find the probability that, at least, one of the light bulbs of this type has a lifetime of more than 900 hours.

In the classical case we can define the random variables $X_1 \sim \varepsilon(\frac{1}{800}), X_2 \sim \varepsilon(\frac{1}{1000})$ and $X_3 \sim \varepsilon(\frac{1}{1200})$. Note that the corresponding cdfs, F_1, F_2 and F_3 are a particular case of a cdf according to the developed theory. Furthermore, the idea of modelling the case in which three light bulbs work together is considering the set $X = [0, \infty[\times\{0, 1, 2\}$ and $\leq$ as the lexicographical order in X. It holds that X is a separable LOTS and that $DM(X) \setminus \phi(X) = \{X\}$. The function $F : X \to [0, 1]$ defined by $F(x, y) = \frac{1}{3}(F_1(x) + F_2(x) + F_3(x))$ is monotonically non-decreasing, right τ-continuous and $\sup F(A) = \inf F(A^u)$, for each $A \in DM(X)$. Since $DM(X) \setminus \phi(X)$ is countable, Theorem 3.8 lets us ensure that there exists a probability measure on X such that its cdf is F. Hence, it is possible for us to know the probability we want by calculating $1 - F(900, y)$, for any $y \in \{0, 1, 2\}$.

5. Conclusions

This work continues the research line started in [11] and further developed in [12]. Indeed, in [12] the authors proposed an open question in order to look for conditions to ensure that a function satisfying the properties of a cdf on a separable linearly ordered topological space is, indeed, the cdf of a probability measure on that space. That question seems to be natural since the main goal of this research line is extending the classical theory of distribution functions to a more general context: the case in which we work with a separable linearly ordered topological space. In this paper, authors answer that question through Theorem 3.8. Before stating that result, authors prove some propositions which consist of necessary conditions for a cdf which is defined on a separable LOTS. Some of that conditions, together with the basic properties of a cdf (proven in [11]), are sufficient conditions to ensure that there exists a probability measure on the Borel σ-algebra of the space, such that its cdf is the function satisfying the properties. What is more, that probability measure is unique. Once we have proven the main theorem of the paper, two corollaries have arisen (see Corollary 3.11 and Corollary 3.16). In them, we give conditions to guarantee that from a function satisfying the properties of F_- and G we can get a probability measure whose cdf gives us F_- and G, respectively.

Finally, we show some examples in which the one-to-one relationship between a probability measure and F, F_- or G holds.

References

[1] L. Arthur Steen, Jr. J. Arthur Seedbach, Counterexamples in topology, (2nd edition), Springer, New York, 1978.
[2] J.W. Baker, J.S. Pym, H.L. Vasudeva, Totally ordered measure spaces and their Lp algebras, Mathematika 29 (1982) 42–54.
[3] B. Lafuerza-Guillén, P. Harikrishnan, Probabilistic Normed Spaces, (1st edition), Imperial College Press, London, 2014.
[4] G. Bezhanishvili, P.J. Morandi, Order compactifications of totally ordered spaces: revisited, Order 28 (2011) 577–592.
[5] T.D. Blackmore, Derivations from totally ordered semigroup algebras into their duals, Canad. Math. Bull. 40 (1997) 133–142.
[6] V.I. Bogachev, Measures on topological spaces, J. Math. Sci. (N. Y.) 91) (1998) 3033–3156.
[7] V.I. Bogachev, Measure Theory, (1st edition), Springer Science & Business Media, 2007.
[8] D.S. Bridges, G.B. Mehta, Representations of Preferences Orderings, (1st edition), Springer Verlag, Berlin, 1995.
[9] D. Dushnik, E.W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600–610.
[10] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Lecture Notes in Pure and Appl. Math. 77, (1st edition), Marcel Dekker, New York, 1982.
[11] J.F. Gálvez-Rodríguez, M.A. Sánchez-Granero, The distribution function of a probability measure on a linearly] ordered topological space, Mathematics 7 (2019) 864.
[12] J.F. Gálvez-Rodríguez, M.A. Sánchez-Granero, The distribution function of a probability measure on the Dedekind-MacNeille completion, Topology Appl. (2019) 107010.
[13] P.R. Halmos, Measure Theory, (1st edition), Springer-Verlag, New York, 1974.
[14] Z. Kamali, M.L. Bami, The Bochner-Schoenberg-Eberlein property for totally ordered semigroup algebras, Fourier Anal. Appl. 22 (2016) 1225–1234.
[15] R. Kaufman, Ordered sets and compact spaces, Colloq. Math. 17 (1967) 35–39.
[16] R. Lowen, On the Existence of Natural Non-Topological, Fuzzy Topological Spaces, (1st edition), Heldermann, Berlin, 1985.
[17] D.J. Lutzer, On generalized ordered spaces, Dissertations Math. LXXXIX (1971).
[18] H.M. MacNeille, Partially ordered sets, Tran. Amer. Math. Soc. 42 (1937) 416–460.
[19] S. Roman, Lattices and ordered sets, (1st edition), Springer Science and Business Media, Irvine, 2008.

[20] A. Sapounakis, Measures on totally ordered space, Mathematika 27 (1980) 225–235.
[21] A. Sapounakis, The existence of strong liftings for totally ordered measure space, Pacific J. Math. 106 (1983) 145–151.
[22] A. Sapounakis, Properties of measures on totally ordered spaces, Rend. del Circ. Mat. Palermo 38 (1989) 277–296.
[23] B.S.W. Schröder, Ordered sets, (1st edition), Birkhäuser, Boston, 2003.
[24] R. Suck, A qualitative characterization of the exponential distribution, J. Math. Psych. 42 (1998) 418–431.