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Inequalities Related to Schatten Norm

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Abstract. In this paper, we investigate the known operator inequalities for the p-Schatten norm and obtain some refinements of these inequalities when parameters taking values in different regions. Let $A_1,\cdots,A_n,B_1,\cdots,B_n\in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^*B_j=0$. Then $p\geq 2,\ p\leq\lambda$ and $\mu\geq 2$,

$$\begin{aligned}
&2^{1/p-\mu/4}n^{3/p-\mu/4-1/2}\Big(\sum_{i=1}^n\|A_i\|_p^{4/\mu}+\sum_{i=1}^n\|B_i\|_p^{4/\mu}\Big)^{\mu/4}\\
\leq\ & n^{2/p-1/2}\Big\|\sum_{i=1}^n|A_i|^2+\sum_{i=1}^n|B_i|^2\Big\|_{p/2}^{1/2}\\
\leq\ & n^{2/p-2/\lambda}\Big(\sum_{i,j=1}^n\|A_i\pm B_j\|_p^\lambda\Big)^{1/\lambda}.
\end{aligned}$$

For $0<p\leq 2,\ p\geq\lambda>0$ and $0<\mu\leq 2$, the inequalities are reversed. Moreover, we get some applications of our results.

1. Introduction

Let $B(H)$ be the C^*-algebra of all bounded linear operators acting on a complex separable Hilbert space H. $|A|=(X^*X)^{1/2}$ denotes the absolute value of an operator $A\in B(H)$. If $A\in B(H)$ is compact, let $\{s_j(A)\}_{j=1}^\infty$ be the sequence of decreasingly ordered singular values of A. For $0<p<\infty$, let $\|A\|_p=(tr|A|^p)^{1/p}=(\sum_{j=1}^\infty s_j^p(A))^{1/p}$, where tr is the usual trace function. This defines the Schatten p-norm (quasi-norm, resp.) for $1\leq p<\infty$ $(0<p<1$, resp.) on the set

$$B_p(H)=\{X\in B(H):\|X\|_p<\infty\},$$

which is called the p-Schatten class of $B(H)$ (see [5]). The Schatten p-norms are unitarily invariant and when $p=1$, $\|A\|_1=tr|A|$ is called the trace norm of A.

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There are some classical Clarkson's inequalities for the Schatten p-norms of operators in $B_p(H)$ (See [3]). If $A, B \in B_p(H)$, then

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \leq \|A - B\|_p^p + \|A + B\|_p^p \leq 2(\|A\|_p^p + \|B\|_p^p) \qquad (1.1)$$

for $0 < p \leq 2$ and

$$2(\|A\|_p^p + \|B\|_p^p) \leq \|A - B\|_p^p + \|A + B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p) \qquad (1.2)$$

for $2 \leq p < \infty$. For $p = 2$, by (1.1) and (1.2), we have

$$\|A - B\|_2^2 + \|A + B\|_2^2 = 2(\|A\|_2^2 + \|B\|_2^2),$$

which is called parallelogram law. When $p \neq 2$, the equality $2(\|A\|_p^p + \|B\|_p^p) = \|A - B\|_p^p + \|A + B\|_p^p$ holds if and only if $A^*B = AB^* = 0$, or equivalently $R(A)$ and $R(B)$ are orthogonal. (See [3]).

Hirzallah, Kittaneh and Moslehian etc. have obtained some generalizations of (1.1) and (1.2) to n-tuples of operators and many different conclusions by using various methods such as complex interpolation method, concavity and convexity of certain functions, etc. (See [1, 6 − 10]).

Recently, some refinements of some p-Schatten inequalities have been given by Conde and Moslehian in [4].

Theorem 1.1 ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leq 2,\ p \leq \lambda$ and $0 < \mu \leq 2$,

$$\begin{aligned} 2^{1/2-1/\mu} n^{1-1/\mu} (\sum_{i=1}^n \|A_i\|_p^\mu + \sum_{i=1}^n \|B_i\|_p^\mu)^{1/\mu} &\leq n^{1/2} (\sum_{i=1}^n \|A_i\|_p^2 + \sum_{i=1}^n \|B_i\|_p^2)^{1/2} \\ &\leq n^{2(1/p-1/\lambda)} (\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda)^{1/\lambda}. \qquad (1.3) \end{aligned}$$

For $2 \leq p,\ 0 < \lambda \leq p$ and $2 \leq \mu$, the inequalities are reversed.

