



Approximation Properties of Bivariate Szász Durrmeyer Operators via Dunkl Analogue

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Abstract. In the present article, we construct a new sequence of bivariate Szász-Durrmeyer operators based on Dunkl analogue. We investigate the order of approximation with the aid of modulus of continuity in terms of well known Peetre's K -functional, weighted approximation results, Voronovskaja type theorems and Lipschitz maximal functions. Further, we also discuss here the approximation properties of the operators in Bøgel-spaces by use of mixed-modulus of continuity.

1. Introduction

In the late of nineteenth century 1885, Weierstrass, a German mathematician proposed a prominent and historical theorem termed as Weierstrass approximation theorem [1]. This theorem plays a vital role and motivated several mathematicians to work on approximation theory. But there was a major drawback of the proof of the theorem, that it was very tedious and lengthy. In the year of 1912, Bernstein [2] proposed the polynomials with the aid of binomial distribution, which give the simplest and easiest proof of Weierstrass approximation theorem as follows:

$$B_n(f; x) = \sum_{v=0}^n p_{n,v}(x) f\left(\frac{v}{n}\right), \quad n \in \mathbb{N}, \quad (1)$$

where $p_{n,v}(x) = \binom{n}{v} x^v (1-x)^{n-v}$. He proved that $B_n(f; x) \rightrightarrows f$ for each $f \in C[0, 1]$ where \rightrightarrows stands for uniform convergence. Szász [3] generalized the operators defined by (1) on unbounded interval, i.e., on $[0, \infty)$ as

$$S_n(f; x) = e^{-nx} \sum_{v=0}^{\infty} \frac{(nx)^v}{v!} f\left(\frac{v}{n}\right), \quad n \in \mathbb{N}. \quad (2)$$

Many generalizations were discussed for (2), by the mathematician Acar [see [22], [23]], Acar et al. [see [24], [25]], Mohiuddin et al. ([15], [16]) to achieve the convergence properties by these sequences on positive

2020 *Mathematics Subject Classification.* Primary 41A10, 41A25, 41A35, 41A36.

Keywords. Szász operators; Simultaneous approximation; Peetre's K -functional; Voronovskaja type theorem; Mixed-modulus of continuity; Bøgel functions.

Received: 21 October 2020; Revised: 15 December 2020; Accepted: 29 December 2020

Communicated by Snežana Č. Živković-Zlatanović

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semi axis. Operators (1) and (2) are restricted for continuous functions only. An integral modification of Bernstein operators (1) on an interval $[0,1]$ was proposed by Durrmeyer [4] to study the approximation properties for Lebsgue integrable functions given by

$$D_n(f; x) = n \sum_{\nu=0}^n p_{n,\nu}(x) \int_0^1 p_{n,\nu}(t) f(t) dt. \tag{3}$$

Lately, several mathematician proposed Szász-type operators via various types of generating functions, i.e. Szász types operators via Charlier polynomials [6], Szász types operators via Sheffer polynomials [7]. Many generalizations have been studied in this directions [see [5], [11]-[33]]. The Szász types operators on the basis of Dunkl analogue was proposed by Sucu [8]. Using the generalization of exponential function is given by [9] as

$$e_\mu(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\lambda_\nu(\nu)} \tag{4}$$

where the coefficients $\gamma_\mu(\nu)$ are defined as follows:

For $\nu \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mu > -1/2$

$$\gamma_\mu(2\nu) = \frac{2^{2\nu} \nu! \Gamma(\nu + \mu + 1/2)}{\Gamma(\mu + 1/2)}, \quad \gamma_\mu(2\nu + 1) = \frac{2^{2\nu+1} \nu! \Gamma(\nu + \mu + 3/2)}{\Gamma(\mu + 1/2)}$$

and the recursive relation for γ_μ is defined as

$$\gamma_\mu(\nu + 1) = (\nu + 1 + 2\mu\theta_{\nu+1})\gamma_\mu(\nu), \quad \nu \in \mathbb{N}_0, \tag{5}$$

with θ_ν is defined to be 0 if $\nu \in 2\mathbb{N}$ and 1 if $\nu \in 2\mathbb{N} + 1$. For $f \in C[0, \infty)$, Sucu [8] defined Szász type operators using (4) given by

$$S_n^*(f; x) = \frac{1}{e_\mu(nx)} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\gamma_\mu(\nu)} f\left(\frac{\nu + 2\mu\theta_\nu}{n}\right), \tag{6}$$

where $\mu, x \geq 0$ and $n \in \mathbb{N}$. One can notice that for $\mu = 0$, the operators (6) reduce to the operators (2). Wafi et al. [10] introduced Durrmeyer type modification of the operators defined by (6) to approximate the Lebesgue integrable function as follows:

For every $f \in C_\beta[0, \infty) = \{f \in C[0, \infty) : f(t) = O(t^\beta), \text{ as } t \rightarrow \infty\}$, we define

$$T_n^*(f; x) = \frac{1}{e_\mu(x)} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\gamma_\mu(\nu)} \frac{1}{B(\nu + 2\mu\theta_\nu + 1, n)} \int_0^\infty \frac{t^{\nu+2\mu\theta_\nu}}{(1+t)^{\nu+2\mu\theta_\nu+n+1}} f(t) dt, \tag{7}$$

where $\beta > n$ and $B(\nu + 2\mu\theta_\nu + 1, n)$ is known as beta function which is defined as

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m, n > 0. \tag{8}$$

The motive of this article is to introduce bivariate Szász-Durremeyer type operators via Dunkl analogue. In order to get approximation results for these bivariate sequences, we yield results using modulus and mixed-modulus of continuity in Voronovskaja type theorem, K-functional and Lipschitz maximal functions, global approximation results. In addition to this, we also study the application of the GBS type operators with the help of mixed-modulus of continuity and yield results for Bögel continuous functions.

2. Construction of bivariate Szász-Durrmeyer-Operators $H_{n_1, n_2}^*(\cdot; \cdot)$ and their Basic Estimates

let $\mathcal{I}^2 = \{(y_1, y_2) : 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1\}$ and $C(\mathcal{I}^2)$ is the class of all continuous functions on \mathcal{I}^2 equipped with the norm $\|g\|_{C(\mathcal{I}^2)} = \sup_{(y_1, y_2) \in \mathcal{I}^2} |g(y_1, y_2)|$. Then for all $h \in C(\mathcal{I}^2)$ and $n_1, n_2 \in \mathbb{N}$, we construct a new sequences of bivariate Szász-Durrmeyer type operators via Dunkl analogue as follow :

$$H_{n_1, n_2}^*(h; y_1, y_2) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} Q_1^*(n_1, y_1) Q_2^*(n_2, y_2) \int_0^{\infty} \int_0^{\infty} P_1^*(n_1, y_1) P_2^*(n_2, y_2) f(t_1, t_2) dt_1 dt_2 \tag{9}$$

where $Q_i^*(n_i, y_i) = \frac{1}{e_{\mu}(n_i y_i)} \frac{(n_i y_i)^{n_i}}{\lambda_{\mu}(\vartheta_i)} \frac{1}{B(v_i+2\vartheta_i+1, n_i)}$ and $P_i^*(n_i, y_i) = \frac{(t_i)^{y_i+2\vartheta_i \theta_{v_i}}}{(1+t_i)^{\vartheta_i+2\vartheta_i \theta_{v_i}+n_i+1}} \quad i = 1, 2$

Lemma 2.1. [10] Let $e_i(t) = t^i, i = 0, 1, 2$ be the test functions. Then, for the operators T_n^* given by (5), we have

$$\begin{aligned} T_n^*(e_0; x) &= 1, \\ T_n^*(e_1; x) &= \frac{nx + 1}{n - 1}, \quad n > 1, \\ T_n^*(e_2; x) &= \frac{1}{(n - 2)(n - 1)} \left\{ n^2 x^2 + \left(4 + 2\mu \frac{e_{\mu}(-nx)}{e_{\mu}(nx)} \right) nx + 2 \right\}, \quad n > 2. \end{aligned}$$

Lemma 2.2. [10] Let $\psi_x^i(t) = (t - x)^i, i = 0, 1, 2$ be the central moments. Then, for the operators defined by (5), we have

$$\begin{aligned} T_n^*(\psi_x^0; x) &= 1, \\ T_n^*(\psi_x^1; x) &= \frac{x + 1}{n - 1}, \quad n > 1, \\ T_n^*(\psi_x^2; x) &= \frac{1}{(n - 2)(n - 1)} \left\{ (n + 2)x^2 + \left(1 + \mu \frac{e_{\mu}(-nx)}{e_{\mu}(nx)} \right) 2nx + 2 \right\}, \quad n > 2. \end{aligned}$$

