Applications of the Horadam Polynomials Involving λ-Pseudo-Starlike Bi-Univalent Functions Associated with a Certain Convolution Operator

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Abstract. In this article, we introduce and study a new family $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$ of normalized analytic and λ-pseudo-starlike bi-univalent functions by using the Horadam polynomials, which is associated with a certain convolution operator defined in the open unit disk $\mathbb{U}$. We establish the bounds for $|a_2|$ and $|a_3|$, where a_2, a_3 are the initial Taylor-Maclaurin coefficients. Furthermore, we obtain the Fekete-Szegö inequality for functions in the class $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$, which we have introduced here. We indicate several special cases and consequences for our results. Finally, we comment on the recent usages, especially in Geometric Function Theory of Complex Analysis, of the basic (or q-) calculus and also of its trivial and inconsequential (p, q)-variation involving an obviously redundant parameter p.

1. Introduction, Motivation and Preliminaries

We denote by $\mathcal{A}$ the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. According to the Koebe one-quarter theorem [10], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stand for the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief historical account and for several interesting examples of functions in the class Σ, see the pioneering work on this subject by Srivastava *et al.* [47], which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [47], we choose to recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the Koebe function is not a member of Σ.

In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [47], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [8, 12, 29, 36, 39, 40, 45, 48, 53, 54, 61, 62, 64, 65]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1) were obtained in most (if not all) of the recent papers. The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ (see, for example, [36, 48, 53]).

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $f \in \mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [11] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subclasses of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory (see, for example, [41, 46, 52]).

Definition 1. A function $f \in \mathcal{A}$ is said to be λ-pseudo-starlike function of order ρ ($0 \leqq \rho < 1$) in $\mathbb{U}$ if (see [6])

$$\Re\left(\frac{z\big(f'(z)\big)^{\lambda}}{f(z)}\right) > \rho \qquad (z \in \mathbb{U};\ \lambda \geqq 1).$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. We say that the function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big).$$

This subordination is denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

It is well known that, if the function g is univalent in $\mathbb{U}$, then (see [21])

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Recently, Hörçum and Koçer [15] considered the so-called Horadam polynomials $h_n(r)$, which are given by the following recurrence relation (see also Horadam and Mahon [14]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}; \ n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{3}$$

with

$$h_1(r) = a \quad \text{and} \quad h_2(r) = br,$$

for some real constants a, b, p and q. The characteristic equation of the recurrence relation (3) is given by

$$t^2 - prt - q = 0.$$

This equation has the following two real roots:

$$\alpha = \frac{pr + \sqrt{p^2 r^2 + 4q}}{2} \quad \text{and} \quad \beta = \frac{pr - \sqrt{p^2 r^2 + 4q}}{2}.$$

Remark 1. By selecting particular values of a, b, p and q, the Horadam polynomial $h_n(r)$ reduces to several known polynomials. Some of these special cases are recorded below.

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$.

2. Taking $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(r)$.

3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$.

4. Taking $a = b = p = 2$ and $q = 1$, we find the Pell-Lucas polynomials $Q_n(r)$.

5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.

6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their extensions and generalizations, are potentially important in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials, see [13, 14, 18, 19].

The generating function of the Horadam polynomials $h_n(r)$ is given as follows (see [15]):

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{4}$$

Srivastava *et al.* [33] have already applied the Horadam polynomials in a context involving analytic and bi-univalent functions. The investigation by Srivastava *et al.* [33] was followed by such works as those by Al-Amoush [2], Wanas and Alina [63] and Abirami *et al.* [1].