Theorem 1.2 ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leq 2,\ p \leq \lambda$ and $0 < \mu \leq 2$,

$$n(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu)^{1/\mu} \leq n^{1/2+1/p-1/\lambda} (\sum_{i=1}^n \|(|A_i|^2 + |B_i|^2)^{1/2}\|_p^\lambda)^{1/\lambda}. \qquad (1.4)$$

For $2 \leq p,\ 0 < \lambda \leq p$ and $2 \leq \mu$, the inequality is reversed.

In this paper, motivated by the above conclusions, we consider some refinements of p-Schatten norm inequalities when p, λ and μ taking values in different regions.

2. Main results

In this section we consider the p-Schatten norm inequalities of (1.3) and (1.4) when parameters taking values in different regions. We start our works with the following lemmas that we will use along the paper.

Fact 1. $M_s(\overline{x}) \leq M_{s'}(\overline{x})$ for $0 < s \leq s'$, where $M_s(\overline{x}) = (\frac{1}{n} \sum_{i=1}^n x_i^s)^{1/s}$, $\overline{x} = (x_1, \cdots, x_n)$ is an n-tuples of non-negative numbers.

Fact 2. $\|T\|_p^2 = \||T|^2\|_{p/2}$ for any $T \in B_p(H)$ with $p > 0$.

Lemma 2.1 ([4]). Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$, then

$$\begin{aligned} \sum_{i,j=1}^n |A_i \pm B_j|^2 &= \sum_{i,j=1}^n |A_i|^2 + |B_j|^2 \pm \sum_{i,j=1}^n A_i^* B_j + B_j^* A_i \\ &= \sum_{i,j=1}^n |A_i|^2 + |B_j|^2. \qquad (2.1) \end{aligned}$$

Lemma 2.2 ([2-3]). If $A_1, \cdots, A_n \in B_p(H)$ for some $p > 0$, and $A_1, \cdots, A_n$ are positive, then for $0 < p \leq 1$,

$$n^{p-1}\sum_{i=1}^{n}\|A_i\|_p^p \leq (\sum_{i=1}^{n}\|A_i\|_p)^p \leq \|\sum_{i=1}^{n}A_i\|_p^p \leq \sum_{i=1}^{n}\|A_i\|_p^p \qquad (2.2)$$

and for $1 \leq p < \infty$ the inequalities are reversed.

They are refinements of Lemma 2.1 in [7]. A commutative version of the previous lemma for scalars is the following:

Let $\overline{x} = (x_1, ..., x_n)$ be an n-tuples of non-negative numbers, then

$$n^{p-1}\sum_{i=1}^{n}x_i^p \leq (\sum_{i=1}^{n}x_i)^p \leq \sum_{i=1}^{n}x_i^p \qquad (2.3)$$

for $0 < p \leq 1$ and

$$\sum_{i=1}^{n}x_i^p \leq (\sum_{i=1}^{n}x_i)^p \leq n^{p-1}\sum_{i=1}^{n}x_i^p \qquad (2.4)$$

for $1 \leq p < \infty$.

Theorem 2.3. Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^{n}A_i^*B_j = 0$. Then for $p \geq 2$, $p \leq \lambda$ and $\mu \geq 2$,

$$\begin{aligned} 2^{1/p-\mu/4}n^{3/p-\mu/4-1/2}(\sum_{i=1}^{n}\|A_i\|_p^{4/\mu} + \sum_{i=1}^{n}\|B_i\|_p^{4/\mu})^{\mu/4} &\leq n^{2/p-1/2}\|\sum_{i=1}^{n}|A_i|^2 + \sum_{i=1}^{n}|B_i|^2\|_{p/2}^{1/2} \\ &\leq n^{2(1/p-1/\lambda)}(\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^{\lambda})^{1/\lambda}. \end{aligned}$$

For $0 < p \leq 2$, $p \geq \lambda > 0$ and $0 < \mu \leq 2$, the inequalities are reversed.