Lemma 2.3. Let $e_{i,j} = y_1^i y_2^j$. Then, for the operator $H_{n_1, n_2}^*(\cdot; \cdot)$, we have

$$\begin{aligned} H_{n_1, n_2}^*(e_{0,0}; y_1, y_2) &= 1, \\ H_{n_1, n_2}^*(e_{1,0}; y_1, y_2) &= \frac{n_1 y_1}{n_1 - 1}, \quad n_1 > 1 \\ H_{n_1, n_2}^*(e_{0,1}; y_1, y_2) &= \frac{n_2 y_2}{n_2 - 1}, \quad n_2 > 1 \\ H_{n_1, n_2}^*(e_{1,1}; y_1, y_2) &= \frac{n_1 y_1}{(n_1 - 1)} \frac{n_2 y_2}{(n_2 - 2)}, \quad n_1, n_2 > 1 \\ H_{n_1, n_2}^*(e_{2,0}; y_1, y_2) &= \frac{1}{(n_1 - 2)(n_1 - 1)} \left\{ n_1^2 y_1^2 + \left(4 + 2\mu \frac{e_{\mu}(-n_1 y_1)}{e_{\mu}(n_1 y_1)} \right) n_1 y_1 + 2 \right\}, \quad n_1 > 2, \\ H_{n_1, n_2}^*(e_{0,2}; y_1, y_2) &= \frac{1}{(n_2 - 2)(n_2 - 1)} \left\{ n_2^2 y_2^2 + \left(4 + 2\mu \frac{e_{\mu}(-n_2 y_2)}{e_{\mu}(n_2 y_2)} \right) n_2 y_2 + 2 \right\}, \quad n_2 > 2. \end{aligned}$$

Proof. In the light of lemma (2.1) and linearly property, we have

$$\begin{aligned} H_{n_1, n_2}^*(e_{0,0}; y_1, y_2) &= H_{n_1, n_2}^*(e_0; y_1, y_2) H_{n_1, n_2}^*(e_0; y_1, y_2), \\ H_{n_1, n_2}^*(e_{1,0}; y_1, y_2) &= H_{n_1, n_2}^*(e_1; y_1, y_2) H_{n_1, n_2}^*(e_0; y_1, y_2), \\ H_{n_1, n_2}^*(e_{0,1}; y_1, y_2) &= H_{n_1, n_2}^*(e_0; y_1, y_2) H_{n_1, n_2}^*(e_1; y_1, y_2), \\ H_{n_1, n_2}^*(e_{1,1}; y_1, y_2) &= H_{n_1, n_2}^*(e_1; y_1, y_2) H_{n_1, n_2}^*(e_1; y_1, y_2), \\ H_{n_1, n_2}^*(e_{2,0}; y_1, y_2) &= H_{n_1, n_2}^*(e_2; y_1, y_2) H_{n_1, n_2}^*(e_0; y_1, y_2), \\ H_{n_1, n_2}^*(e_{0,2}; y_1, y_2) &= H_{n_1, n_2}^*(e_0; y_1, y_2) H_{n_1, n_2}^*(e_2; y_1, y_2), \end{aligned}$$

which proves Lemma (2.3). \square

Lemma 2.4. Let $\Psi_{i,j}^{y_1,y_2}(t,s) = \eta_{i,j}(t,s) = (t - y_1)^i(s - y_2)^j$, $i, j \in \{0, 1, 2\}$ be the central moments. Then from the operators $H_{n_1,n_2}^*(\cdot; \cdot)$ defined by (9) satisfies the following identities

$$\begin{aligned} H_{n_1,n_2}^*(\eta_{0,0}; y_1, y_2) &= 1 \\ H_{n_1,n_2}^*(\eta_{1,0}; y_1, y_2) &= \frac{y_1 + 1}{n_1 - 1}, \quad n_1 > 1, \\ H_{n_1,n_2}^*(\eta_{0,1}; y_1, y_2) &= \frac{y_2 + 1}{n_2 - 1}, \quad n_2 > 1, \\ H_{n_1,n_2}^*(\eta_{1,1}; y_1, y_2) &= \frac{y_1 + 1}{n_1 - 1} \frac{y_2 + 1}{n_2 - 1}, \quad n_1, n_2 > 1, \\ H_{n_1,n_2}^*(\eta_{2,0}; y_1, y_2) &= \frac{1}{(n_1 - 2)(n_1 - 1)} \left\{ (n_1 + 2)y_1^2 + \left(1 + \mu \frac{e_\mu(-n_1 y_1)}{e_\mu(n_1 y_1)} \right) 2n_1 y_1 + 2 \right\}, \quad n_1 > 2. \\ H_{n_1,n_2}^*(\eta_{0,2}; y_1, y_2) &= \frac{1}{(n_2 - 2)(n_2 - 1)} \left\{ (n_2 + 2)y_2^2 + \left(1 + \mu \frac{e_\mu(-n_2 y_2)}{e_\mu(n_2 y_2)} \right) 2n_2 y_2 + 2 \right\}, \quad n_2 > 2. \end{aligned}$$

Proof. Using Lemma (2.2) and linearly property, we have

$$\begin{aligned} H_{n_1,n_2}^*(\eta_{0,0}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_0; y_1, y_2) H_{n_1,n_2}^*(\eta_0; y_1, y_2), \\ H_{n_1,n_2}^*(\eta_{1,0}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_1; y_1, y_2) H_{n_1,n_2}^*(\eta_0; y_1, y_2), \\ H_{n_1,n_2}^*(\eta_{0,1}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_0; y_1, y_2) H_{n_1,n_2}^*(\eta_1; y_1, y_2), \\ H_{n_1,n_2}^*(\eta_{1,1}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_1; y_1, y_2) H_{n_1,n_2}^*(\eta_1; y_1, y_2), \\ H_{n_1,n_2}^*(\eta_{2,0}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_2; y_1, y_2) H_{n_1,n_2}^*(\eta_0; y_1, y_2), \\ H_{n_1,n_2}^*(\eta_{0,2}; y_1, y_2) &= H_{n_1,n_2}^*(\eta_0; y_1, y_2) H_{n_1,n_2}^*(\eta_2; y_1, y_2), \end{aligned}$$

which proves Lemma (2.4). \square

Lemma 2.5. For all $y_1, y_2 \in \mathcal{I}^2$ and sufficiently large $n_1, n_2 \in \mathbb{N}$ the operators $H_{n_1,n_2}^*(\cdot; \cdot)$ satisfy following

$$\begin{aligned} (1) \quad H_{n_1,n_2}^*(\Psi_{y_1,y_2}^{2,0}; y_1, y_2) &= o\left(\frac{1}{n_1}\right) (y_1 + 1)^2 \leq C_1 (y_1 + 1)^2 \text{ as } n_1, n_2 \rightarrow \infty; \\ (2) \quad H_{n_1,n_2}^*(\Psi_{y_1,y_2}^{0,2}; y_1, y_2) &= o\left(\frac{1}{n_2}\right) (y_2 + 1)^2 \leq C_2 (y_2 + 1)^2 \text{ as } n_1, n_2 \rightarrow \infty; \\ (3) \quad H_{n_1,n_2}^*(\Psi_{y_1,y_2}^{4,0}; y_1, y_2) &= o\left(\frac{1}{n_1^2}\right) (y_1 + 1)^4 \leq C_3 (y_1 + 1)^4 \text{ as } n_1, n_2 \rightarrow \infty; \\ (4) \quad H_{n_1,n_2}^*(\Psi_{y_1,y_2}^{0,4}; y_1, y_2) &= o\left(\frac{1}{n_2^2}\right) (y_2 + 1)^4 \leq C_4 (y_2 + 1)^4 \text{ as } n_1, n_2 \rightarrow \infty. \end{aligned}$$

3. Some approximation results in weighted space and their degree of convergence

Let φ be weight function such that $\varphi(y_1, y_2) = 1 + y_1^2 + y_2^2$ and satisfying $B_\varphi(\mathcal{I}^2) = \{g : |g(y_1, y_2)| \leq C_g \varphi(y_1, y_2), \quad C_g > 0\}$, where $B_\varphi(\mathcal{I}^2)$ is the set of all bounded function on $\mathcal{I}^2 = [0, 1] \times [0, 1]$. Suppose $C^{(m)}(\mathcal{I}^2)$ be the m -times continuously differentiable functions defined on $\mathcal{I}^2 = \{(y_1, y_2) \in \mathcal{I}^2 : y_1, y_2 \in [0, 1]\}$. The equipped norm on B_φ defined by $\|g\|_\varphi = \sup_{y_1, y_2 \in \mathcal{I}^2} \frac{|g(y_1, y_2)|}{\varphi(y_1, y_2)}$. Moreover we have classified here some classes of function as follows:

$$C_\varphi^m(\mathcal{I}^2) = \{g : g \in C_\varphi(\mathcal{I}^2); \text{ such that } \lim_{(y_1, y_2) \rightarrow \infty} \frac{g(y_1, y_2)}{\varphi(y_1, y_2)} = k_g < \infty\};$$

$$C_\varphi^0(\mathcal{I}^2) = \{f : f \in C_\varphi^m(\mathcal{I}^2); \text{ such that } \lim_{(y_1, y_2) \rightarrow \infty} \frac{g(y_1, y_2)}{\varphi(y_1, y_2)} = 0\}.$$

$$C_\varphi(\mathcal{I}^2) = \{g : g \in B_\varphi \cap C_\varphi(\mathcal{I}^2)\}.$$