Recently, Wanas [60] introduced the following convolution operator $W_{\alpha,\beta}^{k,\gamma} : \mathcal{A} \to \mathcal{A}$ defined by

$$W_{\alpha,\beta}^{k,\gamma} f(z) = z + \sum_{n=2}^{\infty} [\Phi_n(k, \alpha, \beta)]^\gamma \ a_n z^n, \tag{5}$$

where

$$\Phi_n(k,\alpha,\beta) = \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \tag{6}$$

$$\left(\alpha \in \mathbb{R};\ \beta \geqq 0\ \text{with}\ \alpha + \beta > 0;\ m, \gamma \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \right).$$

Remark 2. The operator $W_{\alpha,\beta}^{k,\gamma}$ is a generalization of several known operators considered in earlier investigations which are being recalled below.

1. For $k = 1$, the operator $W_{\alpha,\beta}^{1,\gamma} \equiv I_{\alpha,\beta}^{\gamma}$ was introduced and studied by Swamy [58].

2. For $k = \beta = 1$, $\gamma = -\mu$, $\Re(\mu) > 1$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the operator $W_{\alpha,1}^{1,-\mu} \equiv J_{\mu,\alpha}$ was investigated by Srivastava and Attiya [35]. The operator $J_{\mu,\alpha}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of the Srivastava-Attiya operator are found in [25, 37, 38, 42, 51] and in the references cited in each of these earlier works.

3. For $k = \beta = 1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,\gamma} \equiv I_{\alpha}^{\gamma}$ was investigated by Cho and Srivastava [9].

4. For $k = \alpha = \beta = 1$, the operator $W_{1,1}^{1,\gamma} \equiv I^{\gamma}$ was considered by Uralegaddi and Somanatha [59].

5. For $k = \alpha = \beta = 1$, $\gamma = -\sigma$ and $\sigma > 0$, the operator $W_{1,1}^{1,-\sigma} \equiv I^{\sigma}$ was introduced by Jung *et al.* [16]. The operator I^{σ} is widely known as the Jung-Kim-Srivastava integral operator.

6. For $k = \beta = 1$, $\gamma = -1$ and $\alpha > -1$, the operator $W_{\alpha,1}^{1,-1} \equiv L_{\alpha}$ was studied by Bernardi [7].

7. For $\alpha = 0$, $k = \beta = 1$ and $\gamma = -1$, the operator $W_{0,1}^{1,-1} \equiv u$ was investigated by Alexander [3].

8. For $k = 1$, $\alpha = 1 - \beta$ and $\beta \geq 0$, the operator $W_{1-\beta,\beta}^{1,\gamma} \equiv D_{\beta}^{\gamma}$ was given by Al-Oboudi [4].

9. For $k = 1$, $\alpha = 0$ and $\beta = 1$, the operator $W_{0,1}^{1,\gamma} \equiv S^{\gamma}$ was considered by Sălăgean [24].

Remark 3. In a recently-published survey-cum-expository review article by Srivastava [29], the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [29, p. 340]). Srivastava [29] also pointed out how the Hurwitz-Lerch Zeta function as well as its *multi-parameter* extension, which is popularly known as the λ-generalized Hurwitz-Lerch Zeta function (see, for details, [26]), have motivated the studies of several other families of extensively- and widely-investigated linear convolution operators which emerge essentially from the Srivastava-Attiya operator [35] (see also [27], [28], [30] and [31]).

2. A Set of Main Results

We begin this section by defining the new subclass $\mathcal{P}_{\Sigma}(\delta, \lambda, k, \gamma, \alpha, \beta, r)$.

Definition 2. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geqq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{P}_{\Sigma}(\delta, \lambda, k, \gamma, \alpha, \beta, r)$ if it satisfies the following subordination conditions:

$$1 + \frac{1}{\delta} \left(\frac{z \left(\left(W_{\alpha,\beta}^{k,\gamma} f(z) \right)' \right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma} f(z)} - 1 \right) \prec \Pi(r,z) + 1 - a$$

and

$$1 + \frac{1}{\delta} \left(\frac{w\left(\left(W_{\alpha,\beta}^{k,\gamma}g(w)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}g(w)} - 1 \right) \prec \Pi(r,w) + 1 - a,$$

where a is a real constant and the function $g = f^{-1}$ is given by (2).

