Filomat 35:13 (2021), 4501–4513
https://doi.org/10.2298/FIL2113501J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Equi-Statistical Convergence of a Sequence of Distribution Functions via Deferred Nörlund Summability Mean and Associated Approximation Theorems

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Abstract. In this paper, the notion of statistical point-wise convergence, equi-statistical convergence and statistical uniform convergence of a sequence of distribution functions via the deferred Nörlund summability mean has been introduced, and accordingly an inclusion relation between these interesting notions is established. Moreover, as an application point of view, a new Korovkin-type approximation theorem is proved via the deferred Nörlund equi-statistical convergence for the sequence of distribution functions. Also, some illustrative examples are considered to justify that the proposed theorem is a nontrivial extension of some well established Korovkin-type approximation theorems for sequence of real-valued functions. Finally, a number of interesting cases are highlighted in support of the definitions and outcomes.

1. Introduction and Motivation

The theory of summability plays a vital role in the convergence analysis of the sequence spaces. Gradually, a new concept has been merged in sequence space called the statistical convergence, and it is more general than the ordinary convergence. This concept was initially introduced and studied independently by two Mathematicians, Fast [10] and Steinhaus [29]. Recently, Et *et. al* [9] has established the μ-deferred statistical convergence and strongly deferred summable functions. Now, several researchers are working on the sequence spaces by using the concept of statistical convergence as well as statistical summability. This concept is also closely related to different fields of pure and applied mathematics, such as, Probability theory, Number theory, Fourier analysis and Measure theory and Differential equations, etc. For more details, see the recent works [3], [4], [11], [15], [19], [27] and [28].

Let $S \subset \mathbb{R}$. Also, let $X_n : S \longrightarrow \mathbb{R} \quad (n \in \mathbb{N})$ be a sequence of random variables defined as

$$X_n(s_i) = x_{ni},$$

where S is the sample space consisting of finite number of elements and $i = 1, 2, \cdots, k$.

2020 *Mathematics Subject Classification*. Primary 40A05, 40G15; Secondary 41A36

Keywords. Sequence of random variables, Equi-statistical convergence, Deferred Nörlund mean, Distribution functions, Positive linear operator, Korovkin-type approximation theorem

Received: 19 October 2020; Revised: 22 January 2021; Revised: 04 May 2021; Accepted: 05 August 2021

Communicated by Hemen Dutta

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A sequence of random variables $(X_n(s))$ converges to a random variable $X(s)$, if

$$\lim_{n\to\infty} X_n(s) = X(s).$$

That is, for every $\epsilon > 0$, there exists a number $n \in \mathbb{N}$ such that

$$|X_n(s) - X(s)| < \epsilon, \quad \forall\, n.$$

Let $\Omega \subseteq \mathbb{N}$ and suppose that

$$\Omega_n = \{j : j \leqq n \quad \text{and} \quad j \in \Omega\},$$

where $|\Omega_n|$ is the cardinality of Ω_n. Then the natural density of Ω is defined by

$$\delta(\Omega) = \lim_{n\to\infty} \frac{|\Omega_n|}{n},$$

provided that the limit exists.

Now, we give the definition of statistical convergence of a sequence of random variables.

Definition 1.1. *A sequence $(X_n(s))$ of random variables is statistically convergent to ν if, for each $\epsilon > 0$,*

$$\Omega_\epsilon = \{j : j \in \mathbb{N} \quad \textit{and} \quad |(X_j(s)) - \nu| \geqq \epsilon\}$$

has natural density zero (see [11], [13]). This means that, for each $\epsilon > 0$,

$$\Omega_\epsilon = \lim_{n\to\infty} \frac{|\Omega_\epsilon|}{n} = 0.$$

Symbolically, it is written as

$$\textit{stat} \lim_{n\to\infty} (X_n(s)) = \nu.$$

Based on Definition 1.1, the following example, provides the idea of statistical convergence in random variables which is different from simple convergence in random variables.

Example 1.2. *A fair coin is tossed once with sample space $S = \{H, T\}$. Let us define a sequence of random variables $(X_n(s))$ on this sample space S given by*

$$X_n(s) = \begin{cases} 1 & (s = H;\ n = m^2,\ m \in \mathbb{N}) \\ \\ 0 & (s = T;\ \text{otherwise}). \end{cases}$$

Here, the sequence of random variables $(X_n(s))$ converges to $X(s)$, where $X(s) = 0$ irrespective of s is a head (H) or tail (T).