Suppose $\omega_\varphi(g; \delta_1, \delta_2)$ is the weighted modulus of continuity for all $g \in C_\varphi^0(\mathcal{I}^2)$ and $\delta_1, \delta_2 > 0$, defined by

$$\omega_\varphi(g; \delta_1, \delta_2) = \sup_{(y_1, y_2) \in [0, 1]} \sup_{0 \leq |\theta_1| \leq \delta_1, 0 \leq |\theta_2| \leq \delta_2} \frac{|g(y_1 + \theta_1, y_2 + \theta_2) - g(y_1, y_2)|}{\varphi(y_1, y_2)\varphi(\theta_1, \theta_2)}. \tag{10}$$

For any $\eta_1, \eta_2 > 0$ one has

$$\omega_\varphi(g; \eta_1\delta_1, \eta_2\delta_2) \leq 4(1 + \eta_1)(1 + \eta_2)(1 + \delta_1^2)(1 + \delta_2^2)\omega_\varphi(g; \delta_1, \delta_2),$$

$$\begin{aligned} |g(t, s) - g(y_1, y_2)| &\leq \varphi(y_1, y_2)\varphi(|t - y_1|, |s - y_2|)\omega_\varphi(g; |t - y_1|, |s - y_2|) \\ &\leq (1 + y_1^2 + y_2^2)(1 + (t - y_1)^2)(1 + (s - y_2)^2)\omega_\varphi(g; |t - y_1|, |s - y_2|). \end{aligned}$$

Theorem 3.1. Let $g \in C_\varphi^0(\mathcal{I}^2)$, then for sufficiently large $n_1, n_2 \in \mathbb{N}$ operator H_{n_1, n_2}^* satisfying the inequality

$$\frac{|H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)|}{(1 + y_1^2 + y_2^2)} \leq \Psi_{y_1, y_2} \left(1 + o(n_1^{-1})\right) \left(1 + o(n_2^{-1})\right) \omega_\varphi\left(g; o\left(n_1^{-\frac{1}{2}}\right), o\left(n_2^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{y_1, y_2} = \left(1 + (y_1 + 1) + C_1(y_1 + 1)^2 + \sqrt{C_3}(y_1 + 1)^3\right) \left(1 + (y_2 + 1) + C_2(y_2 + 1)^2 + \sqrt{C_4}(y_2 + 1)^3\right)$ and $C_1, C_2, C_3, C_4 > 0$.

Proof. For all $\delta_{n_1}, \delta_{n_2} > 0$ we have

$$\begin{aligned} |g(t, s) - g(y_1, y_2)| &\leq 4(1 + y_1^2 + y_2^2) \left(1 + (t - y_1)^2\right) \left(1 + (s - y_2)^2\right) \\ &\times \left(1 + \frac{|t - y_1|}{\delta_{n_1}}\right) \left(1 + \frac{|s - y_2|}{\delta_{n_2}}\right) (1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\ &= 4(1 + y_1^2 + y_2^2)(1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2) \\ &\times \left(1 + \frac{|t - y_1|}{\delta_{n_1}} + (t - y_1)^2 + \frac{1}{\delta_{n_1}} |t - y_1| (t - y_1)^2\right) \\ &\times \left(1 + \frac{|s - y_2|}{\delta_{n_2}} + (s - y_2)^2 + \frac{|s - y_2|}{\delta_{n_2}} (s - y_2)^2\right) \omega_\varphi(g; \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

There for apply operator H_{n_1, n_2}^* and then use Cauchy-Schwarz inequality,

$$\begin{aligned}
 |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq H_{n_1, n_2}^*(|g(\cdot, \cdot) - g(y_1, y_2)|; y_1, y_2) 4(1 + y_1^2 + y_2^2) \\
 &\times H_{n_1, n_2}^*\left(1 + \frac{|t - y_1|}{\delta_{n_1}} + (t - y_1)^2 + \frac{1}{\delta_{n_1}} |t - y_1| (t - y_1)^2; y_1, y_2\right) \\
 &\times H_{n_1, n_2}^*\left(1 + \frac{|s - y_2|}{\delta_{n_2}} + (s - y_2)^2 + \frac{|s - y_2|}{\delta_{n_2}} (s - y_2)^2; y_1, y_2\right) \\
 &\times (1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\
 &= 4(1 + y_1^2 + y_2^2)(1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\
 &\times \left(1 + \frac{1}{\delta_{n_1}} H_{n_1, n_2}^*(|t - y_1|; y_1, y_2) + H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2)\right. \\
 &+ \left.\frac{1}{\delta_{n_1}} H_{n_1, n_2}^*(|t - y_1| (t - y_1)^2; y_1, y_2)\right) \\
 &\times \left(1 + \frac{1}{\delta_{n_2}} H_{n_1, n_2}^*(|s - y_2|; y_1, y_2) + H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2)\right. \\
 &+ \left.\frac{1}{\delta_{n_2}} H_{n_1, n_2}^*(|s - y_2| (s - y_2)^2; y_1, y_2)\right);
 \end{aligned}$$

$$\begin{aligned}
 |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq 4(1 + y_1^2 + y_2^2)(1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\
 &\times \left[1 + \frac{1}{\delta_{n_1}} \sqrt{H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2) + H_{n_1, n_2}^*((t - y_1)^4; y_1, y_2)}\right. \\
 &+ \left.\frac{1}{\delta_{n_1}} \sqrt{H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2)} \sqrt{H_{n_1, n_2}^*((t - y_1)^4; y_1, y_2)}\right] \\
 &\times \left[1 + \frac{1}{\delta_{n_2}} \sqrt{H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2) + H_{n_1, n_2}^*((s - y_2)^4; y_1, y_2)}\right. \\
 &+ \left.\frac{1}{\delta_{n_2}} \sqrt{H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2)} \sqrt{H_{n_1, n_2}^*((s - y_2)^4; y_1, y_2)}\right].
 \end{aligned}$$

In view of Lemma (2.5) and choose $\delta_{n_1} = o\left(n_1^{-\frac{1}{2}}\right)$ and $\delta_{n_2} = o\left(n_2^{-\frac{1}{2}}\right)$, then

$$\begin{aligned}
 |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq 4(1 + y_1^2 + y_2^2)(1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\
 &\times \left[1 + \frac{1}{\delta_{n_1}} \sqrt{o\left(\frac{1}{n_1}\right)(y_1 + 1)^2 + o\left(\frac{1}{n_1}\right)(y_1 + 1)^2}\right. \\
 &+ \left.\frac{1}{\delta_{n_1}} \sqrt{o\left(\frac{1}{n_1}\right)(y_1 + 1)^2} \sqrt{o\left(\frac{1}{n_1}\right)(y_1 + 1)^4}\right] \\
 &\times \left[1 + \frac{1}{\delta_{n_2}} \sqrt{o\left(\frac{1}{n_2}\right)(y_2 + 1)^2 + o\left(\frac{1}{n_2}\right)(y_2 + 1)^2}\right. \\
 &+ \left.\frac{1}{\delta_{n_2}} \sqrt{o\left(\frac{1}{n_2}\right)(y_2 + 1)^2} \sqrt{o\left(\frac{1}{n_2}\right)(y_2 + 1)^4}\right] \\
 &\leq 4(1 + y_1^2 + y_2^2)(1 + \delta_{n_1}^2)(1 + \delta_{n_2}^2)\omega_\varphi(g; \delta_{n_1}, \delta_{n_2}) \\
 &\times \left[1 + (y_1 + 1) + C_1(y_1 + 1)^2 + \sqrt{C_2}(y_1 + 1)^3\right] \left[1 + (y_2 + 1)\right. \\
 &+ \left.C_3(y_2 + 1)^2 + \sqrt{C_4}(y_2 + 1)^3\right].
 \end{aligned}$$

Which completes the proof. \square

Lemma 3.2 ([43, 44]). For the positive sequence of operators $\{L_{n_1, n_2}\}_{n_1, n_2 \geq 1}$, which acting $C_\varphi \rightarrow B_\varphi$ defined earlier then there exists some positive K such that

$$\|L_{n_1, n_2}(\varphi)\|_\varphi \leq K.$$

Theorem 3.3 ([43, 44]). or the positive sequence of operators $\{L_{n_1, n_2}\}_{n_1, n_2 \geq 1}$ acting $C_\varphi \rightarrow B_\varphi$ defined earlier satisfying the following conditions

- (1) $\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}(1) - 1\|_\varphi = 0;$
- (2) $\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}(t) - y_1\|_\varphi = 0;$
- (3) $\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}(s) - y_2\|_\varphi = 0;$
- (4) $\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}((t^2 + s^2)) - (y_1^2 + y_2^2)\|_\varphi = 0.$

Then for all $g \in C_\varphi^0$,

$$\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}g - g\|_\varphi = 0$$

and there exists another function $f \in C_\varphi \setminus C_\varphi^0$, such that

$$\lim_{n_1, n_2 \rightarrow \infty} \|L_{n_1, n_2}f - f\|_\varphi \geq 1.$$

Theorem 3.4. If $g \in C_\varphi^0(\mathcal{I}^2)$, then we have

$$\lim_{n_1, n_2 \rightarrow \infty} \|H_{n_1, n_2}^*(g) - g\|_\varphi = 0.$$

Proof.