Our first main result is asserted by Theorem 1 below.

Theorem 1. *For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geqq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{P}_{\Sigma}(\delta, \lambda, k, \gamma, \alpha, \beta, r)$. Then*

$$|a_2| \leqq \frac{\delta \, |br| \, \sqrt{|br|}}{\sqrt{\left| \left[\delta \left((3\lambda - 1)\Phi_3^{\gamma}(k,\alpha,\beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k,\alpha,\beta) \right) b - p \, (2\lambda - 1)^2 \right] br^2 - qa \, (2\lambda - 1)^2 \right|}}$$

and

$$|a_3| \leqq \frac{\delta \, |br|}{(3\lambda - 1) \left| \Phi_3^{\gamma}(k,\alpha,\beta) \right|} + \frac{\delta^2 b^2 r^2}{(2\lambda - 1)^2 \, \Phi_2^{2\gamma}(k,\alpha,\beta)}.$$

Proof. Let $f \in \mathcal{P}_{\Sigma}(\delta, \lambda, k, \gamma, \alpha, \beta, r)$. Then there are two analytic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{7}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{8}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max \{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{U}),$$

such that

$$\frac{1}{\delta} \left(\frac{z\left(\left(W_{\alpha,\beta}^{k,\gamma}f(z)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}f(z)} - 1 \right) = \Pi(r, u(z)) - a$$

and

$$\frac{1}{\delta} \left(\frac{w\left(\left(W_{\alpha,\beta}^{k,\gamma}g(w)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}g(w)} - 1 \right) = \Pi(r, v(w)) - a.$$

Equivalently, we have

$$\frac{1}{\delta} \left(\frac{z\left(\left(W_{\alpha,\beta}^{k,\gamma}f(z)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}f(z)} - 1 \right) = h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \tag{9}$$

and

$$\frac{1}{\delta} \left(\frac{w\left(\left(W_{\alpha,\beta}^{k,\gamma}g(w)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}g(w)} - 1 \right) = h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots. \tag{10}$$

Combining (7), (8), (9) and (10), we find that

$$\frac{1}{\delta}\left(\frac{z\left(\left(W_{\alpha,\beta}^{k,\gamma}f(z)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}f(z)} - 1\right) = h_2(r)u_1 z + \left[h_2(r)u_2 + h_3(r)u_1^2\right]z^2 + \cdots \tag{11}$$

and

$$\frac{1}{\delta}\left(\frac{w\left(\left(W_{\alpha,\beta}^{k,\gamma}g(w)\right)'\right)^{\lambda}}{W_{\alpha,\beta}^{k,\gamma}g(w)} - 1\right) = h_2(r)v_1 w + \left[h_2(r)v_2 + h_3(r)v_1^2\right]w^2 + \cdots . \tag{12}$$

It is well known that, if

$$\max\{|u(z)|, |v(w)|\} < 1 \qquad (z, w \in \mathbb{U}),$$

then

$$|u_j| \leqq 1 \qquad \text{and} \qquad |v_j| \leqq 1 \qquad (\forall\, j \in \mathbb{N}). \tag{13}$$

Now, by comparing the corresponding coefficients in (11) and (12), and after some simplification, we have

$$\frac{(2\lambda - 1)\Phi_2^{\gamma}(k, \alpha, \beta)}{\delta}\, a_2 = h_2(r)u_1, \tag{14}$$

$$\frac{(3\lambda - 1)\Phi_3^{\gamma}(k, \alpha, \beta)}{\delta}\, a_3 + \frac{(2\lambda(\lambda - 2) + 1)\,\Phi_2^{2\gamma}(k, \alpha, \beta)}{\delta}\, a_2^2$$
$$= h_2(r)u_2 + h_3(r)u_1^2, \tag{15}$$

$$-\frac{(2\lambda - 1)\Phi_2^{\gamma}(k, \alpha, \beta)}{\delta}\, a_2 = h_2(r)v_1 \tag{16}$$

and

$$\frac{(3\lambda - 1)\Phi_3^{\gamma}(k, \alpha, \beta)}{\delta}\,(2a_2^2 - a_3) + \frac{(2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)}{\delta}\, a_2^2$$
$$= h_2(r)v_2 + h_3(r)v_1^2. \tag{17}$$