In the year 2002, Gadjiev and Orhan [12] established some approximation theorems via statistical convergence of a real sequences. Later on, Belen and Mohiuddine [6] studied the generalized statistical convergence and proved some approximation theorems. Subsequently, Braha *et. al* [7] used weighted statistical convergence to prove Korovkin and Voronovskaya type theorems. Furthermore, Baliarsingh *et. al* [2] has established the statistical convergence of difference sequences of fractional order and related Korovkin-type approximation theorems. Recently, a few researchers are working in this direction over the probability space. For some current works, see [1], [14], [22], [23] and [26].

Let us recall the concept of deferred Nörlund $D_a^b(N, p, q)$ summability method.

Let (a_n) and (b_n) be the sequences of non-negative integers such that $a_n < b_n (n \in \mathbb{N})$ and $\lim_{n\to\infty} b_n = \infty$, and let (p_n) and (q_n) be the sequences of non-negative real numbers satisfying

$$P_n = \sum_{m=a_n+1}^{b_n} p_m \quad \text{and} \quad Q_n = \sum_{m=a_n+1}^{b_n} q_m.$$

The convolution of the above sequences is

$$\mathcal{H}_{a_n+1}^{b_n} = \sum_{v=a_n+1}^{b_n} p_v q_{b_n-v}.$$

Now the deferred Nörlund mean $D_a^b(N, p, q)$ of a sequence of random variables is given by

$$\varphi_n = \frac{1}{\mathcal{H}_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_{b_n-m} q_m X_m(s).$$

Motivated essentially by the above mentioned works, the present investigation aims to introduce the concept of statistical point-wise convergence, equi-statistical convergence and statistical uniform convergence of a sequence of distribution functions via the deferred Nörlund summability mean. Also, an inclusion relation has been established by interrelating these beautiful notions. Moreover, as an application point of view, a new Korovkin-type approximation theorem is proved via the deferred Nörlund equi-statistical convergence for the sequence of distribution functions. Also, some illustrative examples are considered to justify that the proposed theorem is a nontrivial extension of some well established Korovkin-type approximation theorems for sequence of real-valued functions. Finally, a number of interesting cases are highlighted in support of the definitions and outcomes.

2. Equi-statistical Convergence for the Sequence of Distribution Functions

A sequence (X_n) of random variables is distribution convergent (or convergence in distribution) to a random variable X, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous.

Let $C(D)$ be the set of all real valued continuous probability density function defined over a compact subset $D \subseteq \mathbb{R}$. Also, let $C(D)$ is a complete normed linear space. For $F_X(x) \in C(D)$, we have

$$\|F_X\|_\infty = \sup_{x\in D} |F_X(x)|.$$

Let $F_{X_n}(x) \in C(D)$.

Let us introduce the following definitions for the proposed study.

Definition 2.1. *A sequence $\{F_{X_n}(x) : n \in \mathbb{N}\}$ of distribution functions is deferred Nörlund statistically point-wise convergent to a distribution function $F_X(x)$ if, for each $\epsilon > 0$ and for every $x \in D$,*

$$\lim_{n\to\infty} \frac{|\{k : k \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \textit{and} \quad p_{b_n-k} q_k |F_{X_k}(x) - F_X(x)| \geqq \epsilon\}|}{\mathcal{H}_{a_n+1}^{b_n}} = 0.$$

Symbolically, it is denoted by

$$F_{X_n} \to F_X \quad (\varphi_n - \textit{stat-point-wise}).$$

Definition 2.2. *A sequence* $\{F_{X_n}(x) : n \in \mathbb{N}\}$ *of distribution functions is said to be deferred Nörlund equi-statistically convergent to a distribution function* $F_X(x)$ *if, for each* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\chi_n(x,\epsilon)}{\mathcal{H}_{a_n+1}^{b_n}} = 0$$

uniformly with respect to $x \in D$, *that is,*

$$\lim_{n\to\infty} \frac{\|\chi_n(x,\epsilon)\|_{C(D)}}{\mathcal{H}_{a_n+1}^{b_n}} = 0,$$

where

$$\chi_n(x,\epsilon) = |\{k : k \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-k}q_k|F_{X_k}(x) - F_X(x)| \geqq \epsilon\}|.$$

Symbolically, it is written as

$$F_{X_n} \to F_X \quad (\varphi_n\text{-equi-stat}).$$

Definition 2.3. *A sequence* $\{F_{X_n}(x) : n \in \mathbb{N}\}$ *of distribution functions is said to be deferred Nörlund statistically-uniform convergent to a distribution function* F_X *if, for each* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k : k \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-k}q_k\|F_{X_k}(x) - F_X(x)\|_{C(D)} \geqq \epsilon\}|}{\mathcal{H}_{a_n+1}^{b_n}} = 0.$$

Symbolically,

$$F_{X_n} \rightrightarrows F_X \quad (\varphi_n - \text{stat-uniform}).$$

Next, in view of Definitions 2.1, 2.2 and 2.3 the following Lemma is trivial.