$$\begin{aligned} \|H_{n_1, n_2}^*(\varphi; y_1, y_2)\|_\varphi &= \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{|H_{n_1, n_2}^*(1 + y_1^2 + y_2^2; y_1, y_2)|}{1 + y_1^2 + y_2^2} \\ &= 1 + \sup_{(y_1, y_2) \in \mathcal{I}^2} \left[\frac{1}{1 + y_1^2 + y_2^2} \left| \left(1 + H_{n_1, n_2}^*(y_1^2; y_1, y_2) + H_{n_1, n_2}^*(y_2^2; y_1, y_2) \right) \right| \right] \\ \|H_{n_1, n_2}^*(\varphi; y_1, y_2)\|_\varphi &= 1 + \left| \frac{n_1^2}{(n_1 - 2)(n_1 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{y_1^2}{1 + y_1^2 + y_2^2} \\ &\quad + \left| \left(4 + 2\mu \frac{e_\mu(-n_1 y_1)}{e_\mu(n_1 y_1)} \right) \frac{n_1}{(n_1 - 2)(n_1 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{y_1}{1 + y_1^2 + y_2^2} \\ &\quad + \left| \frac{2}{(n_1 - 2)(n_1 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{1}{1 + y_1^2 + y_2^2} \\ &\quad + \left| \frac{n_2^2}{(n_2 - 2)(n_2 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{y_2^2}{1 + y_1^2 + y_2^2} \\ &\quad + \left| \left(4 + 2\mu \frac{e_\mu(-n_2 y_2)}{e_\mu(n_2 y_2)} \right) \frac{n_2}{(n_2 - 2)(n_2 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{y_2}{1 + y_1^2 + y_2^2} \\ &\quad + \left| \frac{2}{(n_2 - 2)(n_2 - 1)} \right| \sup_{(y_1, y_2) \in \mathcal{I}^2} \frac{1}{1 + y_1^2 + y_2^2} \end{aligned}$$

$$\begin{aligned} \| H_{n_1, n_2}^*(\varphi; y_1, y_2) \|_\varphi &\leq 1 + \left| \frac{n_1^2}{(n_1 - 2)(n_1 - 1)} \right| + \left| \left(4 + 2\mu \frac{e_\mu(-n_1 y_1)}{e_\mu(n_1 y_1)} \right) \frac{n_1}{(n_1 - 2)(n_1 - 1)} \right| \\ &+ \left| \frac{2}{(n_1 - 2)(n_1 - 1)} \right| + \left| \frac{n_2^2}{(n_2 - 2)(n_2 - 1)} \right| \\ &+ \left| \left(4 + 2\mu \frac{e_\mu(-n_2 y_2)}{e_\mu(n_2 y_2)} \right) \frac{n_2}{(n_2 - 2)(n_2 - 1)} \right| + \left| \frac{2}{(n_2 - 2)(n_2 - 1)} \right|. \end{aligned}$$

Now for all $n_1, n_2 \in \mathbb{N} \setminus \{1, 2\}$, there exists a positive constant K such that

$$\| H_{n_1, n_2}^*(\varphi; y_1, y_2) \|_\varphi \leq K.$$

Therefore, in order to prove Theorem 3.4 it is sufficient from the Lemma 3.2 and Theorem 3.3. Thus we arrive at the prove of Theorem 3.4. \square

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega(g; \delta_{n_1}, \delta_{n_2}) = \sup \{ |g(t, s) - g(y_1, y_2)| : (t, s), (y_1, y_2) \in \mathcal{I}^2 \}$$

with $|t - y_1| \leq \delta_{n_1}$, $|s - y_2| \leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1(g; \delta) = \sup_{0 \leq y_2 \leq 1} \sup_{|x_1 - x_2| \leq \delta} \{ |g(x_1, y_2) - g(x_2, y_2)| \},$$

$$\omega_2(g; \delta) = \sup_{0 \leq y_1 \leq 1} \sup_{|y_1 - y_2| \leq \delta} \{ |g(y_1, y_1) - g(y_1, y_2)| \}.$$

Theorem 3.5. For any $g \in C(\mathcal{I}^2)$ we have

$$| H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2) | \leq 2 \left(\omega_1(g; \delta_{y_1, n_1}) + \omega_2(g; \delta_{n_2, y_2}) \right).$$

Proof. In order to give the prove of Theorem 3.5, in general we use well-known Cauchy-Schwarz inequality. Thus we see that

$$\begin{aligned} | H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2) | &\leq H_{n_1, n_2}^*(|g(t, s) - g(y_1, y_2)| ; y_1, y_2) \\ &\leq H_{n_1, n_2}^*(|g(t, s) - g(y_1, s)| ; y_1, y_2) \\ &+ H_{n_1, n_2}^*(|g(y_1, s) - g(y_1, y_2)| ; y_1, y_2) \\ &\leq H_{n_1, n_2}^*(\omega_1(g; |t - y_1|) ; y_1, y_2) + H_{n_1, n_2}^*(\omega_2(g; |s - y_2|) ; y_1, y_2) \\ &\leq \omega_1(g; \delta_{n_1}) \left(1 + \delta_{n_1}^{-1} H_{n_1, n_2}^*(|t - y_1| ; y_1, y_2) \right) \\ &+ \omega_2(g; \delta_{n_2}) \left(1 + \delta_{n_2}^{-1} H_{n_1, n_2}^*(|s - y_2| ; y_1, y_2) \right) \\ &\leq \omega_1(g; \delta_{n_1}) \left(1 + \frac{1}{\delta_{n_1}} \sqrt{H_{n_1, n_2}^*((t - y_1)^2 ; y_1, y_2) } \right) \\ &+ \omega_2(g; \delta_{n_2}) \left(1 + \frac{1}{\delta_{n_2}} \sqrt{H_{n_1, n_2}^*((s - y_2)^2 ; y_1, y_2) } \right). \end{aligned}$$

If we choose $\delta_{n_1}^2 = \delta_{n_1, y_1}^2 = H_{n_1, n_2}^*((t - y_1)^2 ; y_1, y_2)$ and $\delta_{n_2}^2 = \delta_{n_2, y_2}^2 = H_{n_1, n_2}^*((s - y_2)^2 ; y_1, y_2)$, then we easily to reach our desired results. \square

Here we find convergence in terms of the Lipschitz class for bivariate function. For $M > 0$ and $\rho_1, \rho_2 \in [0, 1]$, Lipschitz maximal function space on $E \times E \subset \mathcal{I}^2$ defined by

$$\begin{aligned} \mathcal{L}_{\rho_1, \rho_2}(E \times E) &= \left\{ g : \sup(1+t)^{\rho_1}(1+s)^{\rho_2} (g_{\rho_1, \rho_2}(t, s) - g_{\rho_1, \rho_2}(y_1, y_2)) \right. \\ &\leq \left. M \frac{1}{(1+y_1)^{\rho_1}} \frac{1}{(1+y_2)^{\rho_2}} \right\}, \end{aligned}$$

where g is continuous and bounded on \mathcal{I}^2 , and

$$g_{\rho_1, \rho_2}(t, s) - g_{\rho_1, \rho_2}(y_1, y_2) = \frac{|g(t, s) - g(y_1, y_2)|}{|t - y_1|^{\rho_1} |s - y_2|^{\rho_2}}; \quad (t, s), (y_1, y_2) \in \mathcal{I}^2. \tag{11}$$

Theorem 3.6. Let $g \in \mathcal{L}_{\rho_1, \rho_2}(E \times E)$, then for any $\rho_1, \rho_2 \in [0, 1]$, there exists $M > 0$ such that

$$\begin{aligned} |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq M \left\{ \left((d(y_1, E))^{\rho_1} + (\delta_{n_1, y_1}^2)^{\frac{\rho_1}{2}} \right) \left((d(y_2, E))^{\rho_2} + (\delta_{n_2, y_2}^2)^{\frac{\rho_2}{2}} \right) \right. \\ &\quad \left. + (d(y_1, E))^{\rho_1} (d(y_2, E))^{\rho_2} \right\}, \end{aligned}$$

where δ_{n_1, y_1} and δ_{n_2, y_2} defined by Theorem 3.5.