Proof. Let $p \geq 2$, $p \leq \lambda$, $\mu \geq 2$. It follows from $M_p(\overline{x}) \leq M_\lambda(\overline{x})$ that

$$\begin{aligned} n^{2(1/p-1/\lambda)}(\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^{\lambda})^{1/\lambda} &= n^{2/p}(\frac{1}{n^2}\sum_{i,j=1}^{n}\||A_i \pm B_j\||_p^{\lambda})^{1/\lambda} \\ &\geq (\sum_{i,j=1}^{n}\||A_i \pm B_j\||_p^{p})^{1/p}. \end{aligned}$$

Applying the well-known fact that $\|T\|_p^2 = \||T|^2\|_{p/2}$ for any $T \in B_p(H)$ with $p > 0$ and Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} (\sum_{i,j=1}^{n}\||A_i \pm B_j\||_p^{p})^{1/p} &= (\sum_{i,j=1}^{n}\||A_i \pm B_j|^2\|_{p/2}^{p/2})^{1/p} \\ &\geq [(n^2)^{1-p/2}\|\sum_{i,j=1}^{n}|A_i \pm B_j|^2\|_{p/2}^{p/2}]^{1/p} \\ &= n^{2/p-1}\|\sum_{i,j=1}^{n}|A_i|^2 + |B_j|^2\|_{p/2}^{1/2} \\ &= n^{2/p-1/2}\|\sum_{i=1}^{n}|A_i|^2 + \sum_{i=1}^{n}|B_i|^2\|_{p/2}^{1/2}. \end{aligned}$$

Using Lemma 2.2, (2.3) and the concavity of the function $f(x) = x^\alpha$ on $[0, +\infty)$ for $0 < \alpha \leq 1$, we obtain

$$\begin{aligned}
& n^{2/p-1/2}\|\sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2\|_{p/2}^{1/2} \\
= \; & n^{2/p-1/2}(\|\sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2\|_{p/2}^{2/\mu})^{\mu/4} \\
\geq \; & n^{2/p-1/2}\{[(\sum_{i=1}^n \||A_i|^2\|_{p/2}^{p/2} + \sum_{i=1}^n \||B_i|^2\|_{p/2}^{p/2})^{2/p}]^{2/\mu}\}^{\mu/4} \qquad \text{by Lemma 2.2} \\
\geq \; & n^{2/p-1/2}\{[(2n)^{2/p-1}(\sum_{i=1}^n \||A_i|^2\|_{p/2} + \sum_{i=1}^n \||B_i|^2\|_{p/2})]^{2/\mu}\}^{\mu/4} \qquad \text{by (2.3)} \\
= \; & n^{2/p-1/2}(2n)^{1/p-1/2}\{(2n)^{2/\mu}[\frac{1}{2n}(\sum_{i=1}^n \||A_i|^2\|_{p/2} + \sum_{i=1}^n \||B_i|^2\|_{p/2})]^{2/\mu}\}^{\mu/4} \\
\geq \; & n^{2/p-1/2}(2n)^{1/p-1/2}(2n)^{1/2}[\frac{1}{2n}(\sum_{i=1}^n \||A_i|^2\|_{p/2}^{2/\mu} + \sum_{i=1}^n \||B_i|^2\|_{p/2}^{2/\mu})]^{\mu/4} \\
= \; & n^{2/p-1/2}(2n)^{1/p-1/2}(2n)^{1/2}(2n)^{-\mu/4}(\sum_{i=1}^n \|A_i\|_p^{4/\mu} + \sum_{i=1}^n \|B_i\|_p^{4/\mu})^{\mu/4} \\
= \; & 2^{1/p-\mu/4}n^{3/p-\mu/4-1/2}(\sum_{i=1}^n \|A_i\|_p^{4/\mu} + \sum_{i=1}^n \|B_i\|_p^{4/\mu})^{\mu/4}.
\end{aligned}$$

Let $0 < p \leq 2$, $p \geq \lambda$ and $0 < \mu \leq 2$. We can prove the inequalities by the same ways.