Lemma 2.4. *The following implications are true*

$$\begin{aligned} F_{X_n} \rightrightarrows F_X(\varphi_n - stat - uniform) &\Rightarrow F_{X_n} \to F_X(\varphi_n - equi - stat) \\ &\Rightarrow F_{X_n} \to F_X(\varphi_n - stat - point - wise). \end{aligned} \tag{1}$$

The inclusions in (1) are strict, that is, the reverse implications in (1) are not always true.

The following examples will justify the strictness of the implications asserted by Lemma 2.4.

Example 2.5. *Let* $\{X_n;\ n \in \mathbb{N}\}$ *be a sequence of random variables and let*

$$p_n = 1, \quad q_n = \frac{1}{n+2}, \quad a_n = n \quad \text{and} \quad b_n = 2n.$$

Let

$$F_{X_n} : [0,1] \to \mathbb{R}$$

be the sequence of continuous real valued probability density functions defined by

$$F_{X_n}(x) = \begin{cases} \frac{\left[\left(\frac{1}{n+2}\right)^2 - x^2\right](n+2)^2}{1+x^2} & \left(0 \leqq x \leqq \frac{1}{n+2}\right) \\ 0 & (\text{otherwise}). \end{cases} \tag{2}$$

Then, for each $\epsilon > 0$,

$$\frac{1}{\mathcal{H}_{a_n+1}^{b_n}}|\{k : k \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-k}q_k|F_{X_k}(x) - F_X(x)| \geqq \epsilon\}| \leqq \frac{1}{\mathcal{H}_{a_n+1}^{b_n}} = 0$$

uniformly on D.
This implies that

$$F_{X_n}(x) \to 0 \quad (\varphi_n - \textit{equi-stat}).$$

However,

$$\sup_{x\in[0,1]} |F_{X_n}(x)| = 1 \quad (\forall\ n \in \mathbb{N}).$$

This yields that

$$F_{X_n}(x) \rightrightarrows 0 \quad (\varphi_n - \textit{uniform-stat})$$

does not hold.

Example 2.6. *This example illustrates that the second inclusion in (1) is strict.*
Indeed, for

$$p_n = 1, \quad q_n = \frac{1}{n+2}, \quad a_n = n \quad \text{and} \quad b_n = 2n.$$

Let

$$F_{X_n} : [0, 1] \to \mathbb{R}$$

be the sequence of continuous real-valued probability density functions defined by

$$F_{X_n}(x) = x^n.$$

Suppose that

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad (x \in [0, 1]).$$

Then

$$F_{X_n}(x) \to F_X(x) \quad (\varphi_n - \textit{point-wise-stat}).$$

If $\epsilon = \frac{1}{3}$, then for all $n \in \mathbb{N}$, there exists $r > n$ such that $r \in [n + 1, 2n]$ and for each $x \in \left(\left[\frac{1}{3}\right]^{\frac{1}{n}}, 1\right)$, it yields

$$|F_{X_r}(x)| = |x^r| > \left|\left(\left[\frac{1}{3}\right]^{\frac{1}{n}}\right)^r\right| > \left|\left(\left[\frac{1}{3}\right]^{\frac{1}{r}}\right)^r\right| = \frac{1}{3}. \tag{3}$$

Clearly, the following condition

$$F_{X_n}(x) \to 0 \quad (\varphi_n - \textit{equi-stat})$$

does not hold.

3. A Korovkin-type Theorem for Sequence of Distribution Functions

In the recent years, quite a few researchers worked toward extending (or generalizing) the Korovkin-type theorems in different fields of pure and applied mathematics. This concept is extremely valuable in Real analysis, Measure Theory, Probability Theory, Summability Theory, Functional analysis, Harmonic Analysis and so on. For further details with several results related to the Korovkin-type theorems and other related developments, one can refer the recent works [5], [8], [15], [16], [17], [18] and [26].