Proof. Take $|y_1 - x_0| = d(y_1, E)$ and $|y_2 - y_0| = d(y_2, E)$. For any $(y_1, y_2) \in \mathcal{I}^2$, and $(x_0, y_0) \in E \times E$ we let $d(y_1, E) = \inf\{|y_1 - y_2| : y_2 \in E\}$. Thus we can write here

$$|g(t, s) - g(y_1, y_2)| \leq M |g(t, s) - g(x_0, y_0)| + |g(x_0, y_0) - g(y_1, y_2)|. \tag{12}$$

Apply H_{n_1, n_2}^* , we obtain

$$\begin{aligned} |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq H_{n_1, n_2}^*(|g(y_1, y_2) - g(x_0, y_0)| + |g(x_0, y_0) - g(y_1, y_2)|) \\ &\leq MH_{n_1, n_2}^*(|t - x_0|^{\rho_1} |s - y_0|^{\rho_2}; y_1, y_2) \\ &\quad + M |y_1 - x_0|^{\rho_1} |y_2 - y_0|^{\rho_2}. \end{aligned}$$

For all $A, B \geq 0$ and $\rho \in [0, 1]$ we know inequality $(A + B)^\rho \leq A^\rho + B^\rho$, thus

$$|t - x_0|^{\rho_1} \leq |t - y_1|^{\rho_1} + |y_1 - x_0|^{\rho_1},$$

$$|s - y_0|^{\rho_2} \leq |s - y_2|^{\rho_2} + |y_2 - y_0|^{\rho_2}.$$

Therefore

$$\begin{aligned} |H_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq MH_{n_1, n_2}^*(|t - y_1|^{\rho_1} |s - y_2|^{\rho_2}; y_1, y_2) \\ &\quad + M |y_1 - x_0|^{\rho_1} H_{n_1, n_2}^*(|s - y_2|^{\rho_2}; y_1, y_2) \\ &\quad + M |y_2 - y_0|^{\rho_2} H_{n_1, n_2}^*(|t - y_1|^{\rho_1}; y_1, y_2) \\ &\quad + 2M |y_1 - x_0|^{\rho_1} |y_2 - y_0|^{\rho_2} H_{n_1, n_2}^*(\mu_{0,0}; y_1, y_2). \end{aligned}$$

On apply Hölder inequality on H_{n_1, n_2}^* , we get

$$\begin{aligned} H_{n_1, n_2}^* (|t - y_1|^{\rho_1} |s - y_2|^{\rho_2}; y_1, y_2) &= \mathcal{U}_{n_1, k}^{\alpha_1} (|t - y_1|^{\rho_1}; y_1, y_2) \mathcal{V}_{n_2, l}^{\alpha_2} (|s - y_2|^{\rho_2}; y_1, y_2) \\ &\leq \left(H_{n_1, n_2}^* (|t - y_1|^2; y_1, y_2) \right)^{\frac{\rho_1}{2}} \left(H_{n_1, n_2}^* (\mu_{0,0}; y_1, y_2) \right)^{\frac{2-\rho_1}{2}} \\ &\times \left(H_{n_1, n_2}^* (|s - y_2|^2; y_1, y_2) \right)^{\frac{\rho_2}{2}} \left(H_{n_1, n_2}^* (\mu_{0,0}; y_1, y_2) \right)^{\frac{2-\rho_2}{2}}. \end{aligned}$$

Thus we can obtain

$$\begin{aligned} |H_{n_1, n_2}^* (g; y_1, y_2) - g(y_1, y_2)| &\leq M \left(\delta_{n_1, y_1}^2 \right)^{\frac{\rho_1}{2}} \left(\delta_{n_2, y_2}^2 \right)^{\frac{\rho_2}{2}} + 2M (d(y_1, E))^{\rho_1} (d(y_2, E))^{\rho_2} \\ &+ M (d(y_1, E))^{\rho_1} \left(\delta_{n_2, y_2}^2 \right)^{\frac{\rho_2}{2}} + L (d(y_2, E))^{\rho_2} \left(\delta_{n_1, y_1}^2 \right)^{\frac{\rho_1}{2}}. \end{aligned}$$

We have complete the proof. \square

Theorem 3.7. *If $g \in C(I^2)$, then for all $(y_1, y_2) \in I^2$, operator H_{n_1, n_2}^* satisfying*

$$|H_{n_1, n_2}^* (g; y_1, y_2) - g(y_1, y_2)| \leq \|g_{y_1}\|_{C(I^2)} \left(\delta_{n_1, y_1}^2 \right)^{\frac{1}{2}} + \|g_{y_2}\|_{C(I^2)} \left(\delta_{n_2, y_2}^2 \right)^{\frac{1}{2}},$$

where δ_{n_1, y_1} and δ_{n_2, y_2} are defined by Theorem 3.5.

Proof. Let $g \in C(I^2)$, and for any fixed $(y_1, y_2) \in I^2$ we have

$$g(t, s) - g(y_1, y_2) = \int_{y_1}^t g_\varrho(\varrho, s) d\varrho + \int_{y_2}^s g_\mu(y_1, \mu) d\mu.$$

On apply H_{n_1, n_2}^*

$$H_{n_1, n_2}^* (g(t, s); y_1, y_2) - g(y_1, y_2) = H_{n_1, n_2}^* \left(\int_{y_1}^t g_\varrho(\varrho, s) d\varrho; y_1, y_2 \right) + H_{n_1, n_2}^* \left(\int_{y_2}^s g_\mu(y_1, \mu) d\mu; y_1, y_2 \right). \tag{13}$$

From the sup-norm on I^2 we can see that

$$\left| \int_{y_1}^t g_\varrho(\varrho, s) d\varrho \right| \leq \int_{y_1}^t |g_\varrho(\varrho, s)| d\varrho \leq \|g_{y_1}\|_{C(I^2)} |t - y_1| \tag{14}$$

$$\left| \int_{y_2}^s g_\mu(y_1, \mu) d\mu \right| \leq \int_{y_2}^s |g_\mu(y_1, \mu)| d\mu \leq \|g_{y_2}\|_{C(I^2)} |s - y_2|. \tag{15}$$

In the view of (13), (14) and (15) we can obtain

$$\begin{aligned} |H_{n_1, n_2}^* (g(y_1, y_2); y_1, y_2) - g(y_1, y_2)| &\leq H_{n_1, n_2}^* \left(\left| \int_{y_1}^t g_\varrho(\varrho, s) d\varrho \right|; y_1, y_2 \right) \\ &+ H_{n_1, n_2}^* \left(\left| \int_{y_2}^s g_\mu(y_1, \mu) d\mu \right|; y_1, y_2 \right) \\ &\leq \|g_{y_1}\|_{C(I^2)} H_{n_1, n_2}^* (|t - y_1|; y_1, y_2) \\ &+ \|g_{y_2}\|_{C(I^2)} H_{n_1, n_2}^* (|s - y_2|; y_1, y_2) \\ &\leq \|g_{y_1}\|_{C(I^2)} \left(H_{n_1, n_2}^* ((t - y_1)^2; y_1, y_2) H_{n_1, n_2}^* (1; y_1, y_2) \right)^{\frac{1}{2}} \\ &+ \|g_{y_2}\|_{C(I^2)} \left(H_{n_1, n_2}^* ((s - y_2)^2; y_1, y_2) H_{n_1, n_2}^* (1; y_1, y_2) \right)^{\frac{1}{2}} \\ &= \|g_{y_1}\|_{C(I^2)} \left(\delta_{n_1, y_1}^2 \right)^{\frac{1}{2}} + \|g_{y_2}\|_{C(I^2)} \left(\delta_{n_2, y_2}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3.8. For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that

$$R_{n_1, n_2}^{\alpha_1, \alpha_2}(f; y_1, y_2) = H_{n_1, n_2}^*(g; y_1, y_2) + f(y_1, y_2) - f\left(\mathcal{U}_{n_1, k}^{\alpha_1}(\mu_{1,0}; y_1, y_2), \mathcal{V}_{n_2, l}^{\alpha_2}(\mu_{0,1}; y_1, y_2)\right).$$

where, from Lemma (2.4), $\mathcal{U}_{n_1, k}^{\alpha_1}(\mu_{1,0}; y_1, y_2) = \frac{y_1+1}{n_1-1}$ $n_1 > 1$ and

$$\mathcal{V}_{n_2, l}^{\alpha_2}(\mu_{0,1}; y_1, y_2) = \frac{y_2+1}{n_2-1} \quad n_2 > 1.$$

Then for all $g \in C'(\mathcal{I}^2)$, $R_{n_1, n_2}^{\alpha_1, \alpha_2}$ satisfying

$$R_{n_1, n_2}^{\alpha_1, \alpha_2}(g; t, s) - g(y_1, y_2) \leq \left\{ \delta_{n_1, y_1}^2 + \delta_{n_2, y_2}^2 + \left(\frac{y_1+1}{n_1-1} - y_1\right)^2 + \left(\frac{y_2+1}{n_2-1} - y_2\right)^2 \right\} \|g\|_{C^2(\mathcal{I}^2)}.$$

Proof. In the light of operator $R_{n_1, n_2}^{\alpha_1, \alpha_2}(f; y_1, y_2)$ and Lemma 2.2, we obtain $R_{n_1, n_2}^{\alpha_1, \alpha_2}(1; y_1, y_2) = 1$, $R_{n_1, n_2}^{\alpha_1, \alpha_2}(t - y_1; y_1, y_2) = 0$ and $R_{n_1, n_2}^{\alpha_1, \alpha_2}(s - y_2; y_1, y_2) = 0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$g(t, s) - g(y_1, y_2) = \frac{\partial g(y_1, y_2)}{\partial y_1}(t - y_1) + \int_{y_1}^t (t - \lambda) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda + \frac{\partial g(y_1, y_2)}{\partial y_2}(s - y_2) + \int_{y_2}^s (s - \psi) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi.$$