It follows from (14) and (16) that

$$u_1 = -v_1 \tag{18}$$

and

$$\frac{2(2\lambda - 1)^2\,\Phi_2^{2\gamma}(k, \alpha, \beta)}{\delta^2}\, a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \tag{19}$$

If we add (15) to (17), we find that

$$\frac{2\left[(3\lambda - 1)\Phi_3^{\gamma}(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right]}{\delta}\, a_2^2$$
$$= h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \tag{20}$$

Upon substituting the value of $u_1^2 + v_1^2$ from (19) into the right-hand side of (20), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(r)(u_2 + v_2)}{2\left[\delta h_2^2(r)\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right) - h_3(r)(2\lambda - 1)^2\right]}. \tag{21}$$

By further computations using (3), (13) and (21), we obtain

$$|a_2| \leqq \frac{\delta |br| \sqrt{|br|}}{\sqrt{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|}}.$$

Next, if we subtract (17) from (18), we can easily see that

$$\frac{2(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)}{\delta}(a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \tag{22}$$

Also, in view of (18) and (19), we find from (22) that

$$a_3 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)} + \frac{\delta^2 h_2^2(r)(u_1^2 + v_1^2)}{2(2\lambda - 1)^2 \Phi_2^{2\gamma}(k, \alpha, \beta)}.$$

Thus, by applying (3), we obtain

$$|a_3| \leqq \frac{\delta |br|}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|} + \frac{\delta^2 b^2 r^2}{(2\lambda - 1)^2 \Phi_2^{2\gamma}(k, \alpha, \beta)}.$$

This completes the proof of Theorem 1. □

In the next theorem, we present the Fekete-Szegö inequality for functions in the class $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$.

Theorem 2. *For* $\delta \in \mathbb{C} \setminus \{0\}$*,* $\lambda \geqq 1$ *and* $r, \mu \in \mathbb{R}$*, let* $f \in \mathcal{A}$ *be in the class* $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$*. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{\delta |br|}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|} \\[2em] \left(|\mu - 1| \leqq \dfrac{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|}{\delta b^2 r^2 (3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right) \\[2em] \dfrac{\delta^2 |br|^3 |\mu - 1|}{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|} \\[2em] \left(|\mu - 1| \geqq \dfrac{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|}{\delta b^2 r^2 (3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right). \end{cases}$$

Proof. It follows from (21) and (22) that

$$a_3 - \mu a_2^2 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)} + (1 - \mu) a_2^2$$

$$= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)}$$

$$+ \frac{\delta^2 h_2^3(r)(u_2 + v_2)(1 - \mu)}{2\left[\delta h_2^2(r)\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right) - h_3(r)(2\lambda - 1)^2\right]}$$

$$= \frac{h_2(x)}{2}\left[\left(\psi(\mu, r) + \frac{\delta}{(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)}\right)u_2 + \left(\psi(\mu, r) - \frac{\delta}{(3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta)}\right)v_2\right],$$

where

$$\psi(\mu, r) = \frac{\delta^2 h_2^2(r)(1 - \mu)}{\delta h_2^2(r)\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right) - h_3(r)(2\lambda - 1)^2}.$$