Let $C(D)$ be the linear space of all real-valued continuous probability density function F_X defined on D, where $D \subseteq \mathbb{R}$ is compact, and suppose that $\mathfrak{A} : C(D) \to C(D)$ be a sequence of random variables of positive linear operators. The operator $\mathfrak{A}$ is positive if, $\mathfrak{A}(F_X; x) \geqq 0$ whenever $x \in [a, b]$. It is also known that $C(D)$ is a complete normed linear space. For $F_X \in D$, the infinite norm of a function F_X is denoted by $\|F_X\|_\infty$, and is given by

$$\|F_X\|_\infty = \sup_{x \in D} |F_X(x)|.$$

This section extends the result of Srivastava *et al.* [25] for the sequence of distribution functions.

Theorem 3.1. *Let $\mathfrak{A}_m$ ($m \in \mathbb{N}$) be the sequence of random variables of positive linear operators from $C(D)$ into itself and let $F_X \in C(D)$. Then*

$$\mathfrak{A}_n(F_X, x) \to F_X \quad (\varphi_n - equi - stat) \quad on \quad D \tag{4}$$

if and only if

$$\mathfrak{A}_n(F_{X_i}, x) \to F_{X_i}(x) \quad (\varphi_n - equi - stat) \quad on \quad D \quad (i = 0, 1, 2) \tag{5}$$

where

$$F_{X_0}(x) = 1, \; F_{X_1}(x) = x, \; F_{X_2}(x) = x^2.$$

Proof. Since each of the functions given by

$$F_{X_i}(x) = x^i$$

is continuous, the implications (4) to (5) is fairly obvious.

In order to complete the proof of Theorem 3.1, it is assumed that (5) holds. Since $F_X \in C(D)$, there exists a constant $\mathcal{K} > 0$ such that

$$|F_X(x)| \leqq \mathcal{K} \quad (\forall \; x \in D)$$

which implies that

$$|F_X(t) - F_X(x)| \leqq 2\mathcal{K} \quad (\forall \; x, t \in D).$$

Clearly, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|F_X(t) - F_X(x)| \leqq \epsilon \quad \text{whenever} \quad |t - x| < \delta. \tag{6}$$

Let us now choose

$$\mu = \mu(t, x) = (t - x)^2.$$

Then if

$$|t - x| \geqq \delta \quad (x, t \in D)$$

then

$$|F_X(t) - F_X(x)| < \frac{2\mathcal{K}}{\delta^2}\mu(t,x). \tag{7}$$

From (6) and (7), it yields

$$|F_X(t) - F_X(x)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\mu(t,x),$$

and this implies that

$$-\epsilon - \frac{2\mathcal{K}}{\delta^2}\mu(t,x) \leqq F_X(t) - F_X(x) \leqq \epsilon + \frac{2\mathcal{K}}{\delta^2}\mu(t,x). \tag{8}$$

Now, since the operator $\mathfrak{A}_m(1,x)$ is monotone and linear, by applying this operator to the inequality in (8), it yields

$$\begin{aligned}\mathfrak{A}_m(1,x)\left(-\epsilon - \frac{2\mathcal{K}}{\delta^2}\mu(t,x)\right) &\leqq \mathfrak{A}_m(1,x)[F_X(t) - F_X(x)]\\ &\leqq \mathfrak{A}_m(1,x)\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\mu(t,x)\right).\end{aligned} \tag{9}$$

One may note here that x is fixed and so $F_X(x)$ is a constant number. Therefore,

$$\begin{aligned}-\epsilon\, \mathfrak{A}_m(1,x) - \frac{2\mathcal{K}}{\delta^2}\mathfrak{A}_m(\mu,x) &\leqq \mathfrak{A}_m(F_X,x) - F_X\mathfrak{A}_m(1,x)\\ &\leqq \epsilon\, \mathfrak{A}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\mathfrak{A}_m(\mu,x)\end{aligned} \tag{10}$$

which in conjunction with the following obvious identity

$$\mathfrak{A}_m(F_X,x) - F_X = [\mathfrak{A}_m(F_X,x) - F_X(x)\mathfrak{A}_m(1,x)] + F_X(x)[\mathfrak{A}_m(1,x) - 1].$$

This yields

$$\mathfrak{A}_m(F_X,x) - F_X < \epsilon\, \mathfrak{A}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\mathfrak{A}_m(\mu,x) + F_X(x)[\mathfrak{A}_m(1,x) - 1]. \tag{11}$$