Corollary 2.4. Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then for $p \geq 2$,

$$\begin{aligned}
2^{1/p-p/4}n^{3/p-p/4-1/2}(\sum_{i=1}^n \|A_i\|_p^{4/p} + \sum_{i=1}^n \|B_i\|_p^{4/p})^{p/4} \; & \leq \; n^{2/p-1/2}\|\sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2\|_{p/2}^{1/2} \\
& \leq \; (\sum_{i,j=1}^n \|A_i \pm B_j\|_p^p)^{1/p}.
\end{aligned}$$

For $0 < p \leq 2$, the inequalities are reversed.
Proof. Motivated by Theorem 2.3, let $\lambda = \mu = p$.

Corollary 2.5. Let $A_1, \cdots, A_n \in B_p(H)$ such that $\sum_{i=1}^n A_i = 0$. Then for $p \geq 2$

$$2^{1/p}n^{3/p-p/4-1/2}(\sum_{i=1}^n \|A_i\|_p^{4/p})^{p/4} \leq 2^{1/2}n^{2/p-1/2}\|\sum_{i=1}^n |A_i|^2\|_{p/2}^{1/2} \leq (\sum_{i,j=1}^n \|A_i \pm A_j\|_p^p)^{1/p}.$$

For $0 < p \leq 2$, the inequalities are reversed.
Proof. $\sum_{i=1}^n A_i = 0$ implies that $\sum_{i,j=1}^n A_i^* A_j = 0$. The statement is a consequence of Corollary 2.4.

Theorem 2.6. Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in B_p(H)$ such that $\Sigma_{i,j=1}^n A_i^* B_j = 0$. Then for $p \geq 2$, $p \leq \lambda$ and $\mu \geq 2$,

$$n(1/n^2 \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu)^{1/\mu} \geq n^{1/2}(\sum_{i=1}^n \|(|A_i|^2 + |B_i|^2)^{1/2}\|_p^\lambda)^{1/\lambda}.$$

For $0 < p \leq 2$, $p \geq \lambda > 0$ and $0 < \mu \leq 2$, the inequality is reversed.

Proof. We suppose that $p \geq 2,\ p \leq \lambda$ and $\mu \geq 2$. Then by Lemma 2.2 and the convexity of the function $f(x) = x^{\alpha}$ on $[0, +\infty)$ for $\alpha \geq 1$

$$
\begin{aligned}
n(1/n^2 \sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^{\mu})^{1/\mu} &= n(\frac{1}{n^2} \sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2}^{\mu/2})^{1/\mu} \\
&\geq n[(\frac{1}{n^2} \sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2})^{\mu/2}]^{1/\mu} \\
&= (\sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2})^{1/2} \\
&\geq \|\sum_{i,j=1}^{n} |A_i \pm B_j|^2\|_{p/2}^{1/2} \qquad \text{by Lemma 2.2} \\
&= \|\sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2\|_{p/2}^{1/2} \\
&= n^{1/2}\|\sum_{i=1}^{n} (|A_i|^2 + |B_i|^2)\|_{p/2}^{1/2} \\
&\geq n^{1/2}(\sum_{i=1}^{n} \||A_i|^2 + |B_i|^2\|_{p/2}^{p/2})^{1/p} \qquad \text{by Lemma 2.2} \\
&= n^{1/2}[\sum_{i=1}^{n} (\|(|A_i|^2 + |B_i|^2)^{1/2}\|_p^{\lambda})^{p/\lambda}]^{1/p} \\
&\geq n^{1/2}[(\sum_{i=1}^{n} \|(|A_i|^2 + |B_i|^2)^{1/2}\|_p^{\lambda})^{p/\lambda}]^{1/p} \quad \text{by (2.4)} \\
&= n^{1/2}(\sum_{i=1}^{n} \|(|A_i|^2 + |B_i|^2)^{1/2}\|_p^{\lambda})^{1/\lambda}.
\end{aligned}
$$

Let $0 < p \leq 2,\ p \geq \lambda > 0$ and $0 < \mu \leq 2$. We can prove the inequality by the same ways.

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