On apply $R_{n_1, n_2}^{\alpha_1, \alpha_2}$, we see that

$$\begin{aligned} &R_{n_1, n_2}^{\alpha_1, \alpha_2}(g(t, s); y_1, y_2) - R_{n_1, n_2}^{\alpha_1, \alpha_2}(g(y_1, y_2)) \\ &= R_{n_1, n_2}^{\alpha_1, \alpha_2}\left(\int_{y_1}^t (t - \lambda) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda; y_1, y_2\right) + R_{n_1, n_2}^{\alpha_1, \alpha_2}\left(\int_{y_2}^s (s - \psi) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi; y_1, y_2\right) \\ &= H_{n_1, n_2}^*\left(\int_{y_1}^t (t - \lambda) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda; y_1, y_2\right) + H_{n_1, n_2}^*\left(\int_{y_2}^s (s - \psi) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi; y_1, y_2\right) \\ &\quad - \int_{y_1}^{\frac{y_1+1}{n_1-1}} \left(\frac{y_1+1}{n_1-1} - \lambda\right) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda \\ &\quad - \int_{y_2}^{\frac{y_2+1}{n_2-1}} \left(\frac{y_2+1}{n_2-1} - \psi\right) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi. \end{aligned}$$

From hypothesis we easily obtain

$$\left| \int_{y_1}^t (t - \lambda) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda \right| \leq \int_{y_1}^t \left| (t - \lambda) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda \right| \leq \|g\|_{C^2(\mathcal{I}^2)} (t - y_1)^2,$$

$$\left| \int_{y_2}^s (s - \psi) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi \right| \leq \int_{y_2}^s \left| (s - \psi) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi \right| \leq \|g\|_{C^2(\mathcal{I}^2)} (s - y_2)^2,$$

$$\begin{aligned} & \left| \int_{y_1}^{\frac{y_1+1}{n_1-1}} \left(\frac{y_1+1}{n_1-1} - \lambda \right) \frac{\partial^2 g(\lambda, y_2)}{\partial \lambda^2} d\lambda \right| \\ & \leq \|g\|_{C^2(I^2)} \left(\frac{y_1+1}{n_1-1} - y_1 \right)^2 \\ & \left| \int_{y_2}^{\frac{y_2+1}{n_2-1}} \left(\frac{y_2+1}{n_2-1} - \psi \right) \frac{\partial^2 g(y_1, \psi)}{\partial \psi^2} d\psi \right| \\ & \leq \|g\|_{C^2(I^2)} \left(\frac{y_2+1}{n_2-1} - y_2 \right)^2. \end{aligned}$$

Thus,

$$\begin{aligned} |R_{n_1, n_2}^{\alpha_1, \alpha_2}(g; t, s) - g(y_1, y_2)| & \leq \left\{ H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2) + H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2) \right. \\ & \quad + \left(\frac{y_1+1}{n_1-1} - y_1 \right)^2 \\ & \quad \left. + \left(\frac{y_2+1}{n_2-1} - y_2 \right)^2 \right\} \|g\|_{C^2(I^2)}. \end{aligned}$$

We complete the proof of desired Theorem 3.8.

□

4. Some approximation results in Bögel space

Take any function $g : I_1 \times I_2 \rightarrow \mathbb{R}$ for a real compact intervals of $I_1 \times I_2$. For all $(t, s), (y_1, y_2) \in I_1 \times I_2$ suppose $\Delta_{(t,s)}^* g(y_1, y_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta_{(t,s)}^* g(y_1, y_2) = g(t, s) - g(t, y_2) - g(y_1, s) + g(y_1, y_2).$$

If at any point $(y_1, y_2) \in I_1 \times I_2$ the function $g : I_1 \times I_2 \rightarrow \mathbb{R}$ defined on $I_1 \times I_2$, then $\lim_{(t,s) \rightarrow (y_1, y_2)} \Delta_{(t,s)}^* g(y_1, y_2) = 0$.

If set of all the space of all Bögel-continuous (B -continuous) denoted by $C_B(I_1 \times I_2)$ on $(y_1, y_2) \in I_1 \times I_2$ and be defined such that $C_B(I_1 \times I_2) = \{g, \text{ such that } g : I_1 \times I_2 \rightarrow \mathbb{R} \text{ is } g, B\text{-bounded on } I_1 \times I_2\}$. Next, the Bögel-differentiable function on $(y_1, y_2) \in I_1 \times I_2$ be $g : I_1 \times I_2 \rightarrow \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s) \rightarrow (y_1, y_2), t \neq y_1, s \neq y_2} \frac{1}{(t - y_1)(s - y_2)} \left(\Delta_{(t,s)}^* \right) = D_B g(y_1, y_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(y_1, y_2)$ and be $D_\varphi(I_1 \times I_2) = \{g, \text{ such that } g : I_1 \times I_2 \rightarrow \mathbb{R} \text{ is } g, B\text{-differentiable on } I_1 \times I_2\}$. Suppose the function g is B -bounded on D and be $g : I_1 \times I_2 \rightarrow \mathbb{R}$, then for all $(t, s), (y_1, y_2) \in I_1 \times I_2$ there exists positive constant M such that $|\Delta_{(t,s)}^* g(y_1, y_2)| \leq M$. The classes of all B -continuous function is called a B -bounded on $I_1 \times I_2$, where $I_1 \times I_2$ is compact subset. Let $B_\varphi(I_1 \times I_2)$ denote the classes of all B -bounded function defined on $I_1 \times I_2$ which equipped the norm on B as $\|g\|_B = \sup_{(t,s), (y_1, y_2) \in I_1 \times I_2} |\Delta_{(t,s)}^* g(y_1, y_2)|$. As we know to approximate the degree for a set of all B -continuous function on positive linear operators, it is essential to use the properties

of mixed-modulus of continuity. So we let for all $(t, s), (y_1, y_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ and be defined such as:

$$\omega_B(g; \delta_1, \delta_2) = \sup\{\Delta_{(t,s)}^* g(y_1, y_2) : |t - y_1| \leq \delta_1, |s - y_2| \leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0, 1] \times [0, 1]$, we suppose the classes of all B -continuous function defined on \mathcal{I}^2 denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on \mathcal{I}^2 be $C(\mathcal{I}^2)$. For further details on space of Bögél functions related to this paper we propose the article [40, 41].

Let $(y_1, y_2) \in \mathcal{I}^2$ and $n_1, n_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators H_{n_1, n_2}^* . Thus we suppose

$$K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(t, s); y_1, y_2) = H_{n_1, n_2}^* \left(g(t, y_2) + g(y_1, s) - g(t, s); y_1, y_2 \right). \tag{16}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(t, s); y_1, y_2) = \sum_{k, l=0}^{\infty} Q_1^*(n_1, y_1) Q_2^*(n_2, y_2) \int_0^{\infty} \int_0^{\infty} \mathcal{P}_1^*(n_1, y_1) P_2^*(n_2, y_2) \tag{17}$$

where $P_{y_1, y_2}(t, s) = (g(t, y_2) + g(y_1, s) - g(t, s))$.

Theorem 4.1. For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that

$$| K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(t, s); y_1, y_2) - g(y_1, y_2) | \leq 4\omega_B(g; \delta_{n_1, y_1}, \delta_{n_2, y_2}),$$

where δ_{n_1, y_1} and δ_{n_2, y_2} are defined by Theorem 3.5.

Proof. Let $(t, s), (y_1, y_2) \in \mathcal{I}^2$. For all $n_1, n_2 \in \mathbb{N}$ and $\delta_{n_1}, \delta_{n_2} > 0$, it follows that

$$\begin{aligned} | \Delta_{(y_1, y_2)}^* g(t, s) | &\leq \omega_B(g; |t - y_1|, |s - y_2|) \\ &\leq \left(1 + \frac{t - y_1}{\delta_{n_1}} \right) \left(1 + \frac{s - y_2}{\delta_{n_2}} \right) \omega_B(g; \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

From (16) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned} | K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(t, s); y_1, y_2) - g(y_1, y_2) | &\leq H_{n_1, n_2}^* \left(| \Delta_{(y_1, y_2)}^* g(t, s) |; y_1, y_2 \right) \\ &\leq \left(H_{n_1, n_2}^*(\phi_{0,0}; y_1, y_2) + \frac{1}{\delta_{n_1}} \left(H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2) \right)^{\frac{1}{2}} \right. \\ &\quad + \frac{1}{\delta_{n_2}} \left(H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2) \right)^{\frac{1}{2}} \\ &\quad + \frac{1}{\delta_{n_1}} \left(H_{n_1, n_2}^*((t - y_1)^2; y_1, y_2) \right)^{\frac{1}{2}} \\ &\quad \times \left. \frac{1}{\delta_{n_2}} \left(H_{n_1, n_2}^*((s - y_2)^2; y_1, y_2) \right)^{\frac{1}{2}} \right) \omega_B(g; \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

In the view of Theorem 3.5 we easily get our results.