Next, in order to estimate $\mathfrak{A}_m(\mu,x)$, let us write

$$\begin{aligned}\mathfrak{A}_m(\mu,x) &= \mathfrak{A}_m((t-x)^2,x) = \mathfrak{A}_m(t^2 - 2xt + x^2,x)\\ &= \mathfrak{A}_m(t^2,x) - 2x\, \mathfrak{A}_m(t,x) + x^2\mathfrak{A}_m(1,x)\\ &= [\mathfrak{A}_m(t^2,x) - x^2] - 2x[\mathfrak{A}_m(t,x) - 2x] + x^2[\mathfrak{A}_m(1,x) - 1].\end{aligned}$$

By using (11), it yields

$$\begin{aligned}\mathfrak{A}_m(F_X,x) - F_X(x) &< \epsilon\mathfrak{A}_m(1,x) + \frac{2\mathcal{K}}{\delta^2}\{[\mathfrak{A}_m(t^2,x) - x^2] - 2x[\mathfrak{A}_m(t,x) - x]\\ &\quad + x^2[\mathfrak{A}_m(1,x) - 1] + F_X(x)[\mathfrak{A}_m(1,x) - 1]\}\\ &= \epsilon[\mathfrak{A}_m(1,x) - \epsilon] + \epsilon + \frac{2\mathcal{K}}{\delta^2}\{[\mathfrak{A}_m(t^2,x) - x^2] - 2x[\mathfrak{A}_m(t,x) - x]\\ &\quad + x^2[\mathfrak{A}_m(1,x) - 1] + F_X(x)[\mathfrak{A}_m(1,x) - 1]\}.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, one can write

$$\begin{aligned}\mathfrak{A}_m(F_X,x)-F_X(x) &\leq \left(\epsilon+\frac{2\mathcal{K}}{\delta^2}+\mathcal{K}\right)|\mathfrak{A}_m(1,x)-1| \\ &\quad+\frac{4k}{\delta^2}|\mathfrak{A}_m(t,x)-x|+\frac{2\mathcal{K}}{\delta^2}|\mathfrak{A}_m(t^2,x)-x^2| \\ &\leqq \mathcal{M}(|\mathfrak{A}_m(1,x)-1|+|\mathfrak{A}_m(t,x)-x|+|\mathfrak{A}_m(t^2,x)-x^2|). \end{aligned} \tag{12}$$

This implies that

$$\begin{aligned}p_{b_n-m}q_m\|\mathfrak{A}_m(F_X,x)-F_X(x)\|C(D) &\leqq \mathcal{B}\, p_{b_n-m}q_m|\mathfrak{A}_m(1,x)-1| \\ &\quad+\mathcal{B}p_{b_n-m}q_m|\mathfrak{A}_m(t,x)-x|+\mathcal{B}p_{b_n-m}q_m|\mathfrak{A}_m(t^2,x)-x^2| \end{aligned} \tag{13}$$

where

$$\mathcal{B}=\max\left\{\epsilon+\frac{2\mathcal{K}}{\delta^2}+\mathcal{K},\frac{4\mathcal{K}}{\delta^2},\frac{2\mathcal{K}}{\delta^2}\right\}.$$

Now, for a given $r > 0$, we can choose $\epsilon > 0$ such that $0 < \epsilon < r$. Then upon setting

$$\mathfrak{R}_m(x,r)=|\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-m}q_m|\mathfrak{A}_m(F_X,x)-F_X(x)| \geqq r\}|$$

and

$$\mathfrak{R}_{i,m}(x,r)=\left|\left\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-m}q_m|\mathfrak{A}_m(F_{X_i},x)-F_{X_i}(x)| \geqq \frac{r-\epsilon}{3k}\right\}\right|,$$

we easily find from (13) that

$$\mathfrak{R}_m(x,r) \leqq \sum_{i=0}^{2}\mathfrak{R}_{i,m}(x,r).$$

Hence,

$$\frac{\|\ \mathfrak{R}_m(x,r)\ \|_{C(D)}}{\mathcal{H}_{a_n+1}^{b_n}} \leqq \sum_{i=0}^{2}\frac{\|\ \mathfrak{R}_{i,m}(x,r)\ \|_{C(D)}}{\mathcal{H}_{a_n+1}^{b_n}}. \tag{14}$$

Finally by using the above assumption the implication in (5) and by Definition 2.2 the right-hand side of (14) is tends to zero as $n \to \infty$. Consequently, it yields

$$\frac{\|\ \mathfrak{R}_m(x,r)\ \|_{C(D)}}{\mathcal{H}_{a_n+1}^{b_n}}=0. \tag{15}$$