Thus, according to (3), we have

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{\delta |br|}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|} & \left(0 \leqq |\psi(\mu, r)| \leqq \dfrac{\delta}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right) \\[4mm] |br| \cdot |\psi(\mu, r)| & \left(|\psi(\mu, r)| \geqq \dfrac{\delta}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right), \end{cases}$$

which, after simple computation, yields

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \dfrac{\delta |br|}{(3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|} & \left(|\mu - 1| \leqq \dfrac{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|}{\delta b^2 r^2 (3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right) \\[6mm] \dfrac{\delta^2 |br|^3 |\mu - 1|}{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|} & \left(|\mu - 1| \geqq \dfrac{\left|\left[\delta\left((3\lambda - 1)\Phi_3^\gamma(k, \alpha, \beta) + (2\lambda(\lambda - 2) + 1)\Phi_2^{2\gamma}(k, \alpha, \beta)\right)b - p(2\lambda - 1)^2\right]br^2 - qa(2\lambda - 1)^2\right|}{\delta b^2 r^2 (3\lambda - 1)\left|\Phi_3^\gamma(k, \alpha, \beta)\right|}\right). \end{cases}$$

We have thus completed the proof of Theorem 2. □

3. Corollaries and Consequences

Our main results (Theorem 1 and Theorem 2) can be specialized to deduce a number of known or new results as their corollaries and consequences dealing with the initial Taylor-Maclaurin coefficient inequalities and the Fekete-Szegö inequalities. We choose to record here one example in which, by putting $\mu = 1$ in Theorem 2, we are led to the following corollary.

Corollary. *For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geqq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$. Then*

$$\left| a_3 - a_2^2 \right| \leqq \frac{\delta \, |br|}{(3\lambda - 1) \left| \Phi_3^\gamma(k, \alpha, \beta) \right|}.$$

Remark 4. By taking particular values of the parameters δ, λ, γ, a, b, p and q, in our main results (Theorem 1 and Theorem 2), we can derive a number of known results. Some of these special cases are recorded below.

1. If we put $\gamma = 0$ and $\delta = \lambda = 1$ in Theorem 1 and Theorem 2, we have the results for the well-known class $S_\Sigma^*(r)$ of bi-starlike functions which was studied recently by Srivastava *et al.* [33].

2. If we put $\gamma = 0$, $\delta = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in Theorem 1 and Theorem 2, we have the results for the class $\mathcal{LB}_\Sigma(t)$ of λ-Pseudo bi-starlike functions which was considered recently by Magesh and Bulut [20].

3. If we put $\gamma = 0$, $\delta = \lambda = a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in Theorem 1 and Theorem 2, we obtain the results for the class $S_\Sigma^*(t)$ of bi-starlike functions which was considered recently by Altınkaya and Yalçin [5].

4. Concluding Remarks and Observations

In our present investigation, we have introduced and studied a (presumably new) class $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$ of normalized analytic and λ-pseudo-starlike bi-univalent functions in the open unit disk $\mathbb{U}$ by applying the Horadam polynomials. This λ-pseudo-starlike bi-univalent function class is associated with a certain convolution operator. We have established the bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to the λ-pseudo-starlike bi-univalent function class which we have defined in this article. Furthermore, we have successfully solved the Fekete-Szegö problem for functions in the same class $\mathcal{P}_\Sigma(\delta, \lambda, k, \gamma, \alpha, \beta, r)$ λ-pseudo-starlike bi-univalent functions. We have also considered a number of special cases and consequences for our main results (Theorem 1 and Theorem 2).

We conclude our investigation by remarking that, in order to motivate further researches on the subject-matter of this article, we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications on the subjects which we have discussed here. One of these directions for further researches should be motivated by a recently-published survey-cum-expository review article by Srivastava [29]. With this point in view, the attention of the interested reader is drawn toward the possibility of investigating the basic (or q-) extensions of the results which are presented in this paper. However, as already pointed out by Srivastava [29], their further extensions using the so-called (p, q)-calculus will be rather trivial and inconsequential variations of the suggested extensions which are based upon the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [29, p. 340] and [32, pp. 1511–1512]). With a view to aiding the interested reader, we choose to cite several recent developments (see [17], [22], [23], [34], [44], [55] and [57]) on various usages of the basic (or q-) calculus in Geometric Function Theory of Complex Analysis.

Conflicts of Interest: The authors declares that they have no conflicts of interest.

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