□

If we let $x = (t, s)$, $y = (y_1, y_2) \in \mathcal{I}^2$, then the Lipschitz function in terms of B -continuous functions defined by

$$Lip_M^\xi = \left\{ g \in C(\mathcal{I}^2) : |\Delta_{(y_1, y_2)}^* g(x, y)| \leq M \|x - y\|^\xi, \right\}$$

where M is a positive constant, $0 < \xi \leq 1$, and Euclidean norm $\|x - y\| = \sqrt{(t - y_1)^2 + (s - y_2)^2}$.

Theorem 4.2. For all $g \in Lip_M^\xi$ operator $K_{n_1, n_2}^{\alpha_1, \alpha_2}$ satisfying

$$|K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(x, y); y_1, y_2) - g(y_1, y_2)| \leq M \{\delta_{n_1, y_1}^2 + \delta_{n_2, y_2}^2\}^{\frac{\xi}{2}},$$

where δ_{n_1, y_1} and δ_{n_2, y_2} are defined by Theorem 3.5.

Proof. We easily see that

$$\begin{aligned} K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(x, y); y_1, y_2) &= H_{n_1, n_2}^* (g(y_1, y) + g(x, y_2) - g(x, s); y_1, y_2) \\ &= H_{n_1, n_2}^* (g(y_1, y_2) - \Delta_{(y_1, y_2)}^* g(x, s); y_1, y_2) \\ &= g(y_1, y_2) - H_{n_1, n_2}^* (\Delta_{(y_1, y_2)}^* g(x, s); y_1, y_2). \end{aligned}$$

Therefore,

$$\begin{aligned} |K_{n_1, n_2}^{\alpha_1, \alpha_2}(g(x, y); y_1, y_2) - g(y_1, y_2)| &\leq H_{n_1, n_2}^* (|\Delta_{(y_1, y_2)}^* g(x, y)|; y_1, y_2) \\ &\leq M H_{n_1, n_2}^* (\|x - y\|^\xi; y_1, y_2) \\ &\leq M \{H_{n_1, n_2}^* (\|x - y\|^2; y_1, y_2)\}^{\frac{\xi}{2}} \\ &\leq M \{H_{n_1, n_2}^* ((t - y_1)^2; y_1, y_2) + H_{n_1, n_2}^* ((s - y_2)^2; y_1, y_2)\}^{\frac{\xi}{2}}. \end{aligned}$$

□

Theorem 4.3. If $g \in D_\varphi(\mathcal{I}^2)$ and $D_B g \in B(\mathcal{I}^2)$, then

$$\begin{aligned} |K_{n_1, n_2}^{\alpha_1, \alpha_2}(g; y_1, y_2) - g(y_1, y_2)| &\leq C \left\{ 3 \|D_B g\|_\infty + \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}) \right\} (y_1 + 1)(y_2 + 1) \\ &\quad + \left\{ 1 + \sqrt{C_2}(y_1 + 1) + \sqrt{C_1}(y_2 + 1) \right\} \\ &\quad \times \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})(y_1 + 1)(y_2 + 1), \end{aligned}$$

where $\delta_{n_1}, \delta_{n_2}$ defined by Theorem 3.5 and C is any positive constant.

Proof. Suppose $\rho \in (y_1, t)$, $\xi \in (y_2, s)$ and

$$\Delta_{(y_1, y_2)}^* g(t, s) = (t - y_1)(s - y_2) D_B g(\rho, \xi),$$

$$D_B g(\rho, \xi) = \Delta_{(y_1, y_2)}^* D_B g(\rho, \xi) + D_B g(\rho, y) + D_B g(x, \xi) - D_B g(y_1, y_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned} |K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Delta_{(y_1, y_2)}^* g(t, s); y_1, y_2)| &= |K_{n_1, n_2}^{\alpha_1, \alpha_2}((t - y_1)(s - y_2)D_B g(\rho, \xi); y_1, y_2)| \\ &\leq K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| |s - y_2| \|\Delta_{(y_1, y_2)}^* D_B g(\rho, \xi)\|; y_1, y_2) \\ &+ K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| |s - y_2| (|D_B g(\rho, y_2)| \\ &+ |D_B g(y_1, \xi)| + |D_B g(y_1, y_2)|); y_1, y_2) \\ &\leq K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| |s - y_2| \\ &\times \omega_{mixed}(D_B g; |\rho - y_1|, |\xi - y_2|); y_1, y_2) \\ &+ 3 \|D_B g\|_\infty K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| |s - y_2|; y_1, y_2). \end{aligned}$$

Here ω_{mixed} , is mixed-modulus of continuity and it follows that

$$\begin{aligned} \omega_{mixed}(D_B g; |\rho - y_1|, |\xi - y_2|) &\leq \omega_{mixed}(D_B g; |t - y_1|, |s - y_2|) \\ &\leq (1 + \delta_{n_1}^{-1} |t - y_1|)(1 + \delta_{n_2}^{-1} |s - y_2|) \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned} |K_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &= |\Delta_{(y_1, y_2)}^* g(t, s); y_1, y_2| \\ &\leq 3 \|D_B g\|_\infty \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}((t - y_1)^2(s - y_2)^2; y_1, y_2)\right)^{\frac{1}{2}} \\ &+ \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| |s - y_2|; y_1, y_2) \right. \\ &+ \delta_{n_1}^{-1} K_{n_1, n_2}^{\alpha_1, \alpha_2}((t - y_1)^2 |s - y_2|; y_1, y_2) \\ &+ \delta_{n_2}^{-1} K_{n_1, n_2}^{\alpha_1, \alpha_2}(|t - y_1| (s - y_2)^2; y_1, y_2) \\ &+ \left. \delta_{n_1}^{-1} \delta_{n_2}^{-1} K_{n_1, n_2}^{\alpha_1, \alpha_2}((t - y_1)^2 (s - y_2)^2; y_1, y_2) \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})\right); \end{aligned}$$

$$\begin{aligned} |K_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &= |\Delta_{(y_1, y_2)}^* g(t, s); y_1, y_2| \\ &\leq 3 \|D_B g\|_\infty \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,2}; y_1, y_2)\right)^{\frac{1}{2}} \\ &+ \left\{ \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,2}; y_1, y_2)\right)^{\frac{1}{2}} \right. \\ &+ \delta_{n_1}^{-1} \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{4,2}; y_1, y_2)\right)^{\frac{1}{2}} + \delta_{n_2}^{-1} \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,4}; y_1, y_2)\right)^{\frac{1}{2}} \\ &+ \left. \delta_{n_1}^{-1} \delta_{n_2}^{-1} K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,2}; y_1, y_2) \right\} \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}). \end{aligned}$$

Which follows that

$$\begin{aligned}
 |K_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &= 3 \|D_B g\|_\infty \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,0}; y_1, y_2) K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{0,2}; y_1, y_2) \right)^{\frac{1}{2}} \\
 &+ \left\{ \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,0}; y_1, y_2) K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{0,2}; y_1, y_2) \right)^{\frac{1}{2}} \right. \\
 &+ \delta_{n_1}^{-1} \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{4,0}; y_1, y_2) K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{0,2}; y_1, y_2) \right)^{\frac{1}{2}} \\
 &+ \delta_{n_2}^{-1} \left(K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,0}; y_1, y_2) K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{0,4}; y_1, y_2) \right)^{\frac{1}{2}} \\
 &+ \left. \delta_{n_1}^{-1} \delta_{n_2}^{-1} K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{2,0}; y_1, y_2) K_{n_1, n_2}^{\alpha_1, \alpha_2}(\Psi_{y_1, y_2}^{0,2}; y_1, y_2) \right\} \\
 &\times \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}).
 \end{aligned}$$

From Lemma (2.5), we demonstrate

$$\begin{aligned}
 |K_{n_1, n_2}^*(g; y_1, y_2) - g(y_1, y_2)| &\leq 3 \|D_B g\|_\infty \left(\sqrt{C_1 C_2} (y_1 + 1)(y_2 + 1) \right) \\
 &+ \left\{ \left(\sqrt{C_1 C_2} (y_1 + 1)(y_2 + 1) \right) \right. \\
 &+ \delta_{n_1}^{-1} \left(\sqrt{C_2} \sqrt{O\left(\frac{1}{n_1}\right)} (y_1 + 1)^2 (y_2 + 1) \right) \\
 &+ \delta_{n_2}^{-1} \left(\sqrt{C_1} \sqrt{O\left(\frac{1}{n_2}\right)} (y_2 + 1)^2 (y_1 + 1) \right) \\
 &+ \left. \delta_{n_1}^{-1} \delta_{n_2}^{-1} \left(\sqrt{O\left(\frac{1}{n_1}\right)} \sqrt{O\left(\frac{1}{n_2}\right)} (y_1 + 1)(y_2 + 1) \right) \right\} \\
 &\times \omega_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2}).
 \end{aligned}$$

Which complete the proof of Theorem 4.3. \square

5. Conclusion and Remarks

These types of generalization, that is, Bivariate Szász operators is a new generalization. In this, manuscript our investigation is to generalize the Szász Durrmeyer operators based on Dunkl analogue [46] by introducing the bivariate functions. We study the bivariate properties of Szász Durrmeyer operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre’s K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we also construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by use of mixed-modulus of continuity.