Therefore, the implication (4) holds. The proof of Theorem 3.1 is thus completed. □

Example 3.2. *Let $D = [0, 1]$ and consider the classical Bernstein polynomial on $C[0, 1]$.*

$$\mathfrak{B}_r^*(F_X,x)=(1+F_{X_r}(x))\sum_{k=0}^{r}F_X\left(\frac{k}{r}\right)\binom{r}{k}x^r(1-x)^{r-k}, \quad x\in[0,1]. \tag{16}$$

Then, it immediately yields

$$\mathfrak{B}_r^*(F_{X_0}, x) = (1 + F_{X_r}(x))F_{X_0}(x)$$
$$\mathfrak{B}_r^*(F_{X_1}, x) = (1 + F_{X_r}(x))F_{X_1}(x)$$
$$\mathfrak{B}_r^*(F_{X_2}, x) = (1 + F_{X_r}(x))\left(F_{X_2}(x) + \frac{x(1-x)}{r}\right).$$

Here $(F_{X_r})_{r\in\mathbb{N}}$ *being the sequence of distribution functions given by Example 2.5 and also* $F_{X_r} \to 0 \quad (\varphi_n - \textit{equi-stat})$, *it is concluded that*

$$\mathfrak{B}_r^*(F_{X_i}, x) \to F_{X_i}(x) \quad (\varphi_n - \textit{equi-stat}). \tag{17}$$

So, by Theorem 3.1, it immediately yields

$$\mathfrak{B}_r^*(F_X, x) \to F_X(x) \quad (\varphi_n - \textit{equi-stat}) \tag{18}$$

on $[0, 1]$ *for all* $F_X \in C[0, 1]$.

Moreover, since the sequence (F_{X_n}) *of distribution functions is* (φ_n)*-equi-statistical convergent but it is not* φ_n*−equi-statistical convergent under the choice of real sequences. Thus the result of Srivastava et al. [25] does not work under the operators defined by (16).*

4. Rate of Equi-statistical Convergence for Sequence of Distribution Functions

In this section, the rate of the deferred Nörlund equi-statistical convergence of a sequence of random variables of positive linear operators defined on $C(D)$ under the modulus of continuity is discussed.

Definition 4.1. *Let* (u_n) *be a positive non-increasing sequence. A sequence* (F_{X_n}) *of distribution functions is* (φ_n)*-equi-statistically convergent to a distribution function* F_X *with the rate* $o(u_n)$ *if, for each* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\Upsilon_n(x, \epsilon)}{u_n \mathcal{H}_{a_n+1}^{b_n}} = 0$$

uniformly with respect to $x \in D$ *or, equivalently, if*

$$\lim_{n\to\infty} \frac{\|\Upsilon_n(x, \epsilon)\|_{C(D)}}{u_n \mathcal{H}_{a_n+1}^{b_n}} = 0$$

where

$$\Upsilon_m(x, r) = |\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-m}q_m|\mathfrak{A}_m(F_X, x) - F_X(x)| \geqq r\}|.$$

Symbolically,

$$F_{X_n} - F_X = o(u_n) \quad (\varphi_n - \textit{equi-stat}) \text{ on } D.$$

Next, a result is considered and proved in the form of Lemma 4.2 as follows.

Lemma 4.2. *Let the sequences* (F_{X_n}) *and* (G_{X_n}) *belonging to* $C(D)$ *satisfy the following conditions:*

$$F_{X_n} - F_X = o(s_n) \quad (\varphi_n - \textit{equi-stat})$$

and

$$G_{X_n} - G_X = o(c_n) \quad (\varphi_n - \textit{equi-stat}) \text{ on } D.$$

Then, the following assertions hold:

(i) $[F_{X_n}(x) + G_{X_n}(x)] - [F_X(x) + G_X(x)] = o(d_n) \quad (\varphi_n - equi - stat)$

(ii) $[F_{X_n}(x) - F_X(x)][G_{X_n}(x) - G_X(x)] = o(s_n c_n) \quad (\varphi_n - equi - stat)$

(iii) $\lambda[F_{X_n}(x) - F_X(x)] = o(s_n)(\varphi_n - equi - stat)$

(iv) $\sqrt{|F_{X_n}(x) - F_X(x)|} = o(s_n)(\varphi_n - equi - stat)$

where $d_n = \max\{s_n, c_n\}$.

Proof. In order to prove the assertion (i) of Lemma 4.2, let us consider the following sets. For $\epsilon > 0$ and $x \in D$

$$\mathfrak{E}_n(x,\epsilon) = |\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \\ p_{b_n-m}q_m|(F_{X_n}(x) + G_{X_n}(x)) - (F_X + G_X)(x)| \geqq \epsilon\}|,$$

$$\mathfrak{E}_{0,n}(x,\epsilon) = |\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-m}q_m|F_{X_n}(x) - F_X(x)| \geqq \frac{\epsilon}{2}\}|$$

and

$$\mathfrak{E}_{1,n}(x,\epsilon) = |\{m : m \leqq \mathcal{H}_{a_n+1}^{b_n} \quad \text{and} \quad p_{b_n-m}q_m|G_{X_n}(x) - G_X(x)| \geqq \frac{\epsilon}{2}\}|.$$

Clearly,

$$\mathfrak{E}_n(x,\epsilon) \subseteq \mathfrak{E}_{0,n}(x,\epsilon) \cup \mathfrak{E}_{1,n}(x,\epsilon). \tag{19}$$

Moreover, since

$$d_n = \max\{s_n, c_n\} \tag{20}$$

by applying the assertion (4) of Theorem 3.1, it yields

$$\frac{\|\mathfrak{E}_n(x,\epsilon)\|_{C(D)}}{d_n\mathcal{H}_{a_n+1}^{b_n}} \leqq \frac{\|\mathfrak{E}_{0,n}(x,\epsilon)\|_{C(D)}}{d_n\mathcal{H}_{a_n+1}^{b_n}} + \frac{\|\mathfrak{E}_{1,n}(x,\epsilon)\|_{C(D)}}{c_n\mathcal{H}_{a_n+1}^{b_n}}. \tag{21}$$

Also, by applying the assertion (5) of Theorem 3.1, it easily yields

$$\frac{\|\mathfrak{E}_n(x,\epsilon)\|_{C(D)}}{d_n\mathcal{H}_{a_n+1}^{b_n}} = 0. \tag{22}$$

This proves the assertion (i) of Lemma. The other assertions (ii) to (iv) of Lemma 4.2 are similar to (i), so it is not difficult to prove these assertions along the similar lines. This evidently completes the proof of Lemma 4.2. □

Recalling that the modulus of continuity of a distribution function $F_X \in C(D)$ which is defined by

$$\omega(F_X, \delta) = \sup_{|t-x| \leqq \delta(x, t \in D)} |F_X(t) - F_X(x)|. \tag{23}$$

Theorem 4.3. *Let $D \subset \mathbb{R}$ be compact set and $\mathfrak{A}_m : C(D) \to C(D)$ be the sequence of random variables of positive linear operators. Suppose that the following conditions are satisfied.*

(i) $\mathfrak{A}_m(F_{X_0}, x) - F_{X_0} = O(s_n) \quad (\varphi_n - \text{ equi-stat }) \quad on \quad D$

(ii) $\omega(F_X, \delta_n) = O(c_n) \quad (\varphi_n - \text{equi-stat}) \quad on \quad D$

where

$$\delta_n(x) = \sqrt{\mathfrak{A}_m(\mu^2 : x)} \quad \text{and} \quad \mu(t) = t - x.$$

Then, for all $F_X \in C(D)$, the following assertion holds:

$$|\mathfrak{A}_m(F_X; x) - F_X(x)| = O(d_n) \quad (\varphi_n - equi - stat) \quad on \quad D \tag{24}$$

where d_n is given by (20).

Proof. Let $f \in C[D]$ and $x \in D$. Then it is well known that

$$|\mathfrak{A}_m(F_X; x) - F_X(x)| \leqq \mathcal{N}|\mathfrak{A}_m(F_{X_0}; x) - F_{X_0}(x)| + (\mathfrak{A}_m(F_{X_0}; x) + \sqrt{\mathfrak{A}_m(F_{X_0}; x)})\omega(F_X, \delta_n)$$

where

$$\mathcal{N} = (\|F_X\|_{C(D)}, F_{X_0}).$$

This yields

$$\begin{aligned} |\mathfrak{A}_m(F_X; x) - F_X(x)| \leq\ & \mathcal{N}|\mathfrak{A}_m(F_{X_0}; x) - F_{X_0}(x)| + 2\omega(F_X, \delta_n) \\ & + \omega(F_X, \delta_n)|\mathfrak{A}_m(F_{X_0}; x) - F_{X_0}(x)| \\ & + \omega(F_X, \delta_n)\sqrt{|\mathfrak{A}_m(F_{X_0}; x) - F_{X_0}(x)|}. \end{aligned} \tag{25}$$

Finally, in view of the conditions (i) and (ii) of Theorem 4.3 in conjunction with Lemma 4.2, the last inequality (25) leads to the assertion (24) of Theorem 4.3. This completes the proof of Theorem 4.3. □

5. Concluding Remarks and Observations

In this concluding section of the investigation, several further remarks and observations concerning the various results are presented.