References

[1] Weierstrass, K.G.: Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen. Sitz. Akad. Berl. , 2, 633–693 (1885)

[2] Bernstein, S.N.: Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités. Commun. Soc. Math. Kharkow. 13(2), 1–2 (1912)

[3] Szász, O.: Generalization of S. Bernstein’s polynomials to the infinite interval. J. Res. Nat. Bur. Stand. 45, 239–245 (1950)

[4] Durrmeyer, J. L.: Une formule d’inversion de la transformée de Laplace-applications à la théorie des moments. Thèse de 3e cycle, Faculté des Sciences de l’Université de Paris, (1967)

- [5] Acar, T., Aral, A., Mohiuddine, S.A.: *On Kantorovich modification of (p, q) -Baskakov operators*. J. Inequal. Appl., **2016**, Article Id: 98 (2016)
- [6] Varma, S., Taşdelen, F.: *Szász type operators involving Charlier polynomials*. Math. Comput. Modelling. 56(5–6), 118–122 (2012)
- [7] Sucu, S., İbikli, E.: *Rate of convergence for Szász type operators including Sheffer polynomials*. Stud. Univ. Babeş-Bolyai Math. 58(1), 55–63 (2013)
- [8] Sucu, S.: *Dunkl analogue of Szász operators*. Appl. Math. Comput. 244, 42–48 (2014)
- [9] Rosenblum, M.: *Generalized Hermite polynomials and the Bose-like oscillator calculus*. Oper. Theory Adv. Appl. 73, 369–396 (1994)
- [10] Wafi, A., Rao, N.: *Szász-Durrmeyer Operator Based on Dunkl Analogue*. Complex Anal Oper Theory 12, 1519–1536 (2018)
- [11] Acar, T., Aral, A., Mohiuddine, S.A.: *On Kantorovich modification of (p, q) -Bernstein operators*. Iran. J. Sci. Technol. Trans. A, Sci., **42**, 1459–1464 (2018)
- [12] Mohiuddine, S.A., Acar, T., Alotaibi, A.: *Construction of a new family of Bernstein-Kantorovich operators*. Math. Methods Appl. Sci., **40**, 7749–7759 (2017)
- [13] Mohiuddine, S.A., Acar, T., Alotaibi, A.: *Durrmeyer type (p, q) -Baskakov operators preserving linear functions*. J. Math. Inequal., **12**, 961–973 (2018)
- [14] Mohiuddine, S.A., Acar, T., Alghamdi, M.A.: *Genuine modified Bernstein-Durrmeyer operators*. J. Inequal. Appl. **2018**, Article id: 104 (2018)
- [15] Mohiuddine S. A., *Approximation by bivariate generalized Bernstein-Schurer operators and associated GBS operators*, Adv. Difference Equ. (2020) 2020:676.
- [16] Mohiuddine S. A., Özger F., *Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α* , Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM (2020) 114:70.
- [17] Mursaleen, M., Ansari, K.J., Khan, A.: *On (p, q) -analogue of Bernstein operators*. Appl. Math. Comput., **266**, 874–882 (2015) [MR3377604], Appl. Math. Comput. **278**, 70–71 (2016)
- [18] Mursaleen, M., Nasiruzzaman, M., Nurgali, A.: *Some approximation results on Bernstein-Schurer operators defined by (p, q) -integers*. J. Inequal. Appl. **2015**, Article Id: 249 (2015)
- [19] Mursaleen, M., Khan, F., Khan, A.: *Approximation by Kantorovich type q -Bernstein-Stancu Operators*. Complex Anal. Oper. Theory., **11**(1), 85–107 (2017)
- [20] Mursaleen, M., Nasiruzzaman, M.: *Some approximation properties of bivariate Bleimann-Butzer-Hahn operators based on (p, q) -integers*, Boll. Unione Mat. Ital., DOI 10.1007/s40574-016-0080-2.
- [21] Mursaleen, M., Nasiruzzaman, M., Ashirbayev, N., Abzhapbarov, A.: *Higher order generalization of Bernstein type operators defined by (p, q) -integers*, J. Comput. Anal. Appl., **25**, 817–829 (2018)
- [22] Acar, T.: *Asymptotic Formulas for Generalized Szász-Mirakyan Operators*. Applied Mathematics and Computation, **263**, 2015, 223–239.
- [23] Acar, T.: *(p, q) -Generalization of Szász-Mirakyan operators*, Mathematical Methods in the Applied Sciences, **39** (10), (2016), 2685–2695.
- [24] Acar, T., Ali, A., Gonska, H.: *On Szász-Mirakyan operators preserving*, Mediterranean Journal of Mathematics, **14** (1), 2017.
- [25] Acar, T., Ali, A., Garrancho, Pedro., Cardenas-Morales, Daniel.: *Szasz-Mirakyan type operators which fix exponentials*, Results in Mathematics, **72** (3), 2017, 1393–140.
- [26] Acar, T., Mursaleen, M., Mohiuddine, S.A.: *Stancu type (p, q) -Szász-Mirakyan-Baskakov operators*. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., **67**(1), 116–128 (2018)
- [27] Acar, T., Aral, A., Mohiuddine, S.A.: *Approximation by bivariate (p, q) -Bernstein-Kantorovich operators*. Iran. J. Sci. Technol. Trans. Sci., **42**, 655–662 (2018)
- [28] Acar, T., Kajla, A.: *Degree of Approximation for bivariate generalized Bernstein type operators*. Results in Mathematics., **7379** (2018) (in press)
- [29] Acar, T., Mohiuddine, S.A., Mursaleen, M.: *Approximation by (p, q) -Baskakov-Durrmeyer-Stancu Operators*. Complex Anal. Oper. Theory., **12** 1453–1468 (2018)
- [30] Ali, A., Erbay, H.: *Parametric generalization of Baskakov operators*. Math. Commun., **24**, 119–131 (2019)
- [31] Alotaibi, A., Nasiruzzaman, M., Mursaleen, M.: *A Dunkl type generalization of Szász operators via post-quantum calculus*, J. Inequal. Appl., **2018**, Article Id: 287 (2018)
- [32] Alotaibi, A.: *Approximation on a class of Phillips operators generated by q -analogue*, J. Inequal. Appl., **2020**, Article Id: 121 (2020)
- [33] Alotaibi, A., Mursaleen, M.: *Approximation of Jakimovski-Leviatan-Beta type integral operators via q -calculus*. AIMS Mathematics., **5**(4), 3019–3034 (2020)
- [34] Acar, T., Ali, A., Vijay, Gupta.: *On Approximation Properties of a New type Bernstein-Durrmeyer Operators*, Mathematica Slovaca, **65** (5), 2015, 1107–1122
- [35] Acar, T., Gulsum, Ulusoy.: *Approximation Properties of Generalized Szász-Durrmeyer Operators*, Periodica Mathematica Hungarica, **72** (1), 2016, 64–75.
- [36] Acar, T.: *Quantitative q -Voronovskaya and q -Grüss-Voronovskaya-type results for q -Szász Operators*, Georgian Mathematical Journal, **23** (4), 2016, 459–468.
- [37] Acar, T.: *Rate of Convergence for Ibragimov-Gadjiev-Durrmeyer operators*, Demonstratio Mathematica, **50** (1), 2017, 119–129.
- [38] Acar, T., Ana, Maria, Acu., Nesibe, Manav.: *Approximation of functions by genuine Bernstein-Durrmeyer type operators*, J. Math. Inequalities, **12** (4), 2018, 975–987.
- [39] Acar, T., Ana, Maria, Acu., Carman-Voileta, Muraru., Voichita, Adriana, Radu.: *Some approximation properties by a class of bivariate operators*, Mathematical Methods in the Applied Sciences, **42** (16), 2019, 5551–5565.
- [40] Bögel, K.: *Mehrdimensionale Differentiation von Funktionen mehrerer Veränderlichen*, J. Reine Angew. Math., **170**, 197–217 (1934)
- [41] Bögel, K.: *Über die mehrdimensionale Differentiation*, Jahresber. Dtsch. Math. Ver., **65**, 45–71 (1935)
- [42] Cai, Q., Lian, B., Zhou, G.: *Approximation properties of λ -Bernstein operators*. J. Inequal. Appl., **2018**, Article Id: 61 (2018).
- [43] Gadžiev, A.D.: *Positive linear operators in weighted spaces of functions of several variables*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk, no., **4**(1), 32–37 (1980)

- [44] Gadžiev, A.D., Hacısalihoglu, H.: *Convergence of the Sequences of Linear Positive Operators*, Ankara University, Yenimahalle (1995)
- [45] Khan, A., Sharma, V.: *Statistical approximation by (p, q) -analogue of Bernstein-Stancu Operators*. *Azerbaijan Journal of Mathematics.*, **8**(2), 2218–6816 (2018)
- [46] Nasiruzzaman, M, Rao, N, Wazir, S, Kumar, R.: *Approximation on parametric extension of Baskakov-Durrmeyer operators on weighted spaces*. *J. Inequal. Appl.*, **2019**, Article Id: 103 (2019)
- [47] Wafi, A., Rao, N.: *Szasz–Gamma Operators Based on Dunkl Analogue*, *Iran J Sci Technol Trans Sci.* **43**(1), 213–223 (2019)