Remark 5.1. *Let $(F_{X_n})_{n\in\mathbb{N}}$ be a sequence of functions given in Example* 2.5. *Then, since*

$$F_{X_n} \to 0 \quad (t_n - equi - stat)$$

on $[0, 1]$, *it yields*

$$\mathfrak{A}_m(F_{X_i}; x) \to F_{X_i}(x) \quad (\varphi_n - equi - stat) \text{ on } [0, 1] \quad (i = 0, 1, 2). \tag{26}$$

Therefore, by applying Theorem 3.1,

$$\mathfrak{A}_m(F_X, x) \to F_X(x) \quad (\varphi_n - \text{equi-stat}) \text{ on } [0, 1] \tag{27}$$

for all $f \in C(D)$.

However, since (F_{X_n}) is not φ_n-statistically uniform convergent to the function $F_X(= 0)$ on the interval $[0, 1]$ *and also since (F_{X_n}) is not uniformly convergent to the function $F_X(= 0)$ on the interval* $[0, 1]$, *the classical Korovkin-type theorem does not work for the operators defined by* (16). *Therefore, this application clearly shows that Theorem 3.1 is a non-trivial generalization of the classical and statistical versions of the Korovkin-type theorems [17] and [25].*

Remark 5.2. *Let us replace the conditions (i) and (ii) in Theorem 4.3 by the following conditions*

$$\mathfrak{A}_m(F_{X_i};x) - F_{X_i} = 0(s_{n_i}) \quad (\varphi_n - \textit{equi-stat}) \quad on \quad D\ (i = 0,1,2). \tag{28}$$

Then, since

$$\mathfrak{A}_m(\phi^2;x) = \mathfrak{A}_m(F_{X_2}(t);x) - 2x\mathfrak{A}_m(F_{X_1}(t);x) + x^2\mathfrak{A}_m(F_{X_0}(t);x),$$

one can write

$$\mathfrak{A}_m(\phi^2;x) \leqq \kappa \sum_{i=0}^{2} |\mathfrak{A}_m(F_{X_i};x) - F_{X_i}|, \tag{29}$$

where

$$\kappa = 1 + 2\|F_{X_1}\|_{C(D)} + \|F_{X_2}\|_{C(D)}.$$

It now follows from (28), (29) *and Lemma 4.2, that*

$$\delta_n = \sqrt{\mathfrak{A}_m(\phi^2)} = O(u_n) \quad (\varphi_n - \textit{equi-stat}) \quad on \quad D, \tag{30}$$

where

$$O(u_n) = \max\{s_{n_0}, s_{n_1}, s_{n_2}\}.$$

Hence, it clearly yields

$$\omega(F_X, \delta) = o(u_n) \quad (\varphi_n - equi - stat) \quad on \quad D.$$

By using (30) *in Theorem 4.3, one can immediately see for all $f \in C(D)$,*

$$\mathfrak{A}_m(F_X;x) - F_X(x) = O(u_n) \quad (t_n - \textit{qui-stat}) \quad on \quad D.$$

Therefore, if instead of conditions (i) and (ii), the condition (28) *is used in Theorem 4.3, then the rates of the deferred Nörlund equi-statistical convergence of the sequence* ($\mathfrak{A}_m$) *of random variables of positive linear operators in Theorem 3.1 can fairly be obtained.*

Remark 5.3. *In random graph theory (see [20, 21]) in the sense that almost convergence means convergence with probability* 1*, whereas in probability convergence the probability is not necessarily* 1*. Mathematically, a sequence of random variables $\{X_n\}$ is probability convergent (converges in probability) to a random variable X if* $\lim_{n\to\infty} P(|Xn - X| \geqq \epsilon) = 0$*, for all $\epsilon > 0$ (arbitrarily small); and almost convergent to X if* $P(\lim_{n\to\infty} X_n = X) = 1$ *(see [24]).